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ON THE SOLVABILITY OF A BOUNDARY VALUE PROBLEM WITH DIRICHLET AND POINCARE CONDITIONS IN THE ANGULAR DOMAIN FOR ONE CLASS OF NONLINEAR SECOND ORDER HYPERBOLIC SYSTEMS

Abstract. Darboux type problem with Dirichlet and Poincare boundary conditions for one class of nonlinear second order hyperbolic systems is considered. The questions of existence and nonexistence, uniqueness and smoothness of global solution of this problem are investigated.

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In the plane of the variables x and t we consider a nonlinear second order hyperbolic system of type

$$Lu:\ u_{tt}-u_{xx}+A(x,t)u_x+B(x,t)u_t+C(x,t)u+f(x,t,u)=F(x,t), \tag{1}$$

where A, B, C are given real $n\times n$-matrices, $f=(f_1,\dots,f_n)$ is a given nonlinear with respect to u real vector-function, $F=(F_1,\dots,F_n)$ is a given and $u=(u_1,\dots,u_n)$ is an unknown real vector-function, $n\geq 2$.

By D_T we denote a triangular domain lying inside the characteristic angle $\{(x,t)\in R^2:\ t>|x|\}$ and bounded by the characteristic segment $\gamma_{1,t}:\ x=t,\ 0\leq t\leq T$, and segments $\gamma_{2,t}:\ x=0,\ 0\leq t\leq T$, $\gamma_{3,t}:\ t=T,\ 0\leq x\leq T$, of time and spatial type, respectively.

For the system (1), we consider a boundary value problem: find in the domain D_T a solution $u=u(x,t)$ of that system, satisfying on segments $\gamma_{1,T}$ and $\gamma_{2,T}$ the Dirichlet and Poincare conditions, respectively,

$$u\big|_{\gamma_{1,T}}=\varphi, \tag{2}$$

$$(\mu_1v_x+\mu_2v_t)\big|_{\gamma_{2,T}}=0, \tag{3}$$

where $\varphi=(\varphi_1,\dots,\varphi_n)$ is a given real vector-function and μ_i, $i=1,2$, are given real $n\times n$-matrices. In the case of $T=\infty$ we have $D_\infty:=t>|x|$, $x>0$, and $\gamma_{1,\infty}:\ x=t,\ 0\leq t\leq\infty$, $\gamma_{2,\infty}:\ x=0,\ 0\leq t\leq\infty$.

Definition 1. Let $A,B,C,F,f\in C(\overline{D}_T\times\mathbb{R}^n)$ and $\varphi\in C^1(\varphi_{1,T})$, $\mu_i\in C(\gamma_{2,T})$, $i=1,2$. We call a vector-function u a generalized solution of the problem $(1),(2),(3)$ of the class C in the domain D_T if $u\in C(\overline{D}_T)$ and there exists a sequence of vector-functions

$$u^m\in C^2_0(\overline{D}_T):=\Big\{v\in C^2(\overline{D}_T):\ (\mu_1v_x+\mu_2v_t)\big|_{\gamma_{2,T}}=0\Big\}$$

such that $u^m\to u$ and $Lu^m\to F$ in the space $C(\overline{D}_T)$, $u^m|_{\gamma_{1,T}}\to\varphi$ in the space $C^1(\gamma_{1,T})$, as $m\to\infty$.

It is obvious that a classical solution $u\in C^2(\overline{D}_T)$ of the problem $(1),(2),(3)$ represents a generalized solution of this problem of the class C in the domain D_T in the sense of Definition 1.

Definition 2. Let $A,B,C,F,f\in C(\overline{D}_\infty\times\mathbb{R}^n)$ and $\varphi\in C^1(\gamma_{1,\infty})$, $\mu_i\in C(\gamma_{2,\infty})$, $i=1,2$. We say that the problem $(1),(2),(3)$ is locally solvable in the class C if there exists a number $T_0=T_0(F,\varphi)>0$ such that for $T<T_0$ this problem has a generalized solution of the class C in the domain D_T in the sense of the Definition 1.

Definition 3. Let $A, B, C, F, f \in C(\overline{D}_\infty \times \mathbb{R}^n)$ and $\varphi \in C^1(\gamma_{1,\infty})$, $\mu_i \in C(\gamma_{2,\infty})$, $i = 1, 2$. We say that the problem (1), (2), (3) is globally solvable in the class C if for any $T > 0$ this problem has a generalized solution of the class C in the domain D_T in the sense of Definition 1.

Definition 4. Let $A, B, C, F, f \in C(\overline{D}_\infty \times \mathbb{R}^n)$ and $\varphi \in C^1(\gamma_{1,\infty})$, $\mu_i \in C(\gamma_{2,\infty})$, $i = 1, 2$. A vector-function $u \in C(\overline{D}_\infty)$ is called a global generalized solution of the problem (1), (2), (3) of the class C in the domain D_∞ if for any $T > 0$ the vector-function $u|_{D_T}$ is a generalized solution of the class C in the domain D_T in the sense of Definition 1.

If in the linear case for scalar hyperbolic equations the boundary value problems of Goursat and Darboux type are well studied (see [5–7, 9, 12, 16]), there arise additional difficulties and new effects in passing to hyperbolic systems. This has been first noticed by A. V. Bitsadze [3] who constructed examples of second order hyperbolic systems for which the corresponding homogeneous characteristic problem has a finite number, and in some cases, an infinite of number of linearly independent solutions. Later these problems for linear second order hyperbolic systems have become a subject of study in the works [10, 11]. In this direction it should also be noted the work [4], in which on the simple examples it is revealed the effect of lowest terms on the correctness of these problems. As shown in [1, 2, 13–15], the presence of the nonlinear term in the scalar hyperbolic equation may affect on the correctness of the Darboux problem, when in some cases this problem is globally solvable, and in other cases may arise the so-called blow up solutions. It should be noted that the above-mentioned works do not contain linear terms involving the first order derivatives, since their presence causes difficulties in investigating the problem, and not only of technical character. In this paper, we study the Darboux type problem for nonlinear system (1) with lowest terms of the first order. The results presented here are new in the case when (1) is a scalar hyperbolic equation.

Local solvability of the problem (1), (2), (3) in sense of Definition 2 holds under the additional requirements

$$\det(\mu_2 - \mu_1)\big|_{\gamma_{2,\infty}} \neq 0 \tag{4}$$

and

$$A, B \in C^2(\overline{D_\infty}), \;\; C \in C^1(\overline{D_\infty}), \;\; f \in C^1(\overline{D}_\infty \times \mathbb{R}^n), \;\; \mu_i \in C^1(\gamma_{2,\infty}). \tag{5}$$

Under the conditions given in the Definition 2, if we additionally require that

$$\|f_i(x, t, u)\| \leq M_1 + M_2\|u\|, \;\; (x, t, u) \in \overline{D}_\infty \times \mathbb{R}^n, \;\; i = 1, \dots, n, \tag{6}$$

and

$$\det \mu_1\big|_{\gamma_{2,T}} \neq 0, \;\; (\mu_1^{-1}\mu_2\theta, \theta)\big|_{\gamma_{2,T}} \leq 0 \;\; \forall\, \theta \in \mathbb{R}^n, \tag{7}$$

where $M_j = M_j(T) = \mathit{const} \geq 0$, $j = 1, 2$, $\forall\, T > 0$; $\|u\| = \sum\limits_{i=1}^{n} |u_i|$, $(\,\cdot\,,\,\cdot\,)$ is scalar product in the Euclidean space $\mathbb{R}^n$, then for a generalized solution of the problem (1), (2), (3) of the class C in the domain D_T the a priori estimate

$$\|u\|_{C(\overline{D_T})} \leq c_1\|F\|_{C(\overline{D_T})} + c_2\|\varphi\|_{C^1(\gamma_{1,T})} + c_3, \tag{8}$$

is valid with nonnegative constants $c_i = c_i(M_0, M_1, M_2, T)$, $i = 1, 2, 3$, not depending on u, F, φ and where $c_i > 0$, $i = 1, 2$. Here $M_0 = M_0(A, B, C) = \mathit{const} \geq 0$.

Under the conditions (4)–(7), from the a priori estimate (8) by virtue of Learay–Schauder's theorem there follows the global solvability of the problem (1), (2), (3) in the class C in the sense of Definition 3.

Note also that in the above assumptions (4)–(7) there exists a unique global generalized solution of the problem (1), (2), (3) of the class C in the domain D_∞ in the sense of Definition 4.

Now consider the case when the condition (5) is violated, i.e.,

$$\overline{\lim_{\|u\| \to \infty}} \frac{\|f(x, t, u)\|}{\|u\|} = \infty,$$

and the problem (1), (2), (3) is not globally solvable, in particular, it does not have a global generalized solution of the class C in the domain D_∞ in the sense of Definition 4.

Theorem. *Let* $A = B = C = 0$, $f = f(u) \in C(\mathbb{R}^n)$, $F \in C(\overline{D}_\infty)$, $\varphi = 0$. *There exists numbers* $l_1, \dots, l_n$, $\sum\limits_{i=1}^n |l_i| \neq 0$ *such that*

$$\sum_{i=1}^n l_i f_i(u) \leq c_0 - c_1 \Big| \sum_{i=1}^n l_i u_i \Big|^\beta, \quad \beta = const > 1, \tag{9}$$

where $c_0, c_1 = const$, $c_1 > 0$. *Let the function* $F_0 = \sum\limits_{i=1}^n l_i F_i - c_0$ *satisfies the following conditions:*

$$F_0 \geq 0, \ \ F(x,t)\big|_{t \geq 1} \geq c_2 t^{-k}; \ \ c_2 = const > 0, \ \ 0 \leq k = const \leq 2.$$

Then there exists a finite positive number $T_0 = T_0(F)$ *such that for* $T > T_0$ *the problem* (1), (2), (3) *does not have a generalized solution of the class* C *in the domain* D_T.

Corollary. *Under the conditions of the theorem, although the problem is locally solvable, it does not have a global generalized solution of the class* C *in the domain* D_∞.

Now let us consider one class of vector-functions f satisfying the condition (9):

$$f_i(u_1, \dots, u_n) = \sum_{j=1}^n a_{ij} |u_j|^{\beta_{ij}} + b_i, \ \ i = 1, \dots, n, \tag{10}$$

where $a_{ij} = const > 0$, $b_i = const$, $\beta_{ij} = const > 1$; $i, j = 1, \dots, n$. In this case we can take: $l_1 = l_2 = \cdots = l_n = -1$. Indeed, let us choose $\beta = const$ such that $1 < \beta < \beta_{ij}$; $i, j = 1, \dots, n$. It is easy to verify that $|s|^{\beta_{ij}} \geq |s|^\beta - 1 \ \forall\, s \in (-\infty, \infty)$. Now, using well - known inequality [8]

$$\sum_{i=1}^n |y_i|^\beta \geq n^{1-\beta} \Big| \sum_{i=1}^n y_i \Big|^\beta \quad \forall\, y = (y_1, \dots, y_n) \in \mathbb{R}^n, \ \ \beta = const > 1,$$

we receive

$$\sum_{i=1}^n f_i(u_1, \dots, u_n) \geq a_0 \sum_{i,j=1}^n |u_j|^{\beta_{ij}} + \sum_{i=1}^n b_i \geq a_0 \sum_{i,j=1}^n \big(|u_j|^\beta - 1\big) + \sum_{i=1}^n b_i$$
$$= a_0 n \sum_{j=1}^n |u_j|^\beta - a_0 n^2 + \sum_{i=1}^n b_i \geq a_0 n^{2-\beta} \Big| \sum_{j=1}^n u_j \Big|^\beta + \sum_{i=1}^n b_i - a_0 n^2, \ \ a_0 = \min_{i,j} a_{ij} > 0.$$

Hence we have the inequality (9) in which: $l_1 = l_2 = \cdots = l_n = -1$, $c_0 = a_0 n^2 - \sum\limits_{i=1}^n b_i$, $c_1 = a_0 n^{2-\beta} > 0$. Note that the vector-function f, represented by the equalities (10), also satisfies the condition (9) with $l_1 = l_2 = \cdots = l_n = -1$ for less restrictive conditions when $a_{ij} = const \geq 0$, but $a_{ik_i} > 0$, where $k_1, \dots, k_n$ is any fixed permutation of numbers $1, 2, \dots, n$.

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