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ITERATIVE TECHNIQUES WITH INITIAL TIME DIFFERENCE AND COMPUTER REALIZATION FOR THE INITIAL VALUE PROBLEM FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

Abstract. Some algorithms are given and applied in an appropriate computer environment to solve approximately the initial value problem for scalar nonlinear Caputo fractional differential equations on a finite interval. Various schemes for constructing successive approximations are suggested. They do not use Mittag–Leffler functions and as a result the practical application of the algorithms is easier. Several particular initial value problems for Caputo fractional differential equations are given to illustrate the advantages of the iterative techniques.*

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1 Introduction

Various processes with anomalous dynamics in science and engineering can be formulated mathematically using fractional differential operators because of its memory and hereditary properties [6, 13]. There is only a small number of fractional differential equations, including linear equations with variable coefficients, which can be solved in closed form and this causes some problems in practical applications.

This paper considers an initial value problem for a nonlinear scalar Caputo fractional differential equation on a closed interval. Several iterative techniques combined with the method of lower and upper solutions are applied to find the approximate solution of the given problem. Mild lower and mild upper solutions are defined. Several algorithms for constructing two convergent monotone functional sequences are given and we prove that both sequences converge and their limits are minimal and maximal solutions of the problem. When the right-hand side of the equations are monotone functions with respect to the time variable, the elements of these sequences do not depend on Mittag–Leffler functions and they can be obtained in closed form with the help of an appropriate software such as Wolfram Mathematica.

We note that iterative techniques combined with lower and upper solutions are applied in the literature to approximately solve various problems in ordinary differential equations [11], second order periodic boundary value problems [5], differential equations with maxima [1, 7], difference equations with maxima [3], impulsive integro-differential equations [8], impulsive differential equations with supremum [9], differential equations of mixed type [10], and Riemann–Liouville fractional differential equations [4, 16].

2 Preliminary and auxiliary results

The Caputo fractional derivative of order $q \in (0,1)$ is defined by (see, for example, [13])

$$ {}_{t_0}^{c}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \int\limits_{t_0}^{t} (t-s)^{-q} m'(s)\, ds, \quad t \geq t_0. \tag{2.1}$$

Let t_0 be an arbitrary initial time. Usually we think of the independent variable t as time in differential equations, so we will assume $t_0 \in \mathbb{R}_+$.

Definition 2.1 ([15])**.** We say $m(t) \in C^q([t_0, T], \mathbb{R}^n)$ if $m(t)$ is differentiable (i.e., $m'(t)$ exists), the Caputo derivative ${}_{t_0}^{C}D^q m(t)$ exists and satisfies (1) for $t \in [t_0, T]$.

Consider the initial value problem (IVP) for the nonlinear *Caputo-type fractional differential equation* (FrDE)

$$\begin{aligned} {}_{t_0}^{C}D_t^q x(t) &= f(t, x(t)) \ \text{ for } \ t \in [t_0, t_0 + T], \\ x(t_0) &= x_0, \end{aligned} \tag{2.2}$$

where $q \in (0,1)$, $x_0 \in \mathbb{R}$, $f : [t_0, t_0 + T] \times \mathbb{R} \to \mathbb{R}$, $x : [t_0, t_0 + T] \to \mathbb{R}$.

Any solution $x = x(t)$ of the IVP for FrDE (2.2) satisfies $x \in C^q([t_0, t_0 + T], \mathbb{R})$.

If $x(t)$ is a solution of the IVP for FrDE (2.2), then it satisfies the following Volterra integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int\limits_{t_0}^{t} (t-s)^{q-1} f(s, x(s))\, ds \ \text{ for } \ t \in [t_0, t_0 + T] \tag{2.3}$$

and, conversely, if $x \in C^q([t_0, t_0 + T], \mathbb{R})$ is a solution of (2.3), then it is a solution of the IVP for FrDE (2.2).

Definition 2.2. We say that the function $x \in C([t_0, t_0 + T], \mathbb{R})$ is *a mild solution* of the IVP for FrDE (2.2) if it satisfies equation (2.3).

Remark 2.1. The mild solution $x(t)$ of the IVP for FrDE (2.2) might not have a fractional derivative ${}_{t_0}^{C}D_t^q x(t)$.

Let $\tau_0 \in \mathbb{R}_+$, $\tau_0 \neq t_0$ be a different initial time. Consider the following IVP for FrDE similar to (2.2) but with a different initial time (ITD):

$$ {}_{\tau_0}^{c}D_t^q x(t) = f(t, x(t)) \ \text{ for } \ t \in [\tau_0, \tau_0 + T], \quad x(\tau_0) = x_0. \tag{2.4} $$

The change of the initial time reflects not only the initial condition but also the fractional derivative on the solution.

Example 2.1. Let $t_0 = 0$, $\tau_0 = 1$, $q \in (0,1)$ and consider two initial value problems with initial time difference for scalar Caputo fractional differential equations

$$ {}_{0}^{c}D_t^q x(t) = \frac{t^{1-q}}{\Gamma(2-q)} \ \text{ for } \ t > 0, \quad x(0) = x_0 \tag{2.5} $$

and

$$ {}_{1}^{c}D_t^q x(t) = x\,\frac{t^{-q}}{\Gamma(2-q)} \ \text{ for } \ t > 1, \quad x(1) = x_0. \tag{2.6} $$

Let $f(t,x) = x\,\frac{t^{-q}}{\Gamma(2-q)}$.

The solution of (2.5) with $x_0 = 0$ is $x(t) = t$, $t \geq 0$. The solution of (2.6) with $x_0 = 0$ is given by

$$ \widetilde{x}(t) = \frac{1}{\Gamma(q)} \int_1^t (t-s)^{q-1} s\, ds \neq t = x(t), \ \ t \geq 1. $$

Therefore, the shift of the fractional derivative changes the solution.

Note that for $y(t) = t-1 = x(t-1)$ we get ${}_{1}^{c}D_t^q y(t) = \frac{(t-1)^{1-q}}{\Gamma(2-q)}$, i.e., ${}_{1}^{c}D_t^q y(t) = f(t-1, y(t))$. This result is theoretically proved in the following Lemma.

Lemma 2.1 ([2, Lemma 3.1] (Shift solutions in FrDE))**.** *Let the function $x \in C^q(\mathbb{R}_+, \mathbb{R}^n)$, $a \geq 0$, be a solution of the initial value problem for FrDE*

$$ {}_{a}^{c}D_t^q x(t) = f(t, x(t)) \ \text{ for } \ t > a, \quad x(a) = x_0. \tag{2.7} $$

Then the function $\widetilde{x}(t) = x(t+\eta)$ satisfies the initial value problem for the FrDE

$$ {}_{b}^{c}D_t^q \widetilde{x}(t) = f(t+\eta, \widetilde{x}(t)) \ \text{ for } \ t > b, \quad \widetilde{x}(b) = x_0, \tag{2.8} $$

where $b \geq 0$, $\eta = a - b$.

Remark 2.2. Let $y(t)$ be a solution of the IVP for FrDE (2.4) for $t \geq \tau_0$. Then according to Lemma 2.1, ${}_{t_0}^{c}D_t^q y(t+\eta) = f(t+\eta, y(t+\eta))$ with $\eta = \tau_0 - t_0$.

Let $\theta_0 \in \mathbb{R}_+$, $\theta_0 \neq t_0$, $\theta_0 \neq \tau_0$, be a different initial time. Consider the following IVP for FrDE

$$ {}_{\theta_0}^{c}D_t^q x(t) = f(t, x(t)) \ \text{ for } \ t \in [\theta_0, \theta_0 + T], \quad x(\theta_0) = x_0. \tag{2.9} $$

Note that the IVP for FrDE (2.9) is similar to (2.2) and (2.4) but with different initial times and, proceeding from the above, they may have different solutions in spite of the same initial value.

3 Mild lower and mild upper solutions of FrDE

Following the ideas in [12], we present various types of lower/upper solutions of FrDEs.

Definition 3.1. We say that the function $v \in C([t_0, t_0+T], \mathbb{R})$ is a minimal (maximal) solution of the IVP for FrDE (2.2) if it is a solution of (2.2) and for any solution $u \in C([t_0, t_0+T], \mathbb{R})$ of (2.2) the inequality $v(t) \leq u(t)$ ($v(t) \geq u(t)$) holds on $[t_0, t_0+T]$.

For any point $t_0 \geq 0$ and any function $\xi \in C([t_0, t_0 + T])$ we define the operator Δ by

$$\Delta(t_0, \xi)(t) = \xi(t_0) + \frac{1}{\Gamma(q)} \int\limits_{t_0}^{t} (t - s)^{q-1} f(s, \xi(s))\, ds, \quad t \in [t_0, t_0 + T]. \tag{3.1}$$

Remark 3.1. According to Definition 2.2, any mild solution $x(t)$ of the IVP for FrDE (2.2) is a fixed point of the operator Δ. Any fixed point $\xi \in C([t_0, t_0 + T])$ of the operator Δ is a mild solution the IVP for FrDE (2.2) if $\xi(t_0) = x_0$.

Similar to Definition 2.7 [12], we present the following definition.

Definition 3.2. We say that the function $v \in C([t_0, t_0 + T], \mathbb{R})$ is a mild lower (a mild upper) solution in $[t_0, t_0 + T]$ of the IVP for FrDE (2.2) if

$$\begin{aligned} v(t) &\leq \ (\geq)\ \Delta(t_0, v)(t) \ \text{ for } \ t \in [t_0, t_0 + T], \\ v(t_0) &\leq \ (\geq)\ x_0. \end{aligned} \tag{3.2}$$

Remark 3.2. A mild lower/upper solution of the IVP for FrDE (2.4) and (2.9), respectively, are defined by Definition 3.2 where the initial time point t_0 is replaced by τ_0 and θ_0, respectively.

Lemma 3.1. *Let the function $f \in C([t_0, t_0 + T] \times \mathbb{R}, \mathbb{R})$ be nondecreasing in its second argument, $x(t)$ be a mild solution of the IVP for FrDE (2.2) and $v(t)$ be a mild lower solution on $[t_0, t_0 + T]$ of (2.2) such that $v(t_0) < x_0$. Then $v(t) < x(t)$ on $[t_0, t_0 + T]$.*

Proof. Assume that the claim is not true. Therefore, there exists a point $t^* \in (t_0, t_0 + T)$ such that

$$v(t) < x(t), \ \ t \in [t_0, t^*), \quad v(t^*) = x(t^*) \ \text{ and } v(t) \geq x(t), \ \ t \in (t^*, t^* + \delta),$$

where δ is a small enough positive number. Using the monotonic property of the function f, we obtain

$$\begin{aligned} x(t^*) = v(t^*) \leq v(t_0) + \frac{1}{\Gamma(q)} \int\limits_{t_0}^{t^*} (t - s)^{q-1} f(s, v(s))\, ds \\ < x_0 + \frac{1}{\Gamma(q)} \int\limits_{t_0}^{t^*} (t - s)^{q-1} f(s, x(s))\, ds = x(t^*), \end{aligned} \tag{3.3}$$

which is a contradiction. Therefore, $v(t) < x(t)$ on $[t_0, t_0 + T]$. □

Similar to Lemma 3.1, we have the following result.

Lemma 3.2. *Let the function $f \in C([t_0, t_0 + T] \times \mathbb{R}, \mathbb{R})$ be nondecreasing in its second argument, $x(t)$ be a mild solution of the IVP for FrDE (2.2) and $w(t)$ be a mild upper solution on $[t_0, t_0 + T]$ of (2.2) such that $w(t_0) > x_0$. Then $w(t) > x(t)$ on $[t_0, t_0 + T]$.*

Lemma 3.3. *Let $\theta_0 < t_0$ and the function $f \in C(([\theta_0, \theta_0 + T] \cup [t_0, t_0 + T]) \times \mathbb{R}, \mathbb{R})$ be nondecreasing in both its arguments, $x(t)$ be a mild solution of the IVP for FrDE (2.2) and $v(t)$ be a mild lower solution on $[\theta_0, \theta_0 + T]$ of (2.9) such that $v(\theta_0) < x_0$. Then $v(t - \eta) < x(t)$ on $[t_0, t_0 + T]$, where $\eta = t_0 - \theta_0 > 0$.*

Proof. From Definition 3.2 and Remark 3.2, we have

$$v(t) \leq v(\theta_0) + \frac{1}{\Gamma(q)} \int\limits_{\theta_0}^{t} (t - s)^{q-1} f(s, v(s))\, ds, \quad t \in [\theta_0, \theta_0 + T], \tag{3.4}$$

or

$$v(t-\eta) \le v(\theta_0) + \frac{1}{\Gamma(q)} \int\limits_{\theta_0}^{t-\eta} (t-\eta-s)^{q-1} f(s, v(s))\, ds, \quad t \in [t_0, t_0+T]. \tag{3.5}$$

Applying the substitution $\nu = s+\eta$ to equation (3.5), we obtain

$$v(t-\eta) \le v(t_0-\eta) + \frac{1}{\Gamma(q)} \int\limits_{t_0}^{t} (t-\nu)^{q-1} f(\nu-\eta, v(\nu-\eta))\, d\nu \ \text{ for } t \in [t_0, t_0+T].$$

Define $\widetilde{v}(t) = v(t-\eta) \in C([t_0, t_0+T], \mathbb{R})$. Therefore $\widetilde{v}(t_0) = v(t_0-\eta) = v(\theta_0) < x_0$ and

$$\widetilde{v}(t) \le \widetilde{v}(t_0) + \frac{1}{\Gamma(q)} \int\limits_{t_0}^{t} (t-\nu)^{q-1} f(\nu-\eta, \widetilde{v}(\nu))\, d\nu \le \widetilde{v}(t_0) + \frac{1}{\Gamma(q)} \int\limits_{t_0}^{t} (t-\nu)^{q-1} f(\nu, \widetilde{v}(\nu))\, d\nu,$$

i.e., the function $\widetilde{v}(t)$ is a mild lower solution on $[t_0, t_0+T]$ of the IVP for FrDE (2.2).

According to Lemma 3.1, the inequality $\widetilde{v}(t) < x(t)$ holds on $[t_0, t_0+T]$. □

The proof of the following result is similar to that in Lemma 3.3 so we omit it.

Lemma 3.4. *Let $t_0 < \theta_0$ and the function $f \in C(([\theta_0, \theta_0+T] \cup [t_0, t_0+T]) \times \mathbb{R}, \mathbb{R})$ be nonincreasing in its first argument and nondecreasing in its second argument, $x(t)$ be a mild solution of the IVP for FrDE* (2.2) *and $v(t)$ be a mild lower solution on $[\theta_0, \theta_0+T]$ of* (2.4) *such that $v(\theta_0) > x_0$. Then $v(t-\eta) > x(t)$ on $[t_0, t_0+T]$, where $\eta = t_0 - \theta_0 < 0$.*

Similar to Lemma 3.3, using Lemma 3.2 instead of Lemma 3.1, we have the following results.

Lemma 3.5. *Let $t_0 < \tau_0$ and the function $f \in C(([\tau_0, \tau_0+T] \cup [t_0, t_0+T]) \times \mathbb{R}, \mathbb{R})$ be nondecreasing in both its arguments, $x(t)$ be a mild solution of the IVP for FrDE* (2.2) *and $w(t)$ be a mild upper solution on $[\tau_0, \tau_0+T]$ of* (2.4) *such that $w(\tau_0) > x_0$. Then $w(t+\xi) > x(t)$ on $[t_0, t_0+T]$, where $\xi = \tau_0 - t_0 > 0$.*

Lemma 3.6. *Let $t_0 > \tau_0$ and the function $f \in C(([\tau_0, \tau_0+T] \cup [t_0, t_0+T]) \times \mathbb{R}, \mathbb{R})$ be nonincreasing in its first argument and nondecreasing in its second argument, $x(t)$ be a mild solution of the IVP for FrDE* (2.2) *and $w(t)$ be a mild upper solution on $[\tau_0, \tau_0+T]$ of* (2.4) *such that $w(\tau_0) > x_0$. Then $w(t+\xi) > x(t)$ on $[t_0, t_0+T]$, where $\xi = \tau_0 - t_0 < 0$.*

4 Main results

We study the case when the IVP for FrDE (2.2), defined on $[t_0, t_0+T]$, has a mild lower and a mild upper solutions defined on different intervals. We will call this case the initial time difference (ITD). In the case when the right-hand side of the FrDE is a monotonic function we present two algorithms for constructing successive approximations to the solution of the IVP for FrDE (2.2).

Case 1. The initial time of the mild lower solution is less than the initial time of the mild upper solution.

Theorem 4.1. *Let the following conditions be fulfilled:*

(1) *Let the points $\theta_0, t_0, \tau_0 : \ 0 \le \theta_0 \le t_0 \le \tau_0$ be given and the function $v \in C([\theta_0, \theta_0+T])$ be a mild lower solution of the IVP for FrDE* (2.9) *on the interval $[\theta_0, \theta_0+T]$ such that $v(\theta_0) < x_0$ and the function $w \in C([\tau_0, \tau_0+T])$ be a mild upper solution of the IVP for FrDE* (2.4) *on the interval $[\tau_0, \tau_0+T]$ such that $w(\tau_0) > x_0$. Let, additionally, $v(t-\eta) \le w(t+\xi)$ for $t \in [t_0, t_0+T]$, where $\eta = t_0 - \theta_0 \ge 0, \ \xi = \tau_0 - t_0 > 0$.*

(2) *The function $f \in C(([\theta_0, \theta_0+T] \cup [t_0, t_0+T] \cup [\tau_0, \tau_0+T]) \times \mathbb{R}, \mathbb{R})$ and it is nondecreasing in both its arguments.*

Then there exist two sequences of functions $\{v^{(n)}(t)\}_0^\infty$ *and* $\{w^{(n)}(t)\}_0^\infty$, $t \in [t_0, t_0+T]$, *such that:*

(a) *The sequences are defined by* $v^{(0)}(t) = v(t-\eta)$, $w^{(0)}(t) = w(t+\xi)$ *and for* $n \geq 1$

$$\begin{aligned} v^{(n)}(t) &= \Delta(t_0, v^{(n-1)}) + x_0 - v^{(n-1)}(t_0) \\ &\equiv x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(n-1)}(s))\, ds \;\; \text{for } \; t \in [t_0, t_0+T] \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} w^{(n)}(t) &= \Delta(t_0, w^{(n-1)}) + x_0 - w^{(n-1)}(t_0) \\ &\equiv x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, w^{(n-1)}(s))\, ds \;\; \text{for } \; t \in [t_0, t_0+T]. \end{aligned} \tag{4.2}$$

(b) *The functions* $v^{(n)}(t)$, $n = 0, 1, 2\ldots$, *are mild lower solutions of the IVP for FrDE* (2.2).

(c) *The functions* $w^{(n)}(t)$, $n = 0, 1, 2\ldots$, *are mild upper solutions of the IVP for FrDE* (2.2).

(d) *The sequence* $\{v^{(n)}(t)\}$ *is increasing on* $[t_0, t_0+T]$, *i.e.,* $v^{(k-1)}(t) \leq v^{(k)}(t)$ *for* $t \in [t_0, t_0+T]$, $k = 1, 2, \ldots$.

(e) *The sequence* $\{w^{(n)}(t)\}$ *is decreasing on* $[t_0, t_0+T]$, *i.e.,* $w^{(k-1)}(t) \geq w^{(k)}(t)$ *for* $t \in [t_0, t_0+T]$, $k = 1, 2, \ldots$.

(f) *The inequality*

$$v^{(k)}(t) \leq w^{(k)}(t) \;\; \text{for } \; t \in [t_0, t_0+T], \;\; k = 1, 2, \ldots, \tag{4.3}$$

holds.

(g) *Both sequences converge on* $[t_0, t_0+T]$ *and*

$$V(t) = \lim_{k\to\infty} v^{(n)}(t), \quad W(t) = \lim_{k\to\infty} w^{(n)}(t), \;\; t \in [t_0, t_0+T].$$

(h) *The limit functions* $V(t)$ *and* $W(t)$ *are mild solutions of the IVP for FrDE* (2.2) *on* $[t_0, t_0+T]$.

(i) *For any mild solution* $x(t)$ *of IVP for FrDE* (2.2) *the inequalities* $V(t) \leq x(t) \leq W(t)$ *for* $t \in [t_0, t_0+T]$ *hold, i.e., the functions* $V(t)$, $W(t)$ *are mild minimal and maximal solutions.*

Proof. According to Lemma 3.3 and Lemma 3.5, if there exists a solution $x(t)$ in $[t_0, t_0+T]$ of the IVP for FrDE (2.2), then $v(t-\eta) < x(t) < w(t-\xi)$ for $t \in [t_0, t_0+T]$. We now prove the existence of the solution and will give an algorithm for obtaining it.

Define $v^{(0)}(t) = v(t-\eta)$ and $w^{(0)}(t) = w(t+\xi)$ for $t \in [t_0, t_0+T]$. Then $v^{(0)}(t_0) = v(\theta_0) < x_0$ and $w^{(0)}(t_0) = w(\tau_0) > x_0$.

Then applying the substitution $\nu = s + \eta$, we get

$$\begin{aligned} v^{(0)}(t) &= v(t-\eta) \leq v(\theta_0) + \frac{1}{\Gamma(q)} \int_{\theta_0}^{t-\eta} (t-\eta-s)^{q-1} f(s, v(s))\, ds \\ &= v(t_0-\eta) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-\nu)^{q-1} f(\nu-\eta, v(\nu-\eta))\, d\nu = v^{(0)}(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-\nu)^{q-1} f(\nu-\eta, v^{(0)}(\nu))\, d\nu \\ &\leq v^{(0)}(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-\nu)^{q-1} f(\nu, v^{(0)}(\nu))\, d\nu, \;\; t \in [t_0, t_0+T], \end{aligned} \tag{4.4}$$

Therefore, the function $v^{(0)}(t)$ is a mild lower solution on $[t_0, t_0+T]$ of the IVP for FrDE (2.2). Similarly, we prove that the function $w^{(0)}(t)$ is a mild upper solution on $[t_0, t_0+T]$ of the IVP for FrDE (2.2).

We use induction to prove the properties of sequences of successive approximations.

Let $n = 1$. From equation (4.1) we get $v^{(1)}(t_0) = x_0$ and applying the monotonic properties of the function f, we obtain

$$\begin{aligned} v^{(1)}(t) &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(0)}(s))\, ds \\ &\le v^{(1)}(\theta_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(1)}(s))\, ds, \quad t \in [t_0, t_0+T], \end{aligned} \tag{4.5}$$

i.e., the function $v^{(1)}(t)$ is a mild lower solution of the IVP for FrDE (2.9). Also,

$$\begin{aligned} v^{(0)}(t) = v(t-\eta) &= v^{(0)}(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-\nu)^{q-1} f(\nu-\eta, v^{(0)}(\nu))\, d\nu \\ &\le x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-\nu)^{q-1} f(\nu, v^{(0)}(\nu))\, d\nu = v^{(1)}(t), \quad t \in [t_0, t_0+T]. \end{aligned} \tag{4.6}$$

Assume $v^{(k-1)}(t) \le v^{(k)}(t),\ k \ge 1$ and $v^{(k)}(t)$ is a mild lower solution of the IVP for FrDE (2.9). Then

$$\begin{aligned} v^{(k)}(t) &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(k-1)}(s))\, ds \\ &\le x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(k)}(s))\, ds = v^{(k+1)}(t), \quad t \in [t_0, t_0+T], \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} v^{(k+1)}(t) &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(k)}(s))\, ds \\ &\le v^{(k+1)}(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(k+1)}(s))\, ds, \quad t \in [t_0, t_0+T], \end{aligned} \tag{4.8}$$

i.e., the claims (b)–(e) are true.

By induction we prove the claim (f).

The sequences $\{v^{(n)}(t)\}_0^\infty$ and $\{w^{(n)}(t)\}_0^\infty$ being monotonic and bounded are uniformly convergent on the interval $[t_0, t_0+T]$. Let for $t \in [t_0, t_0+T]$ we denote

$$V(t) = \lim_{n\to\infty} v^{(n)}(t) \ \text{ and } \ W(t) = \lim_{n\to\infty} w^{(n)}(t).$$

According to (b), (c) and (d), the inequalities

$$\begin{gathered} v^{(n)}(t) \le V(t), \ \ t \in [t_0, t_0+T], \quad W(t) \le w^{(n)}(t), \ \ t \in [t_0, t_0+T], \ \ n = 0, 1, 2, \dots, \\ V(t) \le W(t), \ \ t \in [t_0, t_0+T], \end{gathered} \tag{4.9}$$

hold.

Taking the limit in (4.1) and (4.2) we obtain the Volterra fractional integral equations

$$
\begin{aligned}
V(t) &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, V(s))\, ds, \quad t \in [t_0, t_0+T], \\
W(t) &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, W(s))\, ds, \quad t \in [t_0, t_0+T].
\end{aligned}
\tag{4.10}
$$

Therefore, the limit functions $V(t)$ and $W(t)$ are mild solutions of the IVP for FrDE (2.2).

Let $x(t)$ be an arbitrary mild solution of the IVP for FrDE (2.2). According to Lemma 3.3 and Lemma 3.5, it follows that $v^{(0)} = v(t-\eta) < x(t) < w(t+\xi) = w^{(0)}(t)$ on $[t_0, t_0+T]$. Therefore, there exists a number $N \in \mathbb{N} \cup \{0\}$ such that $v^{(N)}(t) \leq x(t) \leq w^{(0)}(t)$ for $t \in [t_0, t_0+T]$. Then applying the monotonicity property of the function f and the choice of N, we obtain

$$
\begin{aligned}
v^{(N+1)}(t) &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(N)}(s))\, ds \\
&\leq x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))\, ds = x(t), \quad t \in [t_0, t_0+T].
\end{aligned}
\tag{4.11}
$$

Therefore, $V(t) \leq x(t)$, $t \in [t_0, t_0+T]$. The rest of the proof is similar and we omit it. □

In the special case when the right side part of the FrDE (2.2) does not depend on x, i.e., $f(t,x) \equiv f(t)$, we have the following result.

Corollary 4.1. *Let condition* (1) *of Theorem 4.1 be satisfied and the function f be nondecreasing. Then for all $n \geq 1$ the equalities $v^{(n)}(t) \equiv w^{(n)}(t) \equiv x(t)$, $t \in [t_0, t_0+T]$, hold, where the successive approximations $v^{(n)}(t)$ and $w^{(n)}(t)$ are defined by* (4.1) *and* (4.2).

Example 4.1. Let $\theta_0 = 0, t_0 = 0, \tau_0 = 2,\ T = 0.9$ and consider the IVP for the scalar Caputo FrDE

$$
{}_0^cD_t^{0.5} x(t) = x + 1 \ \text{ for } \ t \in [0, 0.9], \quad x(t_0) = x_0. \tag{4.12}
$$

Its solution is given by

$$
x(t) = x_0 E_{0.5}(t^{0.5}) + \int_0^t (t-s)^{-0.5} E_{0.5,0.5}((t-s)^{0.5})\, ds \ \text{ for } \ t \in [0, 0.9], \tag{4.13}
$$

where $E_q(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}$ and $E_{\alpha,\beta}(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$ are the Mittag–Leffler functions with one and two parameters, respectively. Note that the integral in (4.13) cannot be solved in closed form and the solution cannot be obtained as an expression of classical functions.

Now we apply the above technique to find approximately the solution as a limit of a sequence of explicit functions.

Let $x_0 = 0$.

The function $v(t) = t^7 - 0.01$ is a mild lower solution on $[0, 0.9]$ of the IVP for FrDE (4.12), since the inequality

$$
t^7 - 0.01 \leq \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5} (1 + s^7 - 0.01)\, ds \ \text{ for } \ t \in [0, 0.9] \tag{4.14}
$$

is satisfied (see Figure 1).

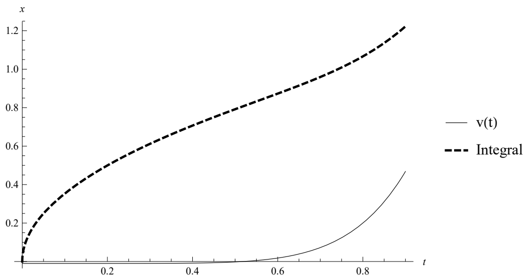

Figure 1. Example 2. Graph of the mild lower solution $t^7 - 0.01$ on $[0, 0.9]$.

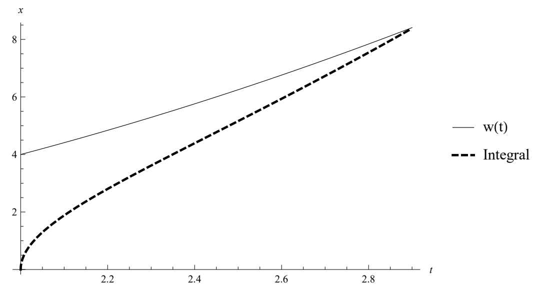

Figure 2. Example 2. Graph of the mild upper solution t^2 on $[2, 2.9]$.

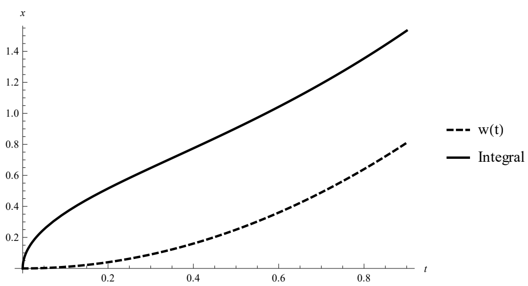

Figure 3. Example 2. Graph of the function t^2 on $[0, 0.9]$.

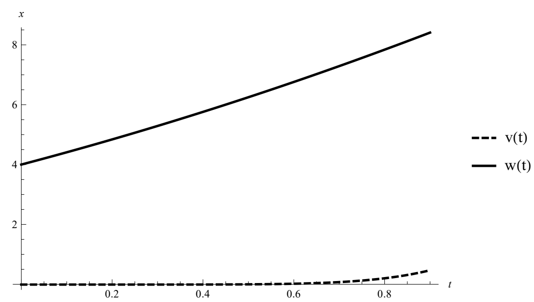

Figure 4. Exampe 2. Graph of the mild lower and mild upper solutions on $[0, 0.9]$.

Consider equation (4.12) with replaced $t_0 = 0$ by $\tau_0 = 2$. The function $w(t) = t^2$ is a mild upper solution on $[2, 2.9]$ of the IVP for FrDE (4.12) with $t_0 = 2$, since the inequality

$$t^2 \geq \frac{1}{\Gamma(0.5)} \int_2^t (t-s)^{-0.5}(1+s^2)\,ds \quad \text{for} \quad t \in [2, 2.9] \tag{4.15}$$

is satisfied (see Figure 2). Note that the function $w(t) = t^2$ is not a mild upper solution on $[0, 0.9]$ of the IVP for FrDE (4.12) (see Figure 3). At the same time, the inequalities $v(t) = t^7 - 0.01 \leq w(t+2) = (t+2)^2$ for $t \in [0, 0.9]$ hold (see Figure 4).

Define the zero lower and upper approximations by $v^{(0)}(t) = v(t) = t^7 - 0.01$ and $w^{(0)}(t) = w(t+2) = (t+2)^2$ for $t \in [0, 0.9]$.

From equation (4.1) we get

$$\begin{aligned}
v^{(1)}(t) &= 0 + \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5}(1+s^7-0.01)\,ds = \frac{1}{\Gamma(0.5)}\Big(0.99\,\frac{2}{3}\,t^{1.5} + \frac{4096}{109395}\,t^{8.5}\Big),\\
v^{(2)}(t) &= \frac{1}{(\Gamma(0.5))^2} \int_0^t (t-s)^{-0.5}\Big(\Gamma(0.5) + 0.99\,\frac{2}{3}\,s^{1.5} + \frac{4096}{109395}\,s^{8.5}\Big)\,ds\\
&= \frac{1}{(\Gamma(0.5))^2}\Big(\Gamma(0.5)\,\frac{2}{3}\,t^{1.5} + 0.129591t^3 + 0.00109083t^{10}\Big), \quad t \in [0, 0.9],
\end{aligned}$$

i.e., the successive approximations are polynomial functions and there is no problem in obtaining them in a closed form with the corresponding integrals.

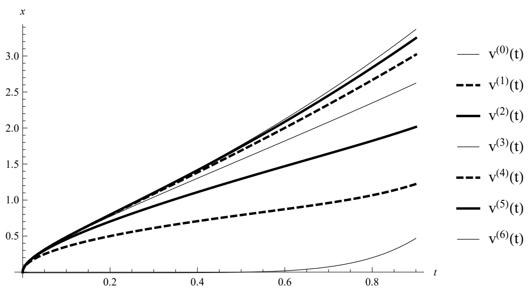

Figure 5. Example 2. Graph of the lower approximations defined by (4.1).

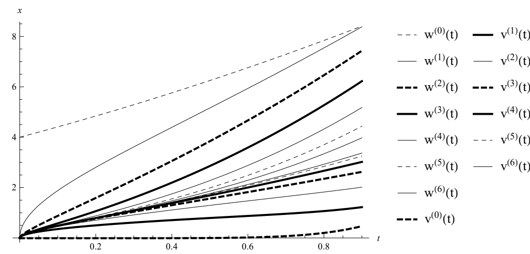

Figure 6. Example 2. Graph of the lower and upper approximations defined by (4.1) and (4.2).

Similarly, we get

$$\begin{aligned}w^{(1)}(t) &= 0 + \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5}(1+(s+2)^2)\, ds \\ &= \frac{1}{\Gamma(0.5)}\Big(\frac{10}{3}\, t^{1.5} + \frac{16}{105}\, t^{3.5} + \frac{16}{15}\, t^{2.5}\Big), \quad t \in [0, 0.9].\end{aligned}$$

Using the software Wolftam Mathematica 10.0 we obtain the successive approximations with the graphs given in Figures 5 and 6.

Then the approximate solution of the IVP for FrDE (4.12) is defined as

$$\begin{aligned}qx(t) = \frac{v^{(6)}(t) + w^{(6)}(t)}{2} &= 1.12838t^{0.5} + t^1 + 0.752253t^{1.5} + 0.5t^2 \\ &+ 0.300901t^{2.5} + 0.499167t^3 + 0.0833333t^4 + 0.00833333t^5 + 0.000694444t^{10}.\end{aligned}$$

Case 2. The initial time of the mild lower solution is greater than the initial time of the mild upper solution.

Theorem 4.2. *Let the following conditions be fulfilled:*

(1) *Let the points θ_0, t_0, τ_0: $0 \le \tau_0 \le t_0 \le \theta_0$ be given and the functions v, w: $v \in C([\theta_0, \theta_0 + T])$, $w \in C([\tau_0, \tau_0 + T])$ be lower and upper solutions of the IVP for NIFrDE* (2.2) *on the intervals $[\theta_0, \theta_0 + T]$ and $[\tau_0, \tau_0 + T]$, respectively. Let, additionally, $v(t + \eta) \le w(t)$ for $t \in [\tau_0, \tau_0 + T]$, where $\eta = \theta_0 - \tau_0 \ge 0$.*

(2) *The function $f \in C(([\theta_0, \theta_0 + T] \cup [t_0, t_0 + T] \cup [\tau_0, \tau_0 + T]) \times \mathbb{R}, \mathbb{R})$ is nonincreasing in its first argument t and it is nondecreasing in its second argument x.*

Then there exist two sequences of functions $\{v^{(n)}(t)\}_0^\infty$ and $\{w^{(n)}(t)\}_0^\infty$, defined by recurrence formulas (4.1) *and* (4.2)*, where $v^{(0)}(t) = v(t + \eta)$, $w^{(0)}(t) = w(t - \xi)$ and the claims* (b)–(i) *of Theorem* 4.1 *are true.*

Example 4.2. Consider the IVP for the scalar Caputo FrDE

$$ {}_1^cD_t^{0.3}x(t) = x^3 - \frac{t}{20} \ \text{ for } \ t \in [1, 3], \quad x(1) = 0. \tag{4.16}$$

Now we apply the above technique to find approximately the solution as a limit of two sequences of explicit functions.

In this case we can find mild lower and mild upper solutions of the IVP for FrDE (4.16) on the interval $[1, 3]$.

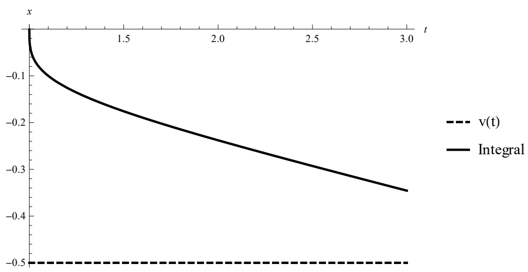

Figure 7. Example 3. Graph of the mild lower solution $v(t) = -0.5$ on $[1,3]$.

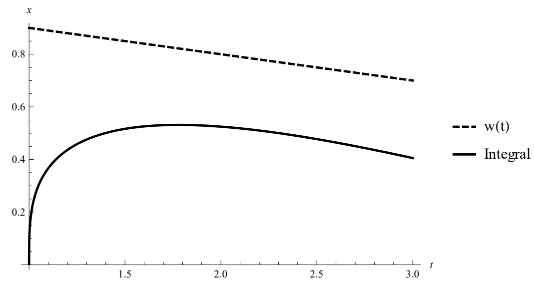

Figure 8. Example 3. Graph of the mild upper solution $w(t) = 1 - 0.1t$ on $[1,3]$.

The function $v(t) = -0.5$ is a mild lower solution on $[1,3]$ of the IVP for FrDE (4.16), since the inequality

$$-0.5 \leq \frac{1}{\Gamma(0.3)} \int_1^t (t-s)^{-0.7}\Big((-0.5)^3 - \frac{s}{20}\Big)\, ds \quad \text{for} \quad t \in [1,3] \tag{4.17}$$

is satisfied (see Figure 7).

The function $w(t) = 1 - 0.1t$ is a mild upper solution on $[1,3]$ of the IVP for FrDE (4.16), since the inequality

$$1 - 0.1t \geq \frac{1}{\Gamma(0.3)} \int_1^t (t-s)^{-0.7}\Big((1-0.1s)^3 - \frac{s}{20}\Big)\, ds \quad \text{for} \quad t \in [1,3] \tag{4.18}$$

is satisfied (see Figure 8).

Therefore, $\theta_0 = t_0 = \tau_0 = 1,\ T = 2$ and $\eta = \xi = 0$.

Define the zero approximation by $v^{(0)}(t) = v(t) = -0.5$, $w^{(0)}(t) = w(t) = 1 - 0.1t$, $t \in [1,3]$.

From equation (4.1) for $t \in [1,3]$ we get

$$\begin{aligned} v^{(1)}(t) &= 0 + \frac{1}{\Gamma(0.3)} \int_1^t (t-s)^{-0.7}\Big((-0.5)^3 - \frac{s}{20}\Big)\, ds \\ &= 0.334273(-2. + t)^{0.3}\Big(-0.998772 + t\big(0.703571 + t(-0.113519 + (0.00599603 - 0.000141416t)t)\big)\Big), \end{aligned}$$

i.e., the successive approximations are the polynomial functions and there is no problem solving in a closed form with the corresponding integrals.

Similarly, we get

$$w^{(1)}(t) = 0 + \frac{1}{\Gamma(0.3)} \int_1^t (t-s)^{-0.7}\Big((1-0.1s)^3 - \frac{s}{20}\Big)\, ds, \quad t \in [1,3].$$

The graphs of the mild lower and upper solutions, obtained by Wolfram Mathematica, which are successive approximations, are given in Figure 9.

Remark 4.1. Both algorithms given above in Theorems 4.1 and 4.2 and illustrated in Examples 4.1, 4.2 could be applied in the special case when both mild upper and mild lower solutions have one and the same initial time, i.e., $\eta = \xi = 0$.

Remark 4.2. From the proof of Theorem 4.1 we see that we use the monotonic property of the function $f(t,x)$ only when $x \in \mathbb{R}:\ v(t) \leq x \leq w(t)$ for $t \in [t_0, t_0 + T]$. Therefore, if a function f satisfies the monotonic property in condition (2) of Theorem 4.1 and Theorem 4.2 when $x \in \mathbb{R}$: $v(t) \leq x \leq w(t)$ for $t \in [t_0, t_0 + T]$, then we may be able to modify f so that Condition 2 is satisfied for all $x \in \mathbb{R}$.

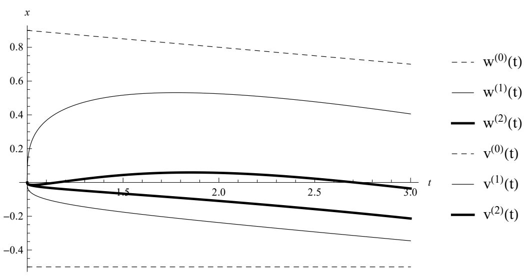

Figure 9. Example 3. Graph of the lower and upper approximations.

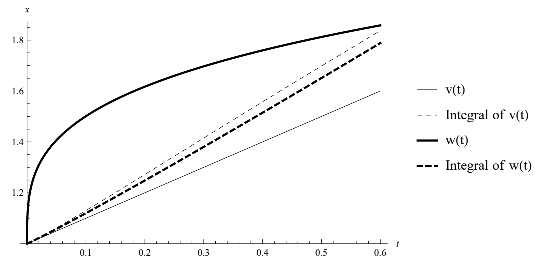

Figure 10. Example 4. Graphs of the mild lower/upper solutions and the correspong integrals on $[0, 0.6]$.

Example 4.3. Consider the IVP for the scalar Caputo FrDE

$$ {}_0^cD_t^{0.3}x(t) = \frac{t}{x(t)} + t^{0.7} \ \text{ for } \ t \in [0, 0.6], \quad x(0) = 1. \tag{4.19} $$

The function $f(t, x) = \frac{t}{x} + t^{0.7}$ is not defined when $x = 0$.

Consider the functions $v(t) = t + 1$, $w(t) = t^{0.3} + 1$, $t \in [0, 0.6]$ and note that

$$ \begin{aligned} &v(t) = t + 1 \le 1 + \frac{1}{\Gamma[0.3]} \int_0^t (t - s)^{0.3-1} \Big(\frac{s}{s+1} + s^{0.7} \Big)\, ds, \quad t \in [0, 0.6], \\ &w(t) = t^{0.3} + 1 \le 1 + \frac{1}{\Gamma[0.3]} \int_0^t (t - s)^{0.3-1} \Big(\frac{s}{s^{0.3}+1} + s^{0.7} \Big)\, ds, \quad t \in [0, 0.6], \\ &v(t) \le w(t), \quad t \in [0, 0.6], \end{aligned} \tag{4.20} $$

hold (see Figure 10).

Therefore, the functions $v(t)$, $w(t)$ are mild lower and upper solutions of the FrDE (4.19) and condition (1) of Theorem 4.2 is satisfied with $t_0 = \tau_0 = \theta_0 = 1$. We can define a function $f_1(t, x) \in C([0, 0.6] \times \mathbb{R}, \mathbb{R})$ by

$$ f_1(t, x) = \begin{cases} f(t, x) = \dfrac{t}{x} + t^{0.7}, & t \in [0.0.6], \ \ x \ge 1, \\ f(t, 1) = t + t^{0.7}, & t \in [0, 0.6], \ \ x < 1, \end{cases} $$

The function $f_1(t, x)$ is nondecreasing in t and nonincreasing in x, i.e., condition (2) of Theorem 4.2 is satisfied and we can construct two sequences of successive approximations to the solution $x(t)$ of the IVP for FrDE (4.19) with $v(t) \le x(t) \le w(t)$. We apply formulas (4.1) and (4.2) with $t_0 = \tau_0 = \theta_0 = 1$ and replace $f(t, x)$ by $f_1(t, x)$.

Remark 4.3. An appropriate iterative scheme for monotonic right side parts for the periodic boundary value problem for the Caputo fractional differential equation is given in [14].

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