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TOPICS ON WIENER REGULARITY FOR ELLIPTIC EQUATIONS AND SYSTEMS

Abstract. This is a survey of results on Wiener's test for the regularity of a boundary point in various nonstandard situations. In particular, higher order elliptic operators, linear elasticity system, Zaremba boundary value problem for the Laplacian are treated.

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რეზიუმე. მიმოხილულია სასაზღვრო წერტილის რეგულარობის ვინერის კრიტერიუმთან დაკავშირებული შედეგები სხვადასხვა არასტანდარტულ სიტუაციაში. კერძოდ, განხილულია მაღალი რიგის ელიფსური ოპერატორები, წრფივი დრეკადობის თეორიის განტოლებათა სისტემა და ზარემბას სასაზღვრო ამოცანა ლაპლასიანისთვის.

Chapter 1

Historical background and structure of the paper

In 1924 Wiener [71] gave his famous criterion for the so called regularity of a boundary point.

A point O at the boundary $\partial\Omega$ of a domain $\Omega \subset \mathbb{R}^n$ is called regular if solutions of the Dirichlet problem for the Laplace equation in Ω with the Dirichlet data, continuous at O, are continuous at this point. (I do not want to explain in which sense the solution is understood — this is not quite trivial and is also due to Wiener [72].)

Before Wiener's result only some special facts concerning the regularity were known. For example, since (by Riemann's theorem) an arbitrary Jordan domain in $\mathbb{R}^2$ is conformally homeomorphic to the unit disc, it follows that any point of its boundary is regular.

As for the n-dimensional case, it was known for years that a boundary point O is regular provided the complement of Ω near O is so thick that it contains an open cone with O as a vertex (Poincaré [62], Zaremba [73]). Lebesgue noticed that the vertex of a sufficiently thin cusp in $\mathbb{R}^3$ is irregular [30]. Therefore it became clear that, in order to characterize the regularity, one should find proper geometric or quasi-geometric terms describing the massiveness of $\mathbb{R}^n \backslash \Omega$ near O.

To this end Wiener introduced the harmonic capacity $\operatorname{cap}(K)$ of a compact set K in $\mathbb{R}^n$, which corresponds to the electrostatic capacity of a body when $n = 3$. Up to a constant factor, the harmonic capacity in the case $n > 2$ is equal to

$$\inf\Big\{ \int\limits_{\mathbb{R}^n} |\operatorname{grad} u|^2 \, dx : \; u \in C_0^\infty(\mathbb{R}^n), \; u > 1 \text{ on } K \Big\}.$$

For $n = 2$ this definition of capacity needs to be altered.

The notion of capacity enabled Wiener to state and prove the following result.

Theorem (Wiener). *The point O at the boundary of the domain $\omega \subset \mathbb{R}^n$, $n \geq 2$, is regular if and only if*

$$\sum_{k \geq 1} 2^{(n-2)k} \operatorname{cap}(B_{2^{-k}} \setminus \Omega) = \infty. \tag{1.0.1}$$

We assume that O is the origin of a coordinate system and use the notation $B_\rho = \{x \in \mathbb{R}^n : \ |x| < \rho\}$. It is straightforward that (1.0.1) can be rewritten in the integral form

$$\int_0 \frac{\operatorname{cap}(B_\sigma \setminus \Omega)}{\operatorname{cap}(B_\sigma)} \frac{d\sigma}{\sigma} = \infty. \tag{1.0.2}$$

Wiener's theorem was the first necessary and sufficient condition characterizing the dependence of properties of solutions on geometric properties of the boundary. The theorem strongly influenced potential theory, partial differential equations, and probability theory. Especially striking was the impact of the notion of the Wiener capacity, which gave an adequate language to answer many important questions. During the years many attempts have been made to extend the range of Wiener's result to different classes of linear equations of the second order, although some of them were successful only in the sufficiency part. I mention here three necessary and sufficient conditions.

First, for uniformly elliptic operators with measurable bounded coefficients in divergence form

$$u \longrightarrow \sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j}. \tag{1.0.3}$$

Littman, Stampacchia and Weinberger [32] proved that the regularity of boundary point is equivalent to the Wiener condition (1.0.1).

Second, in 1982 Fabes, Jerison and Kenig [13] gave an interesting analog of the Wiener criterion for a class of degenerate elliptic operators of the form (1.0.3).

The third criterion for regularity, due to Dal Maso and Mosco [9], concerns the Schrödinger operator

$$u \to -\Delta u + \mu u \ \text{ in } \ \Omega,$$

where μ is a measure. It characterizes both the geometry of Ω and the potential μ near the point O.

It seems worthwhile to mention a recently solved problem, which remained open for twenty five years. I mean the question of the regularity of a boundary point for the non-linear operator

$$u \longrightarrow \operatorname{div}\big(|\operatorname{grad} u|^{p-2} \operatorname{grad} u\big) \ \text{ in } \ \Omega, \tag{1.0.4}$$

where $p > 1$. This differential operator, often called the p-Laplacian, appears in some mechanical applications and is interesting from a pure mathematical point of view.

In 1970 I proved [40] that the following variant of the Wiener criterion is sufficient for the regularity with respect to (1.0.4)

$$\int_0 \Big(\frac{p\text{-cap}(B_\sigma \setminus \Omega)}{p\text{-cap}(B_\sigma)}\Big)^{\frac{1}{p-1}} \frac{d\sigma}{\sigma} = \infty. \tag{1.0.5}$$

Here $1 < p \le n$ and the p-capacity is a modification of the Wiener capacity generated by the p-Laplacian. This result was generalized by Gariepy and Ziemer [16] to a large class of elliptic quasilinear equations

$$\operatorname{div} A(x, u, \operatorname{grad} u) = B(x, u, \operatorname{grad} u).$$

Condition (1.0.5) and its generalizations also turned out to be relevant in studying the fine properties of elements in Sobolev spaces. See, e.g. the book [4].

For a long time it seemed probable that (1.0.5) is also necessary for the regularity with respect to (4), and indeed, for $p \ge n - 1$, Lindqvist and Martio [33] proved this for the operator (1.0.4). Finally, Kilpeläinen and Malý found a proof valid for arbitrary values of $p > 1$ [22].

So far I spoke only about the regularity of a boundary point for second order elliptic equations. However, the topic could be extended to include other equations, systems, boundary conditions and function spaces. In principle, the Wiener criterion suggests the possibility of the complete characterization of properties of domains, equivalent to various solvability and spectral properties of boundary value problems.

The present article is a survey of results on Wiener's test in various nonstandard situations. These results were obtained by myself or together with my collaborators.

In the second chapter, following the paper [49] by V. Maz'ya, I deal with strongly elliptic differential operators of an arbitrary even order $2m$ with constant real coefficients and introduce a notion of the regularity of a boundary point with respect to the Dirichlet problem which is equivalent to that given by N. Wiener in the case $m = 1$. It is shown that a capacitary Wiener-type criterion is necessary and sufficient for the regularity if $n = 2m$. In the case $n > 2m$, the same result is obtained for a subclass of strongly elliptic operators.

In Chapter 3, boundary behaviour of solutions to the polyharmonic equation is considered. First, conditions of weighted positivity of $(-\Delta)^m$ with zero Dirichlet data are studied which, together with results in Chapter 2, give Wiener-type criterion for the space dimensions $n = 2m,\ 2m+1,\ 2m+2$ with $m > 2$ and $n = 4, 5, 6, 7$ with $m = 2$. Second, certain pointwise estimates for polyharmonic Green's function and solutions of the polyharmonic equation are derived for the same n and m. Here I mostly follow my paper [48].

Chapter 4 addresses results by G. Luo and V. Maz'ya [33]. We consider the $3D$ Lamé system and establish its weighted positive definiteness for a certain range of elastic constants. By modifying the general theory developed in Chapter 2, we then show, under the assumption of weighted positive definiteness, that the divergence of the classical Wiener integral for a boundary point guarantees the continuity of solutions to the Lamé system at this point.

In Chapter 5, an analogue of the Wiener criterion for the Zaremba problem is obtained. The results are due to T. Kerimov, V. Maz'ya, and A. Novruzov. They were announced in [20] and published with proofs in [21].

The last Chapter 6 reproduces the papers [39] and [44] by V. Maz'ya, where various capacitary estimates for solutions of the Dirichlet problem, Green's function and the $\mathscr{L}$-harmonic measure for elliptic second order operators in divergent form with measurable bounded coefficients.

Chapter 2

Wiener Test for Higher Order Elliptic Equations

2.1 Introduction

Wiener's criterion for the regularity of a boundary point with respect to the Dirichlet problem for the Laplace equation [71] has been extended to various classes of elliptic and parabolic partial differential equations. These include linear divergence and nondivergence equations with discontinuous coefficients, equations with degenerate quadratic form, quasilinear and fully nonlinear equations, as well as equations on Riemannian manifolds, graphs, groups, and metric spaces (see [32], [13], [9], [33], [22], [34], [3], [4], [26], [66], to mention only a few). A common feature of these equations is that all of them are of second order, and Wiener-type characterizations for higher order equations have been known so far. Indeed, the increase of the order results in the loss of the maximum principle, Harnack's inequality, barrier techniques and level truncation arguments which are ingredients in different proofs related to the Wiener test for the second order equations.

In this chapter Wiener's result is extended to elliptic differential operators $L(\partial)$ of order $2m$ in the Euclidean space $\mathbb{R}^n$ with constant real coefficients

$$L(\partial) = (-1)^m \sum_{|\alpha|=|\beta|=m} a_\alpha \partial^{\alpha+\beta}.$$

We assume without loss of generality that $a_{\alpha\beta} = a_{\beta\alpha}$ and $(-1)^m L(\xi) > 0$ for all nonzero $\xi \in \mathbb{R}^n$. In fact, the results of this paper can be extended to equations with variable (e.g., Hölder continuous) coefficients in divergence form, but we leave aside this generalization to make our exposition more lucid.

We use the notation ∂ for the gradient $(\partial_{x_1}, \dots, \partial_{x_n})$, where ∂_{x_k} is the partial derivative with respect to x_k. By Ω we denote an open set in $\mathbb{R}^n$, and by $B_\rho(y)$ we denote the ball $\{x \in \mathbb{R}^n : \ |x-y| < \rho\}$, where $y \in \mathbb{R}^n$. We write B_ρ instead of $B_\rho(0)$.

Consider the Dirichlet problem

$$L(\partial)u = f, \quad f \in C_0^\infty(\Omega), \ \ u \in \overset{\circ}{H}{}^m(\Omega), \tag{2.1.1}$$

where we use the standard notation $C_0^\infty(\Omega)$ for the space of infinitely differentiable functions in $\mathbb{R}^n$ with compact support in Ω as well as $\overset{\circ}{H}{}^m(\Omega)$ for the completion of $C_0^\infty(\Omega)$ in the energy norm.

Definition 2.1.1. We call the point $0 \in \partial\Omega$ regular with respect to $L(\partial)$ if for any $f \in C_0^\infty(\Omega)$ the solution of (2.1.1) satisfies

$$\lim_{\Omega \ni x \to O} u(x) = 0. \tag{2.1.2}$$

For $n = 2, 3, \dots, 2m - 1$, the regularity is a consequence of the Sobolev imbedding theorem. Therefore we suppose that $n \geq 2m$. In the case of $m = 1$, the above definition of regularity is equivalent to that given by Wiener (see Section 2.6 below).

The following result which coincides with Wiener's criterion in the case of $n = 2$ and $m = 1$, is obtained in Sections 2.8 and 2.9.

Theorem 2.1.1. *Let $2m = n$. Then O is regular with respect to $L(\partial)$ if and only if*

$$\int_0^1 C_{2m}(B_\rho \setminus \Omega)\rho^{-1}\, d\rho = \infty. \tag{2.1.3}$$

Here and elsewhere C_{2m} is the potential-theoretic Bessel capacity of order $2m$ (see Adams and Heard [3] and Adams and Hedberg [4]). The case of $n > 2m$ is more delicate because no result of Wiener's type is valid for all operators $L(\partial)$ (see [53, Chapter 10]). To be more precise, even the vertex of a cone may be irregular with respect to $L(\partial)$) if the fundamental solution of $L(\partial)$,

$$F(x) = F\Big(\frac{x}{|x|}\Big)|x|^{2m-n}, \ \ x \in \mathbb{R}^n \setminus O, \tag{2.1.4}$$

changes its sign. Examples of operators $L(\partial)$ with this property can be found in Maz'ya and Nazarov [52] and Davies [10]. In the sequel, Wiener's type characterization of regularity for $n > 2m$ is given for a subclass of operators $L(\partial)$ called *positive with the weight F*. This means that for all real-valued $u \in C_0^\infty(\mathbb{R}^n \setminus O)$,

$$\int_{\mathbb{R}^n} L(\partial)u(x) \cdot u(x)F(x)\, dx \geq c \sum_{k=1}^m \int_{\mathbb{R}^n} \big|\nabla_k u(x)\big|^2 |x|^{2k-n}\, dx, \tag{2.1.5}$$

where ∇_k is the gradient of order k, that is, where $\nabla_k = \{\partial^\alpha\}$ with $|\alpha| = k$.

In Sections 2.5 and 2.7, we prove the following result.

Theorem 2.1.2. *Let $n > 2m$, and let $L(\partial)$ be positive with weight F. Then O is regular with respect to $L(\partial)$ if and only if*

$$\int_0^1 C_{2m}(B_\rho \setminus \Omega)\rho^{2m-n-1}\, d\rho = \infty. \tag{2.1.6}$$

Note that in a direct analogy with the case of the Laplacian we could say that 0 in Theorems 2.1.1 and 2.1.2 is irregular with respect to $L(\partial)$ if and only if the set $\mathbb{R}^n \setminus \Omega$ is $2m$-thin in the sense of linear potential theory (see [29], [3], [4]).

Since, obviously, the second order operator $L(\partial)$ is positive with the weight F, Wiener's result for F is contained in Theorem 2.1.2. Moreover, one can notice that the same proof with $F(x)$ being replaced by Green's function of the uniformly elliptic operator $u \to -\partial_{x_i}(a_{ij}(x)\partial_{x_j}u)$ with bounded measurable coefficients leads to the main result in [32]. We also note that the pointwise positivity of F follows from (2.1.5), but the converse is not true. In particular, the m-harmonic operator with $2m < n$ satisfies (2.1.5) if and only if $n = 5, 6, 7$ for $m = 2$ and $n = 2m+1, 2m+2$ for $m > 2$ (see [47], where the proof of the sufficiency of (2.1.6) is given for $(-\Delta)^m$ with m and n as above, and also [12] dealing with the sufficiency for noninteger powers of the Laplacian in the intervals $(0,1)$ and $[\frac{n}{2}-1, \frac{n}{2})$.

It is shown in [55] that the vertices of n-dimensional cones are regular with respect to Δ^2 for all dimensions. In Theorem 2.12.1, we consider the Dirichlet problem (2.1.1) for $n \geq 8$ and for the n-dimensional biharmonic operator with 0 being the vertex of an inner cusp. We show that condition (2.1.6), where $m = 2$, guarantees that $u(x) \to 0$ as x approaches O along any nontangential direction. This does not mean, of course, that Theorem 2.1.2 for the biharmonic operator may be extended to higher dimensions, but the domain Ω providing the corresponding counterexample should be more complicated than a cusp.

There are some auxiliary assertions of independent interest proved in this paper which concern the so-called L-capacitary potential U_K of the compact set $K \subset \mathbb{R}^n$, that is, the solution of the variational problem

$$\inf\left\{ \int_{\mathbb{R}^n} L(\partial)u \cdot u\, dx :\ u \in C_0^\infty(\mathbb{R}^n) :\ u = 1 \text{ in vicinity of } K \right\}.$$

We show, in particular, that for an arbitrary operator $L(\partial)$, the potential U_k is subject to the estimate

$$|U_K(y)| \leq c\, \mathrm{dist}(y,K)^{2m-n} C_{2m}(K) \ \text{ for all } \ y \in \mathbb{R}^n \setminus K,$$

where the constant c does not depend on K (see Proposition 2.2.1). The natural analogue of this estimate in the theory of Riesz potentials is quite obvious, and as a matter of fact, our L-capacitary potential is representable as the Riesz potential $F * T$. However, one cannot rely upon the methods of the classical potential theory when studying U_K, because, in general, T is only a distribution and not a positive measure. Among the properties of U_K resulting from the assumption of weighted positivity of $L(\partial)$ are the inequalities $0 < U_K < 2$ on $\mathbb{R}^n \setminus K$, which holds for an arbitrary compact set K of positive capacity C_{2m}. Generally, the upper bound 2 cannot be replaced by 1 if $m > 1$.

In conclusion, it is perhaps worth mentioning that the present paper gives answers to some questions posed in [47].

2.2 Capacities and the L-Capacitary Potential

Let Ω be arbitrary if $n > 2m$ and bounded if $n = 2m$. By Green's m-harmonic capacity $\mathrm{cap}_m(K, \Omega)$ of a compact set $K \subset \Omega$ we mean

$$\inf\Big\{ \sum_{|\alpha|=m} \frac{m!}{\alpha!} \|\partial^\alpha u\|^2_{L_2(\mathbb{R}^n)} : \ u \in C_0^\infty(\Omega), \ u=1 \text{ in vicinity of } K \Big\}. \tag{2.2.1}$$

We omit the reference to Green and write $\mathrm{cap}_m(K)$ if $\Omega = \mathbb{R}^n$. It is well known that $\mathrm{cap}_m(K) = 0$ for all K if $n = 2m$.

Let $n > 2m$. One of the equivalent definitions of the potential-theoretic Riesz capacity of order $2m$ is

$$c_{2m}(K) = \inf\Big\{ \sum_{|\alpha|=m} \frac{m!}{\alpha!} \|\partial^\alpha u\|^2_{L_2(\mathbb{R}^n)} : \ u \in C_0^\infty(\mathbb{R}^n), \ u \geq 1 \text{ on } K \Big\}.$$

The capacities $\mathrm{cap}_m(K)$ and $c_{2m}(K)$ are equivalent; that is, their ratio is bounded and separated from zero by constants depending only on n and m (see [49, Section 9.3.2]).

We use the notation $C_{2m}(K)$ for the potential-theoretic Bessel capacity of order $2m \leq n$ which can be defined by

$$\inf\Big\{ \sum_{0 \leq |\alpha| \leq m} \frac{m!}{\alpha!} \|\partial^\alpha u\|^2_{L_2(\mathbb{R}^n)} : \ u \in C_0^\infty(\mathbb{R}^n), \ u \geq 1 \text{ on } K \Big\}.$$

Here also the replacement of the condition $u \geq 1$ on K by $u = 1$ in a neighborhood of K leads to an equivalent capacity. Furthermore, if $n > 2m$ and $K \subset B_1$, the Riesz and Bessel capacities of K are equivalent.

We use the bilinear form

$$\mathscr{B}(u, v) = \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \partial^\alpha u \cdot \partial^\beta v \, dx. \tag{2.2.2}$$

The solution $U_K \in \overset{\circ}{H}{}^m(\Omega)$ of the variational problem

$$\inf\Big\{ \mathscr{B}(u, u) : \ u \in C_0^\infty(\Omega), \ u = 1 \text{ in a neighbourhood of } K \Big\} \tag{2.2.3}$$

is called Green's L-capacitary potential of the set K with respect to Ω, and the L-capacitary potential of K in the case of $\Omega = \mathbb{R}^n$.

We check that the m-capacitary potential of the unit ball B_1 in $\mathbb{R}^n$, where $n > 2m$, is given for $|x| > 1$ by

$$U_{B_1}(x) = \frac{\Gamma(\frac{n}{2})}{\Gamma(m)\Gamma(-m+\frac{n}{2})} \int\limits_0^{|x|^{-2}} (1-\tau)^{m-1}\tau^{-m-1+\frac{n}{2}}\, d\tau. \tag{2.2.4}$$

This function solves the m-harmonic equation in $\mathbb{R}^n \setminus \overline{B}_1$ because the last integral is equal to

$$2\sum_{j=1}^{m} \frac{(-1)^{m-j}\Gamma(m)}{\Gamma(j)\Gamma(m-j+1)(n-2j)}\, |x|^{2j-n}.$$

Differentiating the integral in (2.2.4), we obtain

$$\partial^k_{|x|} U_{B_1}(x)\Big|_{\partial B_1} = 0 \;\text{ for }\; k = 1, \dots, m-1.$$

The coefficient at the integral in (2.2.4) is chosen to satisfy the boundary condition

$$U_{B_1}(x) = 1 \;\text{ on }\; \partial B_1.$$

Owing to (2.2.4), we see that

$$0 < U_{B_1}(x) < 1 \;\text{ on }\; \mathbb{R}^n \setminus B_1$$

and that U_{B_1} is a decreasing function of $|x|$.

By Green's formula

$$\sum_{|\alpha|=m} \|\partial^\alpha U_{B_1}\|^2_{L_2(\mathbb{R}^n\setminus B_1)} = -\int\limits_{\partial B_1} U_{B_1}(x)\, \frac{\partial}{\partial|x|}\, (-\Delta)^{m-1} U_{B_1}(x)\, ds_x =$$
$$= -\frac{2\Gamma(\frac{n}{2})}{(n-2m)\Gamma(m)\Gamma(-m+\frac{n}{2})} \int\limits_{\partial B_1} \frac{\partial}{\partial|x|}\, (-\Delta)^{m-1}|x|^{2m-n}\, ds_x$$

and by

$$(-\Delta)^{m-1}|x|^{2m-n} = \frac{4^{m-1}\Gamma(m)\Gamma(-1+\frac{n}{2})}{\Gamma(-m+\frac{n}{2})}\, |x|^{2-n},$$

we obtain the value of the m-harmonic capacity of the unit ball:

$$\operatorname{cap}_m B_1 = \frac{4^m}{n-2m}\left(\frac{\Gamma(\frac{n}{2})}{\Gamma(-m+\frac{n}{2})}\right)^2 \omega_{n-1} \tag{2.2.5}$$

with ω_{n-1} denoting the area of B_1.

We recall that the Riesz capacitary measure of order $2m$, $2m < n$, is the normalized area on ∂B_1 (see [29, Chapter 2, Section 3]). Hence, one can verify by direct computation that

$$c_{2m}(B_1) = \frac{2\sqrt{\pi}\,\Gamma(m)\Gamma(m-1+\frac{n}{2})}{\Gamma(m-\frac{1}{2})\Gamma(m-1+\frac{n}{2})}\,\omega_{n-1}. \tag{2.2.6}$$

Lemma 2.2.1. *For any $u \in C_0^\infty(\Omega)$ and any distribution $\Phi \in [C_0^\infty(\Omega)]^*$,*

$$\mathscr{B}(u, u\Phi) = 2^{-1}\int_\Omega u^2 L(\partial)\Phi\,dx + \\ + \int_\Omega \sum_{j=1}^m \sum_{|\mu|=|\mu|=j} \partial^\mu u \cdot \partial^\nu u \cdot \mathscr{P}_{\mu\nu}(\partial)\Phi\,dx, \tag{2.2.7}$$

where $\mathscr{P}_{\mu\nu}(\zeta)$ are homogeneous polynomials of degree $2(m-j)$, $\mathscr{P}_{\mu\nu} = \mathscr{P}_{\nu\mu}$, and $\mathscr{P}_{\alpha\beta}(\zeta) = a_{\alpha\beta}$ for $|\alpha| = |\beta| = m$.

Proof. The left-hand side in (2.2.7) is equal to

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\int_\Omega u\partial^\alpha u \cdot \partial^\beta \Phi\,dx + \\ + \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\Big(\int_\Omega \partial^\alpha u \cdot \partial^\beta u \cdot \Phi\,dx + \\ + \sum_{\beta>\gamma>0} \frac{\beta!}{\gamma!(\beta-\gamma)!}\int_\Omega \partial^\alpha u \cdot \partial^\beta u \cdot \partial^{\beta-\gamma}\Phi\,dx\Big).$$

We have

$$\int_{\mathbb{R}^n} u\partial^\alpha u \cdot \partial^\beta\Phi\,dx = 2^{-1}\int_\Omega \partial^\alpha(u^2)\partial^\beta\Phi\,dx - \\ - 2^{-1}\sum_{\alpha>\gamma>0}\frac{\alpha!}{\gamma!(\alpha-\gamma)!}\int_\Omega \partial^\alpha u \cdot \partial^{\alpha-\gamma}u \cdot \partial^\beta\Phi\,dx.$$

Hence by $a_{\alpha\beta} = a_{\beta\alpha}$, we obtain the identity

$$\mathscr{B}(u, u\Phi) = 2^{-1}\int_\Omega u^2 L(\partial)\Phi\,dx + \\ + \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\sum_{\beta>\gamma>0}\frac{\beta!}{\gamma!(\beta-\gamma)!}\int_\Omega \partial^\gamma u\big(\partial^\alpha u \cdot \partial^{\beta-\gamma}\Phi - 2^{-1}\partial^{\beta-\gamma}u \cdot \partial^\alpha\big)\,dx + \\ + \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\partial^\alpha u \cdot \partial^\beta u \cdot \Phi\,dx.$$

We need to prove that the second term can be written as

$$\int\limits_{\Omega} \sum_{j=1}^{m-1} \sum_{|\mu|=|\nu|=j} \partial^\mu u \cdot \partial^\nu u \cdot \mathscr{P}_{\mu\nu}(\partial)\Phi \, dx.$$

It suffices to establish such a representation for the integral

$$i_{\alpha\beta\gamma} = \int\limits_{\Omega} \partial^\alpha u \cdot \partial^\gamma u \cdot \partial^{\beta-\gamma}\Phi \, dx$$

with $|\alpha| > |\gamma|$. Let $|\alpha|+|\gamma|$ be even. We write $\alpha = \sigma+\tau$, where $|\sigma| = \frac{|\alpha|+|\gamma|}{2}$. After integrating by parts, we have

$$\begin{aligned} i_{\alpha\beta\gamma} = (-1)^{|\tau|} \int\limits_{\Omega} \partial^\sigma u \cdot \partial^{\gamma+\tau} u \cdot \partial^{\beta-\gamma}\Phi \, dx + \\ + (-1)^{|\tau|} \sum_{0\le\delta\le\tau} \frac{\tau!}{\delta!(\tau-\delta)!} \int\limits_{\Omega} \partial^\sigma u \cdot \partial^{\gamma+\delta} u \cdot \partial^{\beta-\gamma+\tau-\delta}\Phi \, dx. \end{aligned}$$

The last integral on the right is in the required form because $|\sigma| = |\gamma|+|\tau| = \frac{|\alpha|+|\gamma|}{2}$. We have $|\gamma| + |\delta| < |\alpha|$ in the remaining terms. Therefore, these terms are subject to the induction hypothesis.

Now let $|\alpha| + |\gamma|$ be odd. Then

$$\begin{aligned} i_{\alpha\beta\gamma} &= (-1)^{|\alpha|} \int\limits_{\mathbb{R}^n} u \partial^\alpha (\partial^\gamma u \cdot \partial^{\beta-\gamma}\Phi) \, dx = \\ &= (-1)^{|\alpha|} \int\limits_{\mathbb{R}^n} \sum_{0\le\delta\le\alpha} \frac{\alpha!}{\delta!(\alpha-\delta)!} \partial^{\gamma+\delta} u \cdot \partial^{\beta-\gamma+\alpha-\delta}\Phi \, dx. \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} i_{\alpha\beta\gamma} &= (-1)^{|\alpha|+|\gamma|} \int\limits_{\mathbb{R}^n} u \sum_{0\le\delta\le\alpha} \frac{\alpha!}{\delta!(\alpha-\delta)!} \partial^\delta u \cdot \partial^\gamma (u \partial^{\beta-\gamma+\alpha-\delta}\Phi) \, dx = \\ &= -\int\limits_{\mathbb{R}^n} u \sum_{0\le\delta\le\alpha} \frac{\alpha!}{\delta!(\alpha-\delta)!} \sum_{0\le\kappa\le\gamma} \frac{\gamma!}{\kappa!(\gamma-\kappa)!} \partial^\delta u \cdot \partial^\kappa u \cdot \partial^{\alpha+\beta-\delta-\kappa}\Phi \, dx. \end{aligned}$$

Hence

$$i_{\alpha\beta\gamma} = -2^{-1} \sum_{\substack{0\le\delta\le\alpha,\ 0\le\kappa\le\gamma \\ |\delta|+|\kappa|<|\alpha|+|\gamma|}} \frac{\alpha!\gamma!}{\delta!(\alpha-\delta)!\kappa!(\gamma-\kappa)!} \int\limits_{\mathbb{R}^n} \partial^\delta u \cdot \partial^\kappa u \cdot \partial^{\alpha+\beta-\delta-\kappa}\Phi \, dx.$$

Every integral on the right is subject to the induction hypothesis. The result follows. □

As in the introduction, by $F(x)$ we denote the fundamental solution $L(\partial)$ in $\mathbb{R}^n$ subject to (2.1.4). Setting $\Phi(x) = F(x-y)$, we conclude that for all $u \in C_0^\infty(\mathbb{R}^n)$,

$$\int\limits_{\mathbb{R}^n} L(\partial)u(x) \cdot u(x) F(x-y)\, dx =$$

$$= 2^{-1} u(y)^2 + \int\limits_{\mathbb{R}^n} \sum_{j=1}^{m} \sum_{|\mu|=|\nu|=j} \partial^\mu u(x) \cdot \partial^\nu u(x) \cdot \mathscr{P}_{\mu\nu}(\partial) F(x-y)\, dx. \quad (2.2.8)$$

Lemma 2.2.2. *Let $\Omega = \mathbb{R}^n$, $2m < n$. For all $y \in \mathbb{R}^n \setminus K$,*

$$U_K(y) = 2^{-1} U_K(y)^2 +$$
$$+ \int\limits_{\mathbb{R}^n} \sum_{m \ge j \ge 1} \sum_{|\mu|=|\nu|=j} \partial^\mu U_K(x) \cdot \partial^\nu U_K(x) \cdot \mathscr{P}_{\mu\nu}(\partial) F(x-y)\, dx, \quad (2.2.9)$$

where the same notation as in Lemma 2.2.1 *is used.*

Proof. We fix an arbitrary point y in $\mathbb{R}^n \setminus K$. Let $\{u_s\}_{s \ge 1}$ be a sequence of functions in $C_0^\infty(\mathbb{R}^n)$ such that $u_s = U_K$ on a neighborhood of y independent of s and $u_s \to U_k$ in $\mathring{H}^m(\mathbb{R}^n)$. Since U_K is smooth on $\mathbb{R}^n \setminus K$ and the function F is smooth on $\mathbb{R}^n \setminus O$ and vanishes at infinity, we can pass to the limit in (2.2.8), where $u = u_s$. This implies

$$\lim_{s\to\infty} \int\limits_{\mathbb{R}^n} L(\partial) U_K(x) \cdot u_s(x) F(x-y)\, dx = 2^{-1} U_K(y)^2 +$$
$$+ \int\limits_{\mathbb{R}^n} \sum_{j=1}^{m} \sum_{|\mu|=|\nu|=j} \partial^\mu U_K(x) \cdot \partial^\nu U_K(x) \cdot \mathscr{P}_{\mu\nu}(\partial) F(x-y)\, dx, \quad (2.2.10)$$

where $L(\partial)U_K$ is an element of the space $H^{-m}(\mathbb{R}^n)$ dual to $\mathring{H}^m(\mathbb{R}^n)$, and the integral on the left is understood in the sense of distributions. Taking into account that $L(\partial)U_K = 0$ on $\mathbb{R}^n \setminus K$ and that u_s can be chosen to satisfy $u_s = 1$ on a neighborhood of K, we write the left-hand side in (2.2.10) as

$$\int\limits_{\mathbb{R}^n} L(\partial) U_K(x) \cdot F(x-y)\, dx = U_K(y). \quad (2.2.11)$$

The result follows. □

Corollary 2.2.1. *Let $2m < n$. For almost all $y \in \mathbb{R}^n$,*

$$\left|\nabla_l U_K(y)\right| \le$$
$$\le c \Big(\left|\nabla_l U_K(y)^2\right| + \int\limits_{\mathbb{R}^n} \sum_{\substack{1 \le r,s \le m \\ r+s > l}} \frac{|\nabla_l U_K(y)|\, |\nabla_s U_K(y)|}{|x-y|^{n-r-s+l}}\, dx \Big), \quad (2.2.12)$$

where $l = 0, 1, \dots, m$.

Proof. Since $\nabla_l U_K$ vanishes almost everywhere on K, it is enough to check (2.2.12) for $y \in \mathbb{R}^n \setminus K$. By (2.2.9), it suffices to estimate

$$\Big| \nabla_l \int\limits_{\mathbb{R}^n} \partial^\mu U_K(x) \cdot \partial^\nu U_K(x) \cdot \mathscr{P}_{\mu\nu}(\partial) F(x-y)\, dx \Big|, \tag{2.2.13}$$

where $|\mu| = |\nu| = j$ and $j = 1, \dots, m$. Let $2j \le l$. Since $\operatorname{ord} \mathscr{P}_{\mu\nu}(\partial) = 2(m-j)$, we have

$$\big| \nabla_l \mathscr{P}_{\mu\nu}(\partial) F(x-y) \big| \le c |x-y|^{-n+2j-i},$$

and we can take

$$c \int\limits_{\mathbb{R}^n} \frac{|\nabla_j U_K(x)|^2}{|x-y|^{n-2j+l}}\, dx \tag{2.2.14}$$

as a majorant for (2.2.13). In the case of $2j > l$, integrating by parts, we estimate (2.2.13) by

$$c \int\limits_{\mathbb{R}^n} \Big| \nabla_{m-j} \big(\partial^\mu U_K(x) \cdot \partial^\nu U_K(x) \big) \Big| \, \big| \nabla_{m-j+l} F(x-y) \big|\, dx \le$$

$$\le c_1 \int\limits_{\mathbb{R}^n} \sum_{i=0}^{m-j} \frac{|\nabla_{i+j} U_K(x)|\, |\nabla_{m-i} U_K(x)|}{|x-y|^{n-m-j+l}}\, dx.$$

Since $m+j \ge 2j > 1$, the sum of the last majorant and (2.2.14) is dominated by the right-hand side in (2.2.12). The proof is complete. □

Proposition 2.2.1. *Let* $\Omega = \mathbb{R}^n$ *and* $2m < n$. *For all* $y \in \mathbb{R}^n \setminus K$, *the following estimate holds:*

$$\big| \nabla_j U_K(y) \big| \le c_j \operatorname{dist}(y, K)^{2m-n-j} \operatorname{cap}_m K, \tag{2.2.15}$$

where $j = 0, 1, \dots$ *and* c_j *does not depend on* K *and* y.

Proof. In order to simplify the notation, we set $y = 0$ and $\delta = \operatorname{dist}(y, K)$. By the well known local estimate for variational solutions of $L(\partial)u = 0$ (see [5, Chapter 3]),

$$\big| \nabla_j u(0) \big|^2 \le c_j \delta^{-n-2j} \int\limits_{B_{\frac{\delta}{2}}} u(x)^2\, dx, \tag{2.2.16}$$

it suffices to prove (2.2.15) for $j = 0$. By (2.2.16) and Hardy's inequality,

$$U_K(0)^2 \le c\delta^{2m-n} \int\limits_{\mathbb{R}^n} U_K(x)^2 \frac{dx}{|x|^{2m}}\, dx \le$$

$$\le c\delta^{2m-n} \int\limits_{\mathbb{R}^n} \big| \nabla_m U_K(x) \big|^2 dx \le c_0 \delta^{2m-n} \operatorname{cap}_m K. \tag{2.2.17}$$

If $\operatorname{cap}_m K \geq c_0^{-1}\delta^{n-2m}$, then estimate (2.2.15) follows from (2.2.17).

Now, let $\operatorname{cap}_m K < c_0^{-1}\delta^{n-2m}$. By virtue of (2.2.17)), we have $U_K(0)^2 \leq |U_K(0)|$. Hence by (2.2.9),

$$|U_K(0)| \leq c\sum_{j=1}^{m}\int\limits_{\mathbb{R}^n} \big|\nabla_j U(x)\big|^2 \frac{dx}{|x|^{n-2(m-j)}}\,.$$

Since by Hardy's inequality all integrals on the right are estimated by the mth integral, we obtain

$$|U_K(0)| \leq c\bigg(\delta^{2m}\sup_{x\in B_{\frac{\delta}{2}}} \big|\nabla_m U_K(x)\big|^2 + \int\limits_{\mathbb{R}^n} \big|\nabla_m U_K(x)\big|^2 \frac{dx}{|x|^{n-2m}}\bigg).$$

We estimate the above supremum using (2.2.16) with $j = 0$ and with u replaced by $\nabla_m\nabla_K$. Then

$$|U_K(0)| \leq c\delta^{2m-n}\bigg(\int\limits_{B_\delta} \big|\nabla_m U_K(x)\big|^2 + \int\limits_{\mathbb{R}^n\setminus B_{\frac{\delta}{2}}} \big|\nabla_m U_K(x)\big|^2\,dx\bigg).$$

The result follows from the definition of U_K. $\square$

By $\mathscr{M}$ we denote the Hardy–Littlewood maximal operator, that is,

$$\mathscr{M}f(x) = \sup_{\rho>0}\frac{n}{\omega_{n-1}\rho^n}\int\limits_{|y-x|<\rho} |f(y)|\,dy.$$

Proposition 2.2.2. *Let $2m < n$ and $0 < \theta < 1$. Also, let K be a compact subset of $\overline{B}_\rho \setminus B_{\theta\rho}$. Then the L-capacitary potential U_K satisfies*

$$\mathscr{M}\nabla_l U_K(0) \leq c_\theta \rho^{2m-l-n}\operatorname{cap}_m K, \tag{2.2.18}$$

where $l = 0, 1, \ldots, m$ and c_θ does not depend on K and ρ.

Proof. Let $r > 0$. We have

$$\int\limits_{B_r} \big|\nabla_l U_K(y)\big|\,dx \leq c\bigg(\int\limits_{B_r\cap B_{\theta\frac{\rho}{2}}} \big|\nabla_l U_K(y)\big|\,dx +$$
$$+\int\limits_{B_r\setminus B_{2\rho}} \big|\nabla_l U_K(y)\big|\,dx + \int\limits_{B_r\cap(B_{2\rho}\setminus B_{\theta\frac{\rho}{2}})} \big|\nabla_l U_K(y)\big|\,dx\bigg).$$

Since $\operatorname{dist}(y,K) \geq c\rho$ for $y \in B_{\theta\frac{\rho}{2}} \cap (B_r \setminus B_{2\rho})$, the first and second integrals on the right do not exceed $cr^n\rho^{2m-l-n}\operatorname{cap}_m K$ in view of (2.2.15). Hence, for $r \leq \theta\frac{\rho}{2}$, the mean value of $|\nabla_k U_K|$ on B_r is dominated by

$c\rho^{2m-l-n}\operatorname{cap}_m K$. Let $r > \theta\,\frac{\rho}{2}$. It follows from Corollary 2.2.1 that the integral

$$I_l(\rho) := \int\limits_{B_{2\rho}\setminus B_{\theta\,\frac{\rho}{2}}} \big|\nabla_l U_K(y)\big|\,dx$$

is majorized by

$$c\Bigg(\int\limits_{B_{2\rho}\setminus B_{\theta\,\frac{\rho}{2}}} \big|\nabla_l U_K(y)^2\big|\,dy +$$

$$+\int\limits_{B_{2\rho}\setminus B_{\theta\,\frac{\rho}{2}}} dy \int\limits_{\mathbb{R}^n} \sum_{\substack{1\le r,s\le m\\ r+s>l}} \frac{|\nabla_r U_K(x)|\,|\nabla_s U_K(x)|}{|x-y|^{n-r-s+l}}\,dx\Bigg) \le$$

$$\le c_1\rho^n \sum_{1\le r,s\le m} \int\limits_{\mathbb{R}^n} \frac{|\nabla_r U_K(x)|\,|\nabla_s U_K(x)|}{(\rho+|x|)^{n-r-s+l}}\,dx \le$$

$$\le c_2\rho^{2m-l} \sum_{1\le r,s\le m} \int\limits_{\mathbb{R}^n} \frac{|\nabla_r U_K(x)|\,|\nabla_s U_K(x)|}{|x|^{2m-r-s}}\,dx.$$

Hence by Hardy's inequality, we obtain

$$I_l(\rho) \le c\rho^{2m-l} \int\limits_{\mathbb{R}^n} \big|\nabla_m U_K(x)\big|^2\,dx \le c\rho^{2m-l}\operatorname{cap}_m K.$$

The proof is complete. □

2.3 Weighted Positivity of $L(\partial)$

Let $2m < n$. It follows from (2.2.8) that the condition of weighted positivity (2.1.5) is equivalent to the inequality

$$\int\limits_{\mathbb{R}^n} \sum_{j=1}^{m} \sum_{|\mu|=|\nu|=j} \partial^\mu u(x)\cdot\partial^\nu u(x)\cdot P_{\mu\nu}(\partial)F(x)\,dx \ge$$

$$\ge c\sum_{k=1}^{m} \int\limits_{\mathbb{R}^n} \frac{|\nabla_k u(x)|^2}{|x|^{n-2k}}\,dx \tag{2.3.1}$$

for all $u \in C_0^\infty(\mathbb{R}^n\setminus O)$. Since the restriction of F to ∂B_1 is a smooth function of the coefficients of $L(\partial)$, the last inequality implies that the set of the operators $L(\partial)$ which are positive with the weight F is open.

Proposition 2.3.1. *Inequality* (2.1.5)*, valid for all* $u \in C_0^\infty(\mathbb{R}^n\setminus O)$*, implies*

$$\mathscr{B}(u, uF) \ge 2^{-1}u(0)^2 + c\sum_{j=1}^{m} \int\limits_{\mathbb{R}^n} \frac{|\nabla_j u(x)|^2}{|x|^{n-2j}}\,dx \tag{2.3.2}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

Proof. Let $u \in C_0^\infty(\mathbb{R}^n)$, $0 < \varepsilon < \frac{1}{2}$ and $\eta_\varepsilon(x) = \eta((\log\varepsilon)^{-1}\log|x|)$, where $\eta \in C_0^\infty(\mathbb{R}^1)$, $\eta(t) = 0$ for $t \geq 2$, and $\eta(t) = 1$ for $t \leq 1$. Clearly, $\eta_\varepsilon(x) = 0$ for $x \in \mathbb{R}^n \setminus B_\varepsilon$, all derivatives of η_ε vanish outside $B_\varepsilon \setminus B_{\varepsilon^2}$, and

$$\big|\nabla_j \eta_\varepsilon(x)\big| \leq c_j |\log\varepsilon|^{-1}|x|^{-j}.$$

By (2.1.5), the bilinear form $\mathscr{B}$ defined by (2.2.2) satisfies

$$\mathscr{B}\big((1-\eta_\varepsilon)u, (1-\eta_\varepsilon)uF\big) \geq c\sum_{j=1}^{m}\int_{\mathbb{R}^n} \big|\nabla_j\big((1-\eta_\varepsilon)u\big)\big|^2 \frac{dx}{|x|^{n-2j}}\,. \tag{2.3.3}$$

Using the just mentioned properties of η_ε, we see that

$$\begin{aligned}\left|\left(\int_{\mathbb{R}^n} \big|\nabla_j\big((1-\eta_\varepsilon)u\big)\big|^2 \frac{dx}{|x|^{n-2j}}\right)^{\frac{1}{2}} - \left(\int_{\mathbb{R}^n} (1-\eta_\varepsilon)^2|\nabla_j u|^2 \frac{dx}{|x|^{n-2j}}\right)^{\frac{1}{2}}\right| &\leq \\ \leq \left(\int_{\mathbb{R}^n} \big|\big[\nabla_j, 1-\eta_\varepsilon\big]u\big|^2 \frac{dx}{|x|^{n-2j}}\right)^{\frac{1}{2}} &\leq \\ \leq c(u)\sum_{k=1}^{j}\int_{\mathbb{R}^n} |\nabla_k \eta_\varepsilon|^2 \frac{dx}{|x|^{n-2j}} &= O\big(|\log\varepsilon|^{-1}\big),\end{aligned}$$

where $[S,T]$ stands for the commutator $ST - TS$. Hence by (2.3.3),

$$\liminf_{\varepsilon\to 0} \mathscr{B}\big((1-\eta_\varepsilon)u, (1-\eta_\varepsilon)uF\big) \geq c\sum_{j=1}^{m}\int_{\Omega} |\nabla_j u|^2 \frac{dx}{|x|^{n-2j}}\,. \tag{2.3.4}$$

Since, clearly,

$$\begin{aligned}\Big|\mathscr{B}\big(\eta_\varepsilon(u-u(0)), \eta_\varepsilon(u-u(0))F\big)\Big| &\leq \\ \leq c\sum_{j=1}^{m}\int_{B_\varepsilon} \frac{|\nabla_j(\eta_\varepsilon(u-u(0)))|^2}{|x|^{n-2j}}\,dx &= O(\varepsilon),\end{aligned}$$

one can replace $(1-\eta_\varepsilon)u$ in the left-hand side of (2.3.4)) by $u - u(0)\eta_\varepsilon$. We use the identity

$$\begin{aligned}\mathscr{B}\big((u-u(0)\eta_\varepsilon), (u-u(0)\eta_\varepsilon)F\big) &= \\ = \mathscr{B}(u,uF) + u(0)^2\big(\mathscr{B}(\eta_\varepsilon,\eta_\varepsilon F) - \mathscr{B}(\eta_\varepsilon, F)\big) &- \\ - u(0)\Big(\mathscr{B}\big(\eta_\varepsilon, (u-u(0))F\big) + \mathscr{B}(u,\eta_\varepsilon F)\Big).\end{aligned}$$

It is straightforward that $|\mathscr{B}(\eta_\varepsilon,(u-u(0)F|+|\mathscr{B}(u,\eta_\varepsilon F)|\le c\varepsilon$. Therefore

$$\liminf_{\varepsilon\to 0}\mathscr{B}\big(\eta_\varepsilon(u-u(0)),\eta_\varepsilon(u-u(0))F\big)= \\ =\mathscr{B}(u,uF)+u(0)^2\big(\mathscr{B}(\eta_\varepsilon,\eta_\varepsilon F)-\mathscr{B}(\eta_\varepsilon,F)\big).$$

Since $\mathscr{B}(\eta_\varepsilon,F)=1$ and since it follows from (2.2.8) that

$$\big|2\mathscr{B}(\eta_\varepsilon,\eta_\varepsilon F)-1\big|\le c\sum_{j=1}^{m}\int\limits_{B_\varepsilon\setminus B_{\varepsilon^2}}|\nabla_j\eta_\varepsilon|^2\,\frac{dx}{|x|^{n-2j}}=O\big(|\log\varepsilon|^{-1}\big),$$

we arrive at (2.3.2). □

Proposition 2.3.2. *The positivity of $L(\partial)$ with the weight F implies $F(x)>0$.*

Proof. Let

$$u_\varepsilon(x)=\varepsilon^{-\frac{n}{2}}\eta\big(\varepsilon^{-1}(x-\omega)\big)|\xi|^{-m}\exp\big(i,(x,\xi)\big),$$

where η is a nonzero function in $C_0^\infty(\mathbb{R}^n)$, ε is a positive number, $\omega\in\partial B_1$, and $\xi\in\mathbb{R}^n$. We put u_ε into the inequality

$$\operatorname{Re}\int\limits_{\mathbb{R}^n}\sum_{j=1}^{m}\sum_{|\mu|=|\mu|=j}\partial^\mu u(x)\cdot\partial^\nu\overline{u(x)}\cdot P_{\mu\nu}(\partial)F(x)\,dx\ge \\ \ge c\sum_{j=1}^{m}\int\limits_{\mathbb{R}^n}\big|\nabla_j u(x)\big|^2\,\frac{dx}{|x|^{n-2j}}$$

which is equivalent to (2.3.1). Taking the limits as $|\xi|\to\infty$, we obtain

$$\sum_{|\alpha|=|\beta|=m}a_{\alpha\beta}\Big(\frac{\xi}{|\xi|}\Big)^{\alpha+\beta}\varepsilon^{-n}\int\limits_{\mathbb{R}^n}\big|\eta(\varepsilon^{-1}(x-\omega))\big|^2F(x)\,dx\le \\ \le c\varepsilon^{-n}\int\limits_{\mathbb{R}^n}\big|\eta(\varepsilon^{-1}(x-\omega))\big|^2\,dx.$$

Now the positivity of F follows by the limit passage as $\varepsilon\to 0$. □

Remark 2.3.1. The positivity of the left-hand side in (2.3.1) is equivalent to the inequality

$$\Re\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\frac{L(i\xi)}{L(i(\xi-\eta))}\,f(\xi)\overline{f(\eta)}\,d\xi\,d\eta>0,$$

valid for all nonzero $f\in C_0^\infty(\mathbb{R}^n)$. The last inequality was studied by S. Eilertsen [12].

2.4 More Properties of the L-Capacitary Potential

Let $L(\partial)$ be positive with the weight F. Then identity (2.2.9) implies that the L-capacitary potential of a compact set K with positive m-harmonic capacity satisfies

$$0 < U_K(x) < 2 \text{ on } \mathbb{R}^n \setminus K. \tag{2.4.1}$$

We show that, in general, the bound 2 in (2.4.1) cannot be replaced by 1.

Proposition 2.4.1. *If $L = \Delta^{2m}$, then there exists a compact set K such that $(U_K - 1)\big|_{\mathbb{R}^n\setminus K}$ changes sign in any neighborhood of a point of K.*

Proof. Let C be an open cone in $\mathbb{R}^n_+ = \{x = (x', x_n) : \ x_n > 0\}$, and let $C_\varepsilon = \{x : \ (\varepsilon^{-1}x', x_n) \in C\}$ with sufficiently small $\varepsilon > 0$. We define the compact set K as $\overline{B}_1 \setminus C_\varepsilon$. Suppose that $U_K(x) - 1$ does not change sign on a δ-neighborhood of the origin. Then either $U_K - 1$ or $1 - U_K$ is a nontrivial nonnegative $2m$-harmonic function on $B_\delta \cap C_\varepsilon$ subject to zero Dirichlet condition on $B_\delta \cap \partial C_\varepsilon$, which contradicts [23, Lemma 1]. The result follows. □

We give a lower pointwise estimate for U_K stated in terms of capacity (cf. the upper estimate (2.2.15)).

Proposition 2.4.2. *Let $n > 2m$, and let $L(\partial)$ be positive with the weight F. If K is a compact subset of B_d and $y \in \mathbb{R}^n \setminus K$, then*

$$U_K(y) \geq c\big(|y| + d\big)^{2m-n} \operatorname{cap}_m K.$$

Proof. Let a be a point in the semiaxis $(2, \infty)$ which is specified later. By (2.3.2),

$$\begin{aligned} U_K(y) &\geq c\big(|y| + ad\big)^{2m-n} \int\limits_{B_{ad}} |\nabla_m u|^2\, dx \geq \\ &\geq c\big(|y| + ad\big)^{2m-n} \bigg(\operatorname{cap}_m K - \int\limits_{\mathbb{R}^n\setminus B_{ad}} |\nabla_m u|^2\, dx\bigg). \end{aligned} \tag{2.4.2}$$

It follows from Proposition 2.2.1 that for $x \in \mathbb{R}^n \setminus B_{ad}$,

$$\big|\nabla_m U_K(x)\big| \leq c_0 \frac{\operatorname{cap}_m K}{(|x| - d)^{n-2m}} \leq 2^{n-2m} c_0 \frac{\operatorname{cap}_m K}{|x|^{n-m}}\,.$$

Hence,

$$\int\limits_{\mathbb{R}^n\setminus B_{ad}} |\nabla_m u|^2\, dx \leq c(\operatorname{cap}_m K)^2 \int\limits_{\mathbb{R}^n\setminus B_{ad}} \frac{dx}{|x|^{2n-2m}} = c_1 \frac{(\operatorname{cap}_m K)^2}{(ad)^{n-2m}}\,,$$

and by (2.4.2),

$$U_K(y) \geq \frac{\operatorname{cap}_m K}{(|y|+d)^{n-2m}} \Big(1 - c\, \frac{\operatorname{cap}_m K}{(ad)^{n-2m}} \Big).$$

Choosing a to make the difference in braces positive, we complete the proof. □

2.5 Poincaré Inequality with m-Harmonic Capacity

The material in this section will be used in the proof of sufficiency in Theorems 2.1.1 and 2.1.2.

We say that a compact subset of the ball $\overline{B}_\rho = \{x : \ |x| \leq \rho\}$ is m-small, $2m \leq n$, if

$$\operatorname{cap}_m(e, B_{2\rho}) \leq 16^{-n} \rho^{n-2m}.$$

In the case $2m > n$, only the empty subset of $\overline{B}_\rho$ will be called m-small.

Let $\overline{u}_\rho$ denote the mean value of u on the ball B_ρ, i.e.

$$\overline{u}_\rho = (\operatorname{mes}_n B_\rho)^{-1} \int\limits_{B_\rho} u(x)\, dx.$$

We introduce the seminorm

$$|||u|||_{m,B_\rho} = \Big(\sum_{j=1}^{m} \rho^{2(j-m)} \|\nabla_j u\|^2_{L_2(B_\rho)} \Big)^{\frac{1}{2}}.$$

Proposition 2.5.1 ([46, 10.1.2])**.** *Let e be a closed subset of the ball $\overline{B}_\rho$.*

(1) *For all $u \in C^\infty(\overline{B}_\rho)$ with $\operatorname{dist}(\operatorname{supp} u, e) > 0$ the inequality*

$$\|u\|_{L_2(B_\rho)} \leq C|||u|||_{m,B_\rho} \tag{2.5.1}$$

is valid, where

$$C^{-2} \geq c\rho^{-n} \operatorname{cap}_m(e, B_\rho)$$

and c depends only on m and n.

(2) *If e is m-small and if inequality* (2.5.1) *holds for all $u \in C^\infty(\overline{B}_\rho)$ with* $\operatorname{dist}(\operatorname{supp} u, e) > 0$*, then the best constant C in* (2.5.1) *satisfies*

$$C^{-2} \leq c\rho^{-n} \operatorname{cap}(e, B_\rho)$$

The second assertion of this proposition will not be used in the sequel and therefore it will not be proved here. Its proof can be found in [46, pp. 405, 406]. In order to check the first assertion we need the following auxiliary result.

Lemma 2.5.1. *Let a be a compact set in $\overline{B}_1$. There exists a constant c depending on n and m and such that*

$$c^{-1}\operatorname{cap}_m(e,B_2)\le$$
$$\le \inf\Big\{\|1-u\|_{H^m(B_1)}:\ u\in C^\infty(\overline{B}_1),\ \operatorname{dist}(\operatorname{supp} u,e)>0\Big\}\le$$
$$\le c\operatorname{cap}_m(e,B_2). \tag{2.5.2}$$

Proof. To obtain the left estimate we need the following well-known assertion.

There exists a linear continuous mapping $A: C^{k-1,1}(\overline{B}_1)\to C^{k-1,1}(\overline{B}_2)$, such that

(i) $Av=v$ on $\overline{B}_1$;

(ii) if $\operatorname{dist}(\operatorname{supp} v,e)>0$, then $\operatorname{dist}(\operatorname{supp} Av,e)>0$;

(iii) the inequality
$$\big\|\nabla_i(Av)\big\|_{L_2(B_2)}\le c\|\nabla_i v\|_{L_2(B_1)} \tag{2.5.3}$$
is valid with $i=0,1,\dots,l$ and c independent of v.

Let $v=A(1-u)$ and let η denote a function in $C_0^\infty(B_2)$ which is equal to 1 in a neighborhood of the ball B_1. Then

$$\operatorname{cap}(e,B_2)\le c\big\|\nabla_l(\eta v)\big\|^2_{L_2(B_2)}\le c\|v\|^2_{H^m(B_2)}. \tag{2.5.4}$$

Now the left estimate in (2.5.2) follows from (2.5.3) and (2.5.4).

Next we derive the right estimate in (2.5.2). Let $w\in C_0^\infty(B_2)$, $w=1$, on a neighborhood of e.

Then
$$\|w\|_{H^m(B_1)}\le c\|\nabla_m w\|_{L_2(B_2)}.$$

Minimizing the last norm, we obtain

$$\inf_u\|1-u\|^2_{H^m(B_1)}\le \inf\|w\|^2_{H^m(B_1)}\le c\operatorname{cap}(e,B_2).$$

Thus the proof is complete. □

Proof of the first assertion of Proposition 2.5.1. It suffices to consider only the case $d=1$ and then use a dilation.

1) Let
$$N=\Bigg(\frac{1}{\operatorname{mes}_n B_1}\int\limits_{B_1}u^2(x)\,dx\Bigg)^{\frac12}.$$

Since $\operatorname{dist}(\operatorname{supp} u,e)>0$, it follows from Lemma 2.5.1 that

$$\operatorname{cap}_m(e,B_2)\le c\|1-N^{-1}u\|^2_{H^m(B_1)}=$$
$$=cN^{-2}|||u|||^2_{m,B_1}+c\|1-N^{-1}u\|^2_{L_2(B_1)},$$

i.e.

$$N^2 \operatorname{cap}_m(e, B_2) \le c|||u|||^2_{m,B_1} + c\|N - u\|^2_{L_2(B_1)}. \tag{2.5.5}$$

Without loss of generality we assume that $\overline{u}_1 \ge 0$. Then

$$\sqrt{\operatorname{mes}_n B_1}\, |N - \overline{u}_1| = \|u\|_{L_2(B_1)} - \|\overline{u}\|_{L_2(B_1)} \le \|u - \overline{u}_1\|_{L_2(B_1)}.$$

Consequently,

$$\|N - u\|_{L_2(B_1)} \le \|N - \overline{u}_1\| + \|u - \overline{u}_1\|_{L_2(B_1)} \le 2\|u - \overline{u}_1\|_{L_2(B_1)}.$$

Hence, by (2.5.5) and the Poincaré inequality

$$\|u - \overline{u}_1\|_{L_2(B_1)} \le \|\nabla u\|_{L_2(B_1)}$$

we obtain

$$\operatorname{cap}(e, B_2)\|u\|^2_{L_2(B_1)} \le c|||u|||^2_{m,B_1},$$

which completes the proof. □

2.6 Proof of Sufficiency in Theorem 2.1.2

In the lemma below and henceforth we use the notation

$$M_\rho(u)\rho^{-n} \int\limits_{\Omega\cap S_\rho} u(x)^2\, dx, \quad S_\rho = \big\{x : \ \rho < |x| < 2\rho\big\}.$$

Lemma 2.6.1. *Let $2m < n$ and let $L(\partial)$ be positive with the weight F. Further, let $u \in \overset{\circ}{H}{}^m(\Omega)$ be a solution of*

$$L(\partial)u = 0 \quad \text{on} \ \ \Omega \cap B_{2\rho}. \tag{2.6.1}$$

Then $\mathscr{B}(u\eta_\rho, u\eta_\rho F_y) \le cM_\rho(u)$ for an arbitrary point $y \in B_\rho$, where

$$\eta_\rho(x) = \eta\Big(\frac{x}{\rho}\Big), \quad \eta \in C_0^\infty(B_2), \quad \eta = 1 \ \ \text{on} \ \ B_{\frac{3}{2}}, \quad F_y(x) = F(x - y).$$

Proof. By the definition of $\mathscr{B}$,

$$\begin{aligned}&\mathscr{B}(u\eta_\rho, u\eta_\rho F_y) - \mathscr{B}(u, u\eta_\rho^2 F_y) = \\ &= \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \int\limits_\Omega \Big([\partial^\alpha, \eta_\rho]u \cdot \partial^\beta(u\eta_\rho F_y) - \partial^\alpha u \cdot [\partial^\beta, \eta_\rho](u\eta_\rho F_y)\Big)\, dx.\end{aligned} \tag{2.6.2}$$

It follows from (2.6.1) that $\mathscr{B}(u, u\eta_\rho^2 F_y) = 0$. The absolute value of the right-hand side in (2.6.2) is majorized by

$$c\sum_{j=0}^m \rho^{2j-n} \int\limits_\Omega \zeta_\rho |\nabla_j u|^2\, dx, \tag{2.6.3}$$

where $\zeta_\rho(x) = \zeta(\frac{x}{\rho})$, $\zeta \in C_0^\infty(S_1)$, and $\zeta = 1$ on $\operatorname{supp}|\nabla_\eta|$. The result follows by the well-known local energy estimate (see [5, Chapter 3])

$$\int\limits_{\Omega} \zeta_\rho |\nabla_j u|^2 \, dx \le c\rho^{-2j} \int\limits_{\Omega\cap S_\rho} u^2 \, dx. \tag{2.6.4}$$

Combining Proposition 2.3.1 and Lemma 2.6.1, we arrive at the following local estimate. □

Corollary 2.6.1. *Let the conditions of Lemma 2.6.1 be satisfied. Then*

$$u(y)^2 + \int\limits_{\Omega\cap B_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{|x-y|^{n-2k}} \, dx \le cM_\rho(u), \quad y \in \Omega \cap B_\rho. \tag{2.6.5}$$

We need the following Poincaré-type inequality proved in Proposition 2.3.2.

Lemma 2.6.2. *Let $u \in \overset{\circ}{H}{}^m(\Omega)$. Then for all $\rho > 0$,*

$$M_\rho(u) \le \frac{c\rho^{n-2m}}{\operatorname{cap}_m(\overline{S}_\rho \setminus \Omega)} \int\limits_{\Omega\cap S_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{\rho^{n-2k}} \, dx. \tag{2.6.6}$$

Corollary 2.6.2. *Let the conditions of Lemma* 2.6.1 *be satisfied. Then for all points $y \in \Omega \cap B_\rho$, the estimate*

$$u(y)^2 + \int\limits_{\Omega\cap B_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{|x-y|^{n-2k}} \, dx \le \frac{c\rho^{n-2m}}{\operatorname{cap}_m(\overline{S}_\rho, \Omega)} \int\limits_{\Omega\cap S_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{\rho^{n-2k}} \, dx$$

holds.

Proof. We combine Corollary 2.6.1 with inequality (2.6.6). □

Lemma 2.6.3. *Let $2m < n$, and let $L(\partial)$ be positive with weight F. Also, let $u \in \overset{\circ}{H}{}^m(\Omega)$ satisfy $L(\partial)u = 0$ on $\Omega \cap B_{2\rho}$. Then, for all $\rho \in (0, R)$,*

$$\sup\left\{|u(p)|^2 : \; p \in \Omega \cap B_\rho\right\} + \int\limits_{\Omega\cap B_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{|x|^{n-2k}} \, dx \le$$

$$\le cM_R(u) \exp\Bigg(- c \int\limits_{\rho}^{R} \operatorname{cap}_m(\overline{B}_\tau \setminus \Omega) \, \frac{d\tau}{\tau^{n-2m+1}} \Bigg). \tag{2.6.7}$$

Proof. Let us use the notation

$$\gamma_m(r) = r^{2m-n}\,\mathrm{cap}_m(\overline{S}_r \setminus \Omega). \tag{2.6.8}$$

It is sufficient to prove (2.6.7) only for $\rho \le \frac{R}{2}$ because in the opposite case the result follows from Corollary 2.6.1. Denote the first and second terms on the left in (2.6.7) by φ_ρ and ψ_ρ, respectively. It follows from Corollary 2.6.2 that for $r \le R$,

$$\varphi_r + \psi_r \le \frac{c}{\gamma_m(r)}\,(\psi_{2r} - \psi_r) \le \frac{c}{\gamma_m(r)}\,(\psi_{2r} - \psi_r + \varphi_{2r} - \varphi_r).$$

This, along with the obvious inequality $\gamma_m(r) \le c$, implies

$$\varphi_r + \psi_r \le c\exp(-c_0\gamma_m(r))(\varphi_{2r} + \psi_{2r}).$$

By setting $r = 2^{-j}R$, $j = 1, 2, \ldots$, we arrive at the estimate

$$\varphi_{2^{-l}R} + \psi_{2^{-l}R} \le c\exp\Big(-c\sum_{j=1}^{l}\gamma_m(2^{-j}R)\Big)(\varphi_R + \psi_R).$$

We choose l so that $l < \log_2(\frac{R}{\rho}) \le l+1$ in order to obtain

$$\varphi_\rho + \psi_\rho \le c\exp\Big(-c_0\sum_{j=1}^{l}\gamma_m(2^{-j}R)\Big)(\varphi_R + \psi_R).$$

Now we notice that by Corollary 2.6.1, $\varphi_R + \psi_R \le cM_R(u)$. Assuming that cap_m is replaced in definition (2.6.8) by the equivalent Riesz capacity c_{2m} and using the subadditivity of this capacity, we see that

$$\varphi_\rho + \psi_\rho \le$$
$$\le cM_R(u)\exp\Big(-c_0\sum_{j=1}^{l}\frac{c_{2m}(\overline{B}_{2^{1-j}R}\setminus\Omega) - c_{2m}(\overline{B}_{2^{-j}R}\setminus\Omega)}{(2^{1-j}R)^{n-2m}}\Big). \tag{2.6.9}$$

Noting that the last sum is equal to

$$-\frac{c_{2m}(\overline{B}_{2^{-l}R}\setminus\Omega)^{n-2m}}{(2^{-l}R)^{n-2m}} + (1-2^{-n+2m})\sum_{j=0}^{l-1}\frac{c_{2m}(\overline{B}_{2^{-j}R}\setminus\Omega)}{(2^{-j}R)^{n-2m}} \ge$$
$$\ge c_1\int_\rho^R \mathrm{cap}_m(\overline{B}_\tau\setminus\Omega)\,\frac{d\tau}{\tau^{n-2m+1}} - c_2,$$

we obtain the result from (2.6.9). □

By (2.6.7) we conclude that (2.1.6) is sufficient for the regularity of O.

2.7 Equivalence of Two Definitions of Regularity

Proposition 2.7.1. *In the case $m = 1$, the regularity in the sense of Definition* 2.1.1 *is equivalent to Wiener's regularity.*

Proof. Let O be regular in the Wiener sense and let u be the solution of (2.1.4) with $m = 1$. We introduce the Newton potential u_f with the density f and note that u_f is smooth in a neighborhood of $\partial\Omega$. Since $v = u - u_f$ is the $H^1(\Omega)$-solution of the Dirichlet problem

$$\begin{aligned} -\Delta u &= 0 \ \text{ on } \ \Omega, \\ v &= -u_f \ \text{ on } \ \partial\Omega, \end{aligned}$$

it follows from Wiener's regularity that e is continuous at O (see [32, Section 3]). Hence O is regular in the sense of Definition 2.1.1.

In order to prove the converse assertion we consider the Dirichlet problem

$$\begin{aligned} -\Delta w = 0 \ \text{ on } \ \Omega, \ \ w \in \overset{\circ}{H}{}^1(\Omega), \\ w(x) = (2n)^{-1}|x|^2 \ \text{ on } \ \partial\Omega. \end{aligned}$$

We show that w is continuous at O provided O is regular in the sense of Definition 2.1.1. In fact, since the function

$$z(x) = w(x) - (2n)^{-1}|x|^2$$

satisfies

$$-\Delta z = 1 \ \text{ on } \ \Omega, \ \ w \in H^1(\Omega),$$

we have

$$z(x) = \int\limits_{\Omega} G(x,s)\, ds,$$

where G is Green's function of the Dirichlet problem. Therefore,

$$z(x) = \int\limits_{\Omega} G(x,s)h(s)\, ds + \int\limits_{\Omega} G(x,s)(1 - h(s))\, ds,$$

where $h \in C_0^\infty(\Omega)$, $0 \le h \le 1$ and $h = 1$ on a domain $\omega, \overline{\omega} \subset \Omega$.

The first integral tends to zero as $x \to 0$ by the regularity assumption. Hence,

$$\limsup_{x\to 0} |z(x)| \le c \int\limits_{\Omega\setminus\omega} \frac{ds}{|x-s|^{n-2}} = O\Big(\big(\operatorname{mes}_n(\Omega\setminus\omega)\big)^{\frac{2}{n}} \Big)$$

for $n > 2$, and

$$\limsup_{x\to 0} |z(x)| \le c_1 \int\limits_{\Omega\setminus\omega} \big|\log\big(c_2|x-s|\big)\big|\, ds = O\Big(\big(\operatorname{mes}_2(\Omega\setminus\omega)\big)^{1-\varepsilon} \Big)$$

for $n = 2$.

Since $\text{mes}_n(\Omega \setminus \omega)$ can be taken arbitrarily small, $z(x) \to 0$ as $x \to 0$. As a result, we find that z satisfies the definition of barrier (see [29, Chapter 4, Section 2]), and by Theorem 4.8 in [29], the regularity of O in the Wiener sense follows. □

2.8 Regularity as a Local Property

We show that the regularity of a point O does not depend on the geometry of Ω at any positive distance from O.

Lemma 2.8.1. *Let $n > 2m$ and let $L(\partial)$ be positive with the weight F. If O is regular for the operator L on Ω, then the solution $u \in \overset{\circ}{H}{}^m(\Omega)$ of*

$$L(\partial)u = \sum_{\{\alpha\colon |\alpha| \le m\}} \partial^\alpha f_\alpha \quad on \;\; \Omega,$$

with $f_\alpha \in L_2(\Omega) \cap C^\infty(\Omega)$ and $f_\alpha = 0$ in a neighborhood of O, satisfies (2.1.2).

Proof. Let $\zeta \in C^\infty(\Omega)$. We represent u as the sum $v + w$, where $w \in \overset{\circ}{H}{}^m(\Omega)$ and

$$L(\partial)u = \sum_{\{\alpha\colon |\alpha| \le m\}} \partial^\alpha(\zeta f_\alpha).$$

By the regularity of O, we have $v(x) = o(1)$ as $x \to 0$. We verify that w can be made arbitrarily small by making the Lebesgue measure of the support of $1 - \zeta$ sufficiently small. Let $f_\alpha = 0$ on B_δ, and let $y \in \Omega$, $|y| < \frac{\delta}{2}$. By the definition of w and by (2.3.2),

$$\sum_{\{\alpha\colon |\alpha| \le m\}} \int_\Omega (1-\zeta) f_\alpha (-\partial)^\alpha (w F_y)\, dx \ge$$

$$\ge 2^{-1} w^2(p) + c \sum_{k=1}^{m} \int_\Omega \frac{|\nabla_k w(x)|^2}{|x-y|^{n-2k}}\, dx,$$

where $F_y(x) = F_y(x - y)$ and c does not depend on Ω. The proof is complete. □

Lemma 2.8.2. *Let O be a regular point for the operator $L(\partial)$ on Ω, and let Ω' be a domain such that $\Omega' \cap B_{2\rho} = \Omega \cap B_{2\rho}$ for some $\rho > 0$. Then O is regular for the operator $L(\partial)$ on Ω'.*

Proof. Let $u \in \overset{\circ}{H}{}^m(\Omega')$ satisfy $L(\partial)u = f$ on Ω' with $f \in C_0^\infty(\Omega')$. We introduce $\eta_\rho(x) = \eta(\frac{x}{\rho})$, $\eta \in C_0^\infty(B_2)$, $\eta = 1$ on $B_{\frac{3}{2}}$. Then $\eta_\rho u \in \overset{\circ}{H}{}^m(\Omega)$ and

$$L(\partial)(\eta_\rho u) = \eta_\rho f + \big[L(\partial), \eta_\rho\big] u \;\; \text{on} \;\; \Omega.$$

Since the commutator $[L(\partial), \eta_\rho]$ is a differential operator of order $2m-1$ with smooth coefficients supported by $B_{2\rho} \setminus \overline{B}_{\frac{3\rho}{2}}$, it follows that

$$L(\partial)(\eta_\rho u) = \sum_{\{\alpha\colon |\alpha| \le m\}} \partial^\alpha f_\alpha \quad \text{on} \quad \Omega,$$

where $f_\alpha \in L_2(\Omega) \cap C^\infty(\Omega)$ and $f_\alpha = 0$ in a neighborhood of O. Therefore, $(\eta_\rho u)(x) = o(1)$ as x tends to O by Lemma 2.8.1 and by the regularity of O with respect to $L(\partial)$ on Ω. □

2.9 Proof of Necessity in Theorem 2.1.2

Let $n > 2m$, and let condition (2.1.6) be violated. We fix a sufficiently small $\varepsilon > 0$ depending on the operator $L(\partial)$ and choose a positive integer N in order to have

$$\sum_{j=N}^{\infty} 2^{(n-2m)j} \operatorname{cap}_m(\overline{B}_{2^{-j}} \setminus \Omega) < \varepsilon. \tag{2.9.1}$$

By Lemma 2.8.2, it suffices to show that O is irregular with respect to the domain $\mathbb{R}^n \setminus K$, where $K = \overline{B}_{2^{-N}} \setminus \Omega$. Denote by U_K the L-capacitary potential of K. By subtracting a cut-off function $\eta \in C_0^\infty(\mathbb{R}^n)$ used in the proof of Lemma 2.8.2 from U_K and noting that η is equal to 1 in a neighborhood of K, we obtain a solution of $Lu = f$ on $\mathbb{R}^n \setminus K$ with $f \in C_0^\infty(\mathbb{R}^n)$ and zero Dirichlet data on $\partial(\mathbb{R}^n \setminus K)$. Therefore, it suffices to show that $U_K(x)$ does not tend to 1 as $x \to 0$. This statement results from (2.9.1) and the inequality

$$\mathscr{M} U_K(0) \le c \sum_{j \ge N} 2^{(n-2m)j} \operatorname{cap}_m(\overline{B}_{2^{-j}} \setminus \Omega), \tag{2.9.2}$$

which is obtained in what follows.

We introduce the L-capacitary potential $U^{(j)}$ of the set

$$K^{(j)} = K \cap \big(\overline{B}_{2^{1-j}} \setminus B_{2^{-1-j}}\big), \quad j = N, N+1, \dots .$$

We also need a partition of unity $\{\eta^{(j)}\}_{j \ge N}$ subordinate to the covering of K by the sets $B_{2^{1-j}} \setminus \overline{B}_{2^{-1-j}}$. One can construct this partition of unity so that $|\nabla_k \eta^{(j)}| \le c_k 2^{kj}$, $k = 1, 2, \dots$. We now define the function

$$V = \sum_{j \ge N} \eta(j) U^{(j)} \tag{2.9.3}$$

satisfying the same Dirichlet conditions as U_K. Let $Q_u(y)$ denote the quadratic form

$$\sum_{k=1}^{m} \int_{\mathbb{R}^n} \frac{|\nabla_k u(x)|^2}{|x-y|^{n-2k}}\, dx,$$

and let $I_\lambda f$ be the Riesz potential $|x|^{\lambda-n} * f$, $0<\lambda<n$. It is standard that $\mathscr{M} I_\lambda f(0) \le c I_\lambda f(0)$ if $f \ge 0$ (see the proof of [29, Theorem 1.11]). Hence,

$$\mathscr{M} Q_u(0) \le c \sum_{k=1}^{m} \int_{\mathbb{R}^n} \big|\nabla_k u(x)\big|^2 \frac{dx}{|x-y|^{n-2k}} \,.$$

This inequality and definition (2.9.3) show that

$$\begin{aligned}\mathscr{M} Q_V(0) &\le \sum_{j\ge N} \sum_{k=0}^{m} \int_{\overline{B}_{2^{1-j}} \setminus B_{2^{-1-j}}} \big|\nabla_k U^{(j)}(x)\big|^2 \frac{dx}{|x|^{n-2k}} \le \\ &\le c \sum_{j\ge N} 2^{(n-2m)j} \int_{\mathbb{R}^n} \big|\nabla_k U^{(j)}(x)\big|^2 \frac{dx}{|x|^{2(m-k)}} \le \\ &\le c \sum_{j\ge N} 2^{(n-2m)j} \int_{\mathbb{R}^n} \big|\nabla_m U^{(j)}(x)\big|^2 \, dx,\end{aligned}$$

the last estimate being based on Hardy's inequality. Therefore,

$$\mathscr{M} Q_V(0) \le c \sum_{j\ge N} 2^{(n-2m)j} \operatorname{cap}_m K^{(j)}. \tag{2.9.4}$$

Furthermore, by Proposition 2.2.2,

$$\mathscr{M} V(0) \le c \sum_{j\ge N} 2^{(n-2m)j} \operatorname{cap}_m K^{(j)}. \tag{2.9.5}$$

We deduce similar inequalities for $W = U_K - V$. Note that W solves the Dirichlet problem with zero boundary data for the equation $L(\partial)W = -L(\partial)V$ on $\mathbb{R}^n \setminus K$. Hence by (2.3.2), we conclude that for $y \in \mathbb{R}^n \setminus K$,

$$\begin{aligned}&2^{-1} W(y)^2 + c Q_W(y) \le \\ &\qquad \le \Big| \int_{\mathbb{R}^n} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \partial^\alpha V(x) \cdot \partial^\beta \big(W(x) F(x-y)\big)\, dx \Big|. \end{aligned} \tag{2.9.6}$$

2.10 Proof of Sufficiency in Theorem 2.1.1

In the case of $n = 2m$, the operator $L(\partial)$ is arbitrary. We introduce a sufficiently large positive constant C subject to a condition specified later. We also need a fundamental solution

$$F(x) = \varkappa \log |x|^{-1} + \Psi\Big(\frac{x}{|x|}\Big) \tag{2.10.1}$$

of $L(\partial)$ in $\mathbb{R}^n$ (see [5]). Here $\varkappa = \mathit{const}$, and we assume that the function Ψ, which is defined up to a constant term, is chosen so that

$$F(x) \ge \varkappa \log\big(4|x|^{-1}\big) + C \quad \text{on } B_2. \tag{2.10.2}$$

Proposition 2.10.1. *Let Ω be an open set in $\mathbb{R}^n$ of diameter d_Ω. Then for all $u \in C_0^\infty$ and $y \in \Omega$,*

$$\int\limits_\Omega L(\partial)u(x) \cdot u(x) F\Big(\frac{x-y}{d_\Omega}\Big)\, dx - 2^{-1}u(y)^2 \geq$$

$$\geq \sum_{j=1}^m \int\limits_\Omega \frac{|\nabla_j u(x)|^2}{|x-y|^{2(m-j)}} \log \frac{4d_\Omega}{|x-t|}\, dx. \quad (2.10.3)$$

Everywhere in this section, by c we denote positive constants independent of Ω.

Proof. It suffices to assume $d_\Omega = 1$. By Lemma 2.2.1, the left-hand side in (2.10.3) is equal to the quadratic form

$$\mathscr{H}_u(y) = \int\limits_\Omega \sum_{j=1}^m \sum_{|\mu|=|\nu|=j} \partial^\mu u \cdot \partial^\nu u \cdot P_{\mu\nu}(\partial) F(x-y)\, dx.$$

By Hardy's inequality,

$$\Big|\mathscr{H}_u(y) - \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \partial^\alpha u(x) \cdot \partial^\beta u(x) \cdot F(x-y)\, dx\Big| \leq$$

$$\leq \sum_{j=1}^{m-1} \int\limits_\Omega \frac{|\nabla_j u(x)|^2}{|x-y|^{2(m-j)}}\, dx \leq c \int\limits_\Omega |\nabla_m u(x)|^2\, dx.$$

Hence, there exist constants c_1 and c_2 such that

$$c_1 \mathscr{H}_u(y) \leq \int\limits_\Omega |\nabla_m u(x)|^2 \log\big(4|x-y|^{-1}\big)\, dx \leq c_2 \mathscr{H}_u(y). \quad (2.10.4)$$

(Here we have used the fact that the constant C in (2.10.2) is sufficiently large in order to obtain the right-hand inequality). By the Hardy-type inequality

$$\int\limits_\Omega \frac{|\nabla_j u(x)|^2}{|x-y|^{2(m-j)}} \log\big(4|x-y|^{-1}\big)\, dx \leq$$

$$\leq c \int\limits_\Omega |\nabla_m u(x)|^2 \log\big(4|x-y|^{-1}\big)\, dx, \quad (2.10.5)$$

we can also write

$$\int\limits_\Omega \frac{|\nabla_j u(x)|^2}{|x-y|^{2(m-j)}} \log\big(4|x-y|^{-1}\big)\, dx \leq c\mathscr{H}_u(y). \quad (2.10.6)$$

Thus the proof is complete. $\square$

Lemma 2.10.1. *Let $n = 2m$, and let $u \in \overset{\circ}{H}{}^m(\Omega)$ be subject to* (2.6.1). *Then for an arbitrary point $y \in B_\rho$, $\rho \le 1$,*

$$u(y)^2 + \mathscr{B}(u\eta_\rho, u\eta_\rho F_{y,\rho}) \le cM_\rho(u),$$

where $\mathscr{B}$, η_ρ and $M_\rho(u)$ are the same as in Lemma 2.5.1, *$F_{y,\rho}(x) = F(\frac{x-y}{2}\rho)$, and F is given by* (2.10.1).

Proof. We majorize the second term by repeating the proof of Lemma 2.5.1. Then the first term is estimated by (2.10.3), where the role of Ω is played by $\Omega \cap B_{2\rho}$, and u is replaced by $u\eta_\rho$. The result follows.

Combining Proposition 2.10.1 with $\Omega \cap B_{2\rho}$ and $u\eta_\rho$ instead of Ω and u, with Lemma 2.10.1, we obtain the following local estimate similar to (2.6.5).

Lemma 2.10.2. *Let the conditions of Lemma* 2.10.1 *be satisfied. Then for all $y \in \Omega \cap B_\rho$, $\rho \le 1$, the estimate*

$$u(y)^2 + \int\limits_{\Omega\cap B_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(y)|^2}{|x-y|^{n-2k}} \log\left(4\rho|x-y|^{-1}\right) dx \le cM_\rho(u) \tag{2.10.7}$$

holds.

We now are in a position to finish the proof of sufficiency in Theorem 2.1.1.

Let $n = 2m$, and let $u \in \overset{\circ}{H}{}^m(\Omega)$ and $L(\partial)u = 0$ on $\Omega \cap B_{2\rho}$. We diminish the right-hand side in (2.10.7) replacing B_ρ by $B_\rho \setminus B_\varepsilon$ with an arbitrarily small $\varepsilon > 0$. The obtained integral is continuous at $y = 0$. Hence,

$$\int\limits_{\Omega\cap B_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{|x|^{n-2k}} \log\left(4\rho|x|^{-1}\right) dx \le cM_\rho(u). \tag{2.10.8}$$

Putting here $\rho = 1$ and $\gamma_m(r) = \mathrm{cap}_m(\overline{S}_r \setminus \Omega, B_{4r})$, we estimate the left-hand side from below by using the estimate

$$M_\rho(u) \le \frac{c}{\gamma_m(r)} \int\limits_{\Omega\cap S_r} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{\rho^{n-2k}} dx$$

proved in Proposition 2.5.1. We have

$$\sum_{j\ge 1} j\gamma_m(2^{-j}) M_{2^{-j}}(u) \le cM_1(u).$$

Hence by (2.10.7)),

$$\sum_{j=1}^{\infty} j\gamma_m(2^{-j}) \sup_{\Omega\cap B_{2^{-j}}} u^2 \le cM_1(u).$$

Suppose that O is irregular. Assuming that

$$\lim_{j\to\infty} \sup_{\Omega\cap B_{2^{-j}}} u^2 > 0,$$

we have

$$\sum_{j=1}^{\infty} j\gamma_m(2^{-j}) < \infty. \tag{2.10.9}$$

Since

$$\operatorname{cap}_m(\overline{S}_r \setminus \Omega, B_{4r}) \geq \operatorname{cap}_m(\overline{S}_r \setminus \Omega) \geq cC_{2m}(\overline{S}_r \setminus \Omega) \;\text{ for }\; r \leq 1$$

(see Section 2.2) and since the Bessel capacity is subadditive, we obtain the estimate

$$\gamma_m(2^{-j}) \geq c\Big(C_{2m}(\overline{B}_{2^{1-j}} \setminus \Omega) - C_{2m}(\overline{B}_{2^{-j}} \setminus \Omega)\Big).$$

Hence and by Abel's summation, we conclude that

$$\sum_{j=1}^{\infty} C_{2m}(\overline{B}_{2^{-j}} \setminus \Omega) < \infty;$$

that is, condition (2.10.9) is violated. The result follows. □

2.11 Proof of Necessity in Theorem 2.1.1

By $G(x,y)$ we denote Green's function of the Dirichlet problem for $L(\partial)$ on the ball B_1. Also, we use the fundamental solution f given by (2.10.1). As is well known and easily checked, for all x and y in $B_{\frac{4}{5}}$,

$$\big|G(x,y) - F(x-y)\big| \leq c, \tag{2.11.1}$$

where c is a constant depending on $L(\partial)$. Hence, there exists a sufficiently small k such that for all y in the ball $B_{\frac{3}{4}}$ and for all x subject to $|x-y| \leq k$,

$$c_1 \log\left(2k|x-y|^{-1}\right) \leq G(x,y) \leq c_2 \log\left(2k|x-y|^{-1}\right), \tag{2.11.2}$$

and for all multi-indices α, β with $|\alpha| + |\beta| > 0$,

$$\big|\partial_x^\alpha \partial_y^\beta G(x,y)\big| \leq c_{\alpha,\beta}|x-y|^{-|\alpha|-|\beta|}. \tag{2.11.3}$$

Moreover, $G(x,y)$ and its derivatives are uniformly bounded for all x and y in B_1 with $|x-y| > k$. By Lemma 2.2.1, for all $u \in C_0^\infty(B_1)$,

$$\int_{B_1} L(\partial)u \cdot uG_y\, dx = 2^{-1}u(y)^2 + \int_{B_1} \sum_{j=1}^{m} \sum_{|\mu|=|\nu|=j} \partial^\mu u \cdot \partial^\nu u \cdot P_{\mu\nu}(\partial)G_y\, dx,$$

where $y \in B_1$ and $G_y(x) = G(x, y)$. Hence, using the same argument as in Lemma 2.2.2, we see that for an arbitrary compact set K in $\overline{B}_1$ and for all $y \in B_1 \setminus K$ the L-capacitary potential with respect to B_1 satisfies

$$U_K(y) = \frac{1}{2} U_K(y)^2 + \int\limits_{B_1} \sum_{j=1}^{m} \sum_{|\mu|=|\nu|=j} \partial^\mu U_K \cdot \partial^\nu U_K \cdot P_{\mu\nu}(\partial) G_y \, dx. \quad (2.11.4)$$

(Note that the notation U_K was used in the case of $n < 2m$ in a different sense.)

Lemma 2.11.1. *Let K be a compact subset of $\overline{B}_{\frac{1}{2}}$. For all $y \in B_1 \setminus K$, the equality*

$$|U_K(y) - 1| \le 1 + c \operatorname{cap}_m(K, B_1) \quad (2.11.5)$$

holds, where (*and in the sequel*) *by c we denote positive constants independent of K.*

Proof. Since $L(\partial) U_K = 0$ on $B_1 \setminus B_{\frac{1}{2}}$ and since U_K satisfies zero Dirichlet conditions on ∂B_1, it is standard that

$$\sup_{B_1 \setminus B_{\frac{3}{4}}} |U_K| \le c \sup_{B_{\frac{3}{4}} \setminus B_{\frac{1}{2}}} |U_K|$$

(see [5, Chapter 3]). Thus we only need to check (2.11.5) for $y \in B_{\frac{3}{4}} \setminus K$. By (2.11.4) and (2.11.3),

$$\big(U_K(y) - 1\big)^2 \le 1 - \int\limits_{B_1} a_{\alpha\beta} \partial^\alpha U_K \cdot \partial^\beta U_K \cdot G_y \, dx +$$
$$+ c \sum_{j=1}^{m-1} \int\limits_{B_1} \big|\nabla_j U_K(x)\big|^2 |x - y|^{2j-n} \, dx.$$

It follows from (2.11.2) and Hardy's inequality

$$\int\limits_{B_1} \big|\nabla_j U_K(x)\big|^2 |x - y|^{2j-n} \, dx \le c \int\limits_{B_1} \big|\nabla_m U_K(x)\big|^2 \, dx, \quad 1 \le j \le m,$$

that

$$\big(U_K(y) - 1\big)^2 \le 1 - c_1 \int\limits_{B_k(y)} \big|\nabla_m U_K(x)\big|^2 \log\big(4k|x - y|^{-1}\big) \, dx +$$
$$+ c \int\limits_{B_1} \big|\nabla_m U_K(x)\big|^2 \, dx \le 1 + c_2 \operatorname{cap}_m(K, B_1),$$

which is equivalent to (2.11.5). $\square$

Lemma 2.11.2. *Let $n = 2m$, and let K be a compact subset of $\overline{B}_1 \setminus B_{\frac{1}{2}}$. Then the L-capacitary potential U_K with respect to B_2 satisfies*

$$\mathscr{M}\nabla_l U_K(0) \le c\ \mathrm{cap}_m(K, B_2) \ \ \textit{for}\ \ l = 0, 1, \dots, m.$$

Proof. It follows from (2.11.4) and (2.10.5) that U_K satisfies the inequalities

$$|U_K(y)| \le c\biggl(U_K(y)^2 + \int\limits_{B_2} \bigl|\nabla_m U_K(x)\bigr|^2 \log\bigl(4|x-y|^{-1}\bigr)\, dx\biggr),$$

$$|\nabla_l U_K(y)| \le c\biggl(|\nabla_l U_K(y)|^2 + \int\limits_{B_2} \sum_{\substack{1<r,s\le m\\ r+s>l}} \frac{|\nabla_r U_K(x)|\,|\nabla_s U_K(x)|}{|x-y|^{n-r-s+l}}\, dx\biggr)$$

(cf. the proof of Corollary 2.2.1). It remains to repeat the proof of Proposition 2.2.1 with the above inequalities playing the role of (2.2.12). □

Lemma 2.11.3. *Let $n = 2m$, and let K be compact subset of $\overline{B}_\delta$, $\delta < 1$, subject to*

$$C_{2m}(K) \le \frac{\varepsilon(m)}{\log(\frac{2}{\delta})}\,, \tag{2.11.6}$$

where $\varepsilon(m)$ is a sufficiently small constant independent of K and δ. Then there exists a constant $c(m)$ such that $\mathrm{cap}_m(K, B_{2\delta}) \le c(m) C_{2m}(K)$.

Proof. Let $\delta^{-1}K$ denote the image of K under the δ^{-1}-dilation. Clearly, $\mathrm{cap}_m(K, B_{2\delta}) = \mathrm{cap}_m(\delta^{-1}K, B_2)$. By using a cutoff function, one shows that $\mathrm{cap}_m(\delta^{-1}K, B_2)$ does not exceed

$$c \inf\biggl\{ \sum_{0\le k\le m} \|\nabla_k u\|^2_{L_2(\mathbb{R}^n)} : \\ u \in C_0^\infty(\mathbb{R}^n),\ n = 1 \text{ in a neighborhood of } \delta^{-1}K \biggr\}.$$

Now we recall that by allowing the admissible functions to satisfy the inequality $U \ge 1$ on K in the last infimum, one arrives at the capacity of $\delta^{-1}K$ equivalent to $C_{2m}(\delta^{-1}K)$. Hence, it is enough to verify that

$$C_{2m}(\delta^{-1}K) \le c C_{2m}(K). \tag{2.11.7}$$

We denote by $P\mu$ the $2m$-order Bessel potential of measure μ and by G_{2m} the kernel of the integral operator P. Let μ_K be the corresponding equilibrium measure of K. Since $K \subset \overline{B}_\delta$ and $\delta < 1$, we obtain for all $y \in K$

except for a subset of K with zero capacity C_{2m},

$$\int_K G_{2m}\big(\delta^{-1}(x-y)\big)\,d\mu_K(x) \geq c\int_K \log\big(\delta|x-y|^{-1}\big)\,d\mu_K(x) \geq$$
$$\geq c\bigg(\int_K \log\big(2|x-y|^{-1}\big)\,d\mu_K(x) - C_{2m}(K)\log(2\delta^{-1})\bigg) \geq$$
$$\geq c\bigg(\int_K G_{2m}(x-y)\,d\mu_K(x) - \varepsilon(m)\bigg) \geq c_0\big(1-\varepsilon(m)\big).$$

Thus, for the measure $\mu^{(\delta)} = c_0^{-1}(1-\varepsilon(m))^{-1}\mu_K(\delta\xi)$ which is supported by $\delta^{-1}K$, we have $P\mu^{(\delta)} \geq 1$ on $\delta^{-1}K$ outside a subset with zero capacity C_{2m}. Therefore,

$$C_{2m}(\delta^{-1}K) \leq \big\langle P\mu^{(\delta)}, \mu^{(\delta)}\big\rangle =$$
$$= c_0^{-2}\big(1-\varepsilon(m)\big)^{-2}\int_K\int_K G_{2m}\big(\delta^{-1}(x-y)\big)\,d\mu_K(x)\,d\mu_K(y), \quad (2.11.8)$$

where $\langle P\mu^{(\delta)}, \mu^{(\delta)}\rangle$ denotes the energy of $\mu^{(\delta)}$. Now we note that

$$G_{2m}\big(\delta^{-1}(x-y)\big) \leq c\log\big(4\delta|x-y|^{-1}\big) <$$
$$< c\log\big(4|x-y|^{-1}\big) \leq c_1 G_{2m}(x-y)$$

for x and y in K. This and (2.11.8), combined with the fact that the energy of μ_K is equal to $C_{2m}(K)$, complete the proof of the lemma. $\square$

Suppose that O is regular with respect to the set Ω. Assuming that

$$\int_0^1 C_{2m}(\overline{B}_r \setminus \Omega)\,\frac{dr}{r} < \infty, \quad (2.11.9)$$

we arrive at a contradiction. We fix a sufficiently small $\varepsilon > 0$ and choose a positive integer N so that

$$\sum_{j=N}^{\infty} C_{2m}(\overline{B}_{2^{-j}} \setminus \Omega) < \varepsilon. \quad (2.11.10)$$

Let $K = \overline{B}_{2^{-N}} \setminus \Omega$, and let U_k denote the L-capacitary potential of K with respect to B_1. We note that using (2.10.3) one can literally repeat the proof of locality of the regularity property given in Lemma 2.10.1. Therefore, O is regular with respect to $B_1 \setminus K$, which implies $U_K(x) \to 1$ as

$x \to O$. It suffices to show that this is not the case. It is well known that (67) implies

$$\sum_{j \ge N} j C_{2m}(K^{(j)}) \le c\varepsilon,$$

where $K^{(j)} = \{x \in K : \ 2^{-1-j} \le |x| \le 2^{1-j}\}$, and c depends only on n. A proof can be found in [19, p. 240] for $m = 1$, and no changes are necessary to apply the argument for $m > 1$. Hence, by Lemma 2.11.3, we obtain

$$\sum_{j \ge N}^{\infty} j \operatorname{cap}_m \left(K^{(j)}, B_{2^{2-j}}\right) \le c\varepsilon. \tag{2.11.11}$$

We use the partition of unity $\{\eta^{(j)}\}_{j \ge N}$ introduced at the beginning of Section 2.9, and by $U^{(j)}$ we denote the L-capacitary potential of $K^{(j)}$ with respect to $B_{2^{2-j}}$. We also need the function V defined by (2.9.3) with the new $U^{(j)}$. Let

$$T^{(j)}(y) = \sum_{k=1}^{m} \int\limits_{B_1} \frac{|\nabla_k U^{(j)}(x)|^2}{|x-y|^{n-2k}} \log \frac{2^{4-j}}{|x-y|} \, dx.$$

By (2.10.5),

$$T^{(j)}(y) = c \int\limits_{B_1} \left|\nabla_m U^{(j)}(x)\right|^2 \log \frac{2^{4-j}}{|x-y|} \, dx,$$

and therefore for $r \le 1$,

$$r^{-n} \int\limits_{B_r} T^{(j)}(y) \, dy \le c \int\limits_{B_2^{2-j}} \left|\nabla_m U^{(j)}(x)\right|^2 \log \frac{2^{4-j}}{r + |x|} \, dx \le$$

$$\le c \log \left(\frac{2^{4-j}}{r}\right) \operatorname{cap}\left(K^{(j)}, B_{2^{2-j}}\right).$$

Hence, bearing in mind that $\operatorname{supp} \eta^{(j)} \subset B_{2^{1-j}} \setminus \overline{B}_{2^{-1-j}}$, we have

$$\mathscr{M}(\eta^{(j)} T^{(j)})(0) \le c \operatorname{cap}_m \left(K^{(j)}, B_{2^{2-j}}\right). \tag{2.11.12}$$

Furthermore, by (2.11.4) and Lemma 2.11.1,

$$\mathscr{M}(\eta^{(j)} U^{(j)})(0) \le$$

$$\le 2^{-1}\Big(1 + c_0 \operatorname{cap}_m \left(k^{(j)}, B_{2^{2-j}}\right)\Big) \mathscr{M}(\eta^{(j)} T^{(j)})(0) + c_1 \mathscr{M}(\eta^{(j)} T^{(j)})(0).$$

Since we may have $\operatorname{cap}_m(k^{(j)}, B_{2^{2-j}}) \le (2c_0)^{-1}$ by choosing a sufficiently small ε, we obtain

$$\mathscr{M}(\eta^{(j)} U^{(j)})(0) \le 4c_1 \mathscr{M}(\eta^{(j)} T^{(j)})(0),$$

and by (2.11.12),

$$\mathscr{M}(\eta^{(j)}U^{(j)})(0) \leq c\ \mathrm{cap}_m\left(K^{(j)}, B_{2^{2-j}}\right), \tag{2.11.13}$$

which implies

$$\mathscr{M}V(0) \leq c \sum_{j \geq N} \mathrm{cap}\left(K^{(j)}, B_{2^{2-j}}\right). \tag{2.11.14}$$

We introduce the function

$$T_u(y) = \sum_{k=1}^{m} \int\limits_{B_1} \frac{|\nabla_k u(x)|^2}{|x-y|^{n-2k}}\ \log\left(4|x-y|^{-1}\right) dy.$$

By (2.10.5),

$$\begin{aligned} T_V(y) &\leq c \int\limits_{B_1} (\nabla_m V(x))^2 \log\left(4|x-y|^{-1}\right) dy \leq \\ &\leq \sum_{j \geq N} \int\limits_{B_1} \left|\nabla_m(\eta^{(j)}U^{(j)})(x)\right|^2 \log\left(4|x-y|^{-1}\right) dx. \end{aligned}$$

Hence, for $r \leq 1$,

$$\begin{aligned} & r^{-n} \int\limits_{B_r} T_V(y)\, dy \leq \\ &\leq c \sum_{j \geq N} \int\limits_{B_{2^{1-j}} \setminus B_{2^{-1-j}}} \left|\nabla_m(\eta^{(j)}U^{(j)})(x)\right|^2 \log \frac{4}{|x|+r}\, dx \leq \\ &\leq c \sum_{j \geq N} j \int\limits_{B_1} \left|\nabla_m(\eta^{(j)}U^{(j)})(x)\right|^2 dx. \end{aligned} \tag{2.11.15}$$

Clearly,

$$\begin{aligned} & \int\limits_{B_1} \left|\nabla_m(\eta^{(j)}U^{(j)})(x)\right|^2 dx \leq \\ &\leq c \int\limits_{B_1} \left|\nabla_m \eta^{(j)}(x)\right|^2 U^{(j)}(x)^2\, dx + c \sum_{k=1}^{m} \int\limits_{B_1} \frac{|\nabla_k U^{(j)}(x)|^2}{|x|^{2(m-k)}}\, dx. \end{aligned} \tag{2.11.16}$$

Owing to Hardy's inequality, each term in the last sum is majorized by

$$c \int\limits_{B_1} \left|\nabla_m U^{(j)}(x)\right|^2 dx = c\, \mathrm{cap}_m(K^{(j)}, B_{2-j}).$$

By Lemma 2.10.2, the first integral in the right-hand side of (2.11.16) is dominated by

$$c2^{2mj} \int\limits_{\operatorname{supp}\eta^{(j)}} U^{(j)}(x)^2\, dx \le c\mathscr{M}(\zeta^{(j)}U^{(j)})(0),$$

where $\zeta^{(j)}$ is a function in $C_0^\infty(B_{2^{1-j}} \setminus \overline{B}_{2^{-1-j}})$ equal to 1 on the support of $\eta^{(j)}$. Now we note that (2.11.13) is also valid with $\eta^{(j)}$ replaced by $\zeta^{(j)}$. Hence,

$$\int\limits_{B_1} \left|\nabla_m(\eta^{(j)}U^{(j)})(x)\right|^2 dx \le c\,\mathrm{cap}_m(K^{(j)}, B_{2^{2-j}}), \tag{2.11.17}$$

which, combined with (2.11.15), yields

$$\mathscr{M}T_V(0) \le c\sum_{j\ge N} j\,\mathrm{cap}(K^{(j)}, B_{2^{2-j}}). \tag{2.11.18}$$

We turn to estimate the function $W = U_K - V$, which solves the Dirichlet problem for the equation

$$L(\partial)W = -L(\partial)V \quad \text{on} \quad B_1 \setminus K. \tag{2.11.19}$$

It follows from (2.10.3) that for $y \in B_1 \setminus K$,

$$\begin{aligned} 2^{-1}W(y)^2 + c\int\limits_{B_1} \big(\nabla_m W(x)\big)^2 \log\big(4|x-y|^{-1}\big)\, dx \le \\ \le \int\limits_{B_1} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\partial^\alpha V(x)\cdot \partial^\beta\big(W(x)F(x-y)\big)\, dx. \end{aligned} \tag{2.11.20}$$

Hence by (2.10.1),

$$\begin{aligned} W(y)^2 + \int\limits_{B_1} \big(\nabla_m W(x)\big)^2 \log\big(4|x-y|^{-1}\big)\, dx \le \\ \le c\Bigg(\int\limits_{B_1} |\nabla_m V(x)|\, |W(x)|\, \frac{dx}{|x-y|^{n-m}} + \\ + \int\limits_{B_1} |\nabla_m V(x)| \sum_{k=1}^{m-1} |\nabla_k W(x)|\, \frac{dx}{|x-y|^{n-m-k}} + \\ + \int\limits_{B_1} |\nabla_m V(x)|\, |\nabla_m W(x)| \log\big(4|x-y|^{-1}\big)\, dx\Bigg). \end{aligned} \tag{2.11.21}$$

Since both $|U_K|$ and $|V|$ are bounded by a constant depending on L, the same holds for $|W|$. Thus, the integral on the right containing $|W|$ is majorized by

$$c\int\limits_{B_1} |\nabla_m V(x)|\,\frac{dx}{|x-y|^{n-m}}.$$

Obviously, two other integrals in the right-hand side of (2.11.21) are not greater than

$$cT_V(y)^{\frac{1}{2}}\bigg(\sum_{k=1}^{m-1}\int\limits_{B_1}\frac{(\nabla_k W(x))^2}{|x-y|^{n-2k}}\,dx+\int\limits_{B_1}\big(\nabla_k W(x)\big)^2\log\frac{4}{|x-y|}\,dx\bigg)^{\frac{1}{2}}.$$

By Hardy's inequality, we can remove the sum in k enlarging the constant c. Hence by (2.11.21),

$$W(y)^2+\int\limits_{B_1}\big(\nabla_m W(x)\big)^2\log\frac{4}{|x-y|}\,dx\le$$
$$\le c\bigg(\int\limits_{B_1}|\nabla_m V(x)|\,\frac{dx}{|x-y|^{n-m}}+T_V(y)\bigg).$$

Thus by $U_K=V+W$, we arrive at

$$U_K(y)^2+c\int\limits_{B_1}\big(\nabla_m U_K(x)\big)^2\log\frac{4}{|x-y|}\,dx\le$$
$$\le c\bigg(V(y)^2+T_V(y)+\int\limits_{B_1}|\nabla_m V(x)|\,\frac{dx}{|x-y|^{n-m}}\bigg).$$

The left-hand side is not less than $c|U_K(y)|$ by (2.11.4). Therefore,

$$\mathscr{M}U_K(0)\le c\bigg(\mathscr{M}V^2(0)+\mathscr{M}T_V(0)+\int\limits_{B_1}|\nabla_m V(x)|\,\frac{dx}{|x|^{n-m}}\bigg).$$

By Lemma 2.11.1, $|V|\le c$. This, along with (2.11.14) and (2.11.18), implies

$$\mathscr{M}V^2(0)+\mathscr{M}T_V(0)\le\sum_{j\ge N} j\,\mathrm{cap}\,\big(K^{(j)},B_{2^{2-j}}\big).$$

It follows from the definition of V and from Lemma 2.11.2 that

$$\int\limits_{B_1}\frac{|\nabla_m V(x)|}{|x|^{n-m}}\,dx\le c\sum_{j\ge N}2^{(n-m)j}\int\limits_{B_{2^{2-j}}}\big|\nabla_m(\eta^{(j)}U^{(j)})(x)\big|\,dx\le$$
$$\le c\sum_{j\ge N}\mathrm{cap}_m(K^{(j)},B_{2^{2-j}}).$$

Finally,

$$\mathscr{M} U_K(0) \le c \sum_{j\ge N} j\, \mathrm{cap}_m(K^{(j)}, B_{2^{2-j}}),$$

and the contradiction required is a consequence of (2.11.12). The necessity of (2.1.3) for the regularity of O follows.

2.12 The Biharmonic Equation in a Domain with Inner Cusp $(n \ge 8)$

Let the bounded domain Ω be described by the inequality $x_n < f(x')$, $x' = (x_1, \dots, x_{n-1})$ on B_1, where f is a continuous function on the ball $\{x' : |x'| < 1\}$, subject to the conditions: $f(0) = 0$, f is smooth for $x' \ne 0$, and $\frac{\partial f}{\partial |x'|}$ is a decreasing function of $|x'|$ which tends to $+\infty$ as $|x'| \to 0$.

These conditions show that at the point O the surface $\partial\Omega$ has a cusp that is directed inside Ω.

Theorem 2.12.1. *Let $n \ge 8$, and let u solve the Dirichlet problem*

$$\Delta^2 u = f, \quad u \in \overset{\circ}{H}{}^2(\Omega),$$

where $f \in C_0^\infty(\Omega)$. If

$$\int_0^1 C_4(\overline{B}_\rho \setminus \Omega)\, \frac{d\rho}{\rho^{n-3}} = \infty, \tag{2.12.1}$$

then $u(x) \to 0$ as x tends to O along any nontangential direction.

Proof. By ν_x we denote the external normal to $\partial\Omega$ at the point $x \in (B_1 \cap \partial\Omega)\setminus O$. We introduce the function family $\{f_\varepsilon\}$ by $f_\varepsilon(x') = (f(x') - \varepsilon)_+ + \varepsilon$. Replacing $x_n < f(x')$ in the definition of Ω by $x_n < f_\varepsilon(x')$, we obtain the family of domains Ω_ε such that $O \in \Omega_\varepsilon$ and $\Omega_\varepsilon \downarrow \Omega$ as $\varepsilon \downarrow 0$.

By the implicit function theorem, the set $E_\varepsilon = \{x : \ x_n = f(x') = \varepsilon\}$ is a smooth $(n-2)$-dimensional surface for sufficiently small ε. In a neighborhood of any point of E_ε, the boundary of Ω_ε is diffeomorphic to a dihedral angle. It follows from our conditions on f that the two hyperplanes , which are tangent to $\partial\Omega$ at any point of the edge E_ε, form a dihedral angle with opening $> \frac{3\pi}{2}$ (from the side of Ω). Then, as is well known, the solution of the Dirichlet problem

$$\Delta^2 u_\varepsilon = f, \quad u_\varepsilon \in \overset{\circ}{H}{}^m(\Omega_\varepsilon),$$

satisfies the estimate

$$\big|\nabla_j u_\varepsilon(x)\big| = O\big(\operatorname{dist}(x, E_\varepsilon)^{-j+\lambda}\big), \tag{2.12.2}$$

where $\lambda > \frac{3\pi}{2}$ (see, e.g., [54, Theorem 10.5] combined with [24, Section 7.1]). The value of λ can be made more precise, but this is irrelevant for us. In fact, we only need (80) to justify the integration by parts in what follows.

By y we denote a point on the semiaxis $x' = 0$, $x_n \leq 0$, at a small distance from O. Let (r, ω) be spherical coordinates centered at y, and let G denote the image of Ω_ε under the mapping $x \to (t, \omega)$, where $t = -\log r$. For $u_\varepsilon(x)$ written in the coordinates (t, ω), we use the notation $v(t, \omega)$. Also, let δ_ω denote the Laplace–Beltrami operator on ∂B_1, and let ∂_t, ∂_t^2, and so on, denote partial derivatives with respect to t. Since

$$\Delta = e^{2t}(\partial_t^2 - (n-2)\partial_t + \delta_\omega),$$

we have $\Delta^2 = e^{4t}\Lambda$, where

$$\begin{aligned}\Lambda = \big((\partial_t + 2)^2 - (n-2)(\partial_t + 2) + \delta_\omega\big)\big(\partial_t^2 - (n-2)\partial_t + \delta_\omega\big) = \\ = \partial_t^4 + 2\partial_t^2\delta_\omega + \delta_\omega^2 - 2(n-4)(\partial_t^3 + \partial_t\delta_\omega) - 2(n-4)\delta_\omega + \\ + (n^2 - 10n + 20)\partial_t^2 + 2(n-2)(n-4)\partial_t.\end{aligned}$$

Consider the integral

$$I_1 = \int\limits_{\Omega_\varepsilon} \Delta^2 u_\varepsilon \cdot \frac{\partial u_\varepsilon}{\partial r}\,\frac{dx}{r^{n-5}} = \int\limits_G \Lambda v \cdot \partial_t v \; dt\, d\omega.$$

Integratig by parts in the right-hand side, we obtain

$$\begin{aligned}I_1 = 2(n-4)\int\limits_G \Big((\partial_t^2 v)^2 + (\operatorname{grad}_\omega \partial_t v)^2 + (n-2)(\partial_t v)^2\Big)\, dt\, d\omega - \\ - \frac{1}{2}\int\limits_{\partial G} \Big((\partial_t v)^2 + 2(\operatorname{grad}_\omega \partial_t v)^2 + (\delta_\omega v)^2\Big)\cos(\nu, t)\, ds.\end{aligned}$$

Since the angle between ν and the vector $x - y$ does not exceed $\frac{\pi}{2}$, we have $\cos(\nu, t) \leq 0$ and therefore,

$$2(n-4)\int\limits_G \Big((\partial_t v)^2 + (\operatorname{grad}_\omega \partial_t v)^2 + (n-2)(\partial_t v)^2\Big)\, dt\, d\omega \leq I_1. \qquad (2.12.3)$$

We make use of another integral

$$I_2 = \int\limits_{\Omega_\varepsilon} \Delta^2 u_\varepsilon \cdot u_\varepsilon \,\frac{dx}{r^{n-4}} = \int\limits_G \Lambda v \cdot v \; dt\, d\omega. \qquad (2.12.4)$$

We remark that $y \in \Omega_\varepsilon$ implies

$$2\int\limits_G \partial_t v \cdot v \; dt\, d\omega = \int\limits_{\partial B_1} \big(v(+\infty, \omega)\big)^2\, d\omega = \omega_{n-1}(u_\varepsilon(y))^2.$$

After integrating by parts in (2.12.4), we obtain

$$\int_G \Big((\partial_t^2 v)^2 + (\delta_\omega v)^2 + 2(\operatorname{grad}_\omega v_t)^2 + 2(n-4)(\operatorname{grad}_\omega v)^2 -$$
$$- (n^2 - 10n + 20)(\partial_t v)^2\Big)\, dt\, d\omega + \omega_{n-1}(n-2)(n-4)(u_\varepsilon(y))^2 \le I_2.$$

Combining this inequality with (2.12.3), we arrive at

$$\int_G \Big(2(n-3)(\partial_t^2 v)^2 + 2(n-2)(\operatorname{grad}_\omega \partial_t v)^2 +$$
$$+ 2(\delta_\omega v)^2 + 4(n-4)(\operatorname{grad}_\omega v)^2 + 8(n-3)(\partial_t v)^2\Big)\, dt\, d\omega +$$
$$+ 2\omega_{n-1}(n-2)(n-4)(u_\varepsilon(y))^2 \le I_1 + 2I_2.$$

Coming back to the coordinates x, we obtain

$$(u_\varepsilon(y))^2 + \int_{\Omega_\varepsilon} \Big((\nabla_2 u_\varepsilon)^2 + \frac{(\nabla u_\varepsilon)^2}{r^2}\Big) \frac{dx}{r^{n-4}} \le$$
$$\le c \int_{\Omega_\varepsilon} f\Big(r\,\frac{\partial u_\varepsilon}{\partial r} + 2u_\varepsilon\Big) \frac{dx}{r^{n-4}}\,. \qquad (2.12.5)$$

Since $u_\varepsilon \to u$ in $H^m(\mathbb{R}^n)$, we can here replace u_ε by u and Ω_ε by Ω.

Now let η_ρ and ζ_ρ be the cutoff functions used in the proof of Lemma 2.5.1. Since $\Delta^2(u\eta_\rho) = f\eta_\rho + [\Delta^2, \eta_\rho]u$ and $f = 0$ near O, we see that for $y_n \in (-\frac{\rho}{2}, 0)$,

$$(u(y))^2 + \int_\Omega \Big(\big(\nabla_2(u\eta_\rho)\big)^2 + \frac{(\nabla(u\eta_\rho))^2}{r^2}\Big) \frac{dx}{r^{n-4}} \le$$
$$\le c \int_{\Omega_\varepsilon} \Big(r\,\frac{\partial(u\eta_\rho)}{\partial r} + 2u\eta_\rho\Big)[\Delta^2, \eta_\rho]u\, \frac{dx}{r^{n-4}}\,.$$

Integrating by parts in the right-hand side, we majorize it by (2.6.3), and therefore it follows from (2.6.4) that

$$\sup_{-\frac{\rho}{2} < y_n < 0} |u(0, y_n)|^2 + \int_{B_\rho} \Big((\nabla_2 u)^2 + \frac{(\nabla u)^2}{r^2}\Big) \frac{dx}{r^{n-4}} < cM_\rho(u). \qquad (2.12.6)$$

We fix a sufficiently small θ and introduce a cone $C_\theta = \{x :\ x_n > 0,\ |x'| \le \theta x_n\}$. Clearly, for all $r \in (0, \rho)$,

$$\sup_{(\partial B_r)\setminus C_\theta} |u|^2 \le c\Big(\big|u(0, -r)\big|^2 + r^2 \sup_{(\partial B_r)\setminus C_\theta} |\nabla u|^2\Big),$$

the function u being extended by zero outside Ω. Hence and by the well-known local estimate

$$r^2 \sup_{(\partial B_r)\setminus C_\theta} |\nabla u|^2 \le c \int\limits_{(B_{2r}\setminus B_{\frac{r}{2}})\setminus C_{\frac{\theta}{2}}} |\nabla u(x)|^2 \, \frac{dx}{|x|^{n-2}} \, ,$$

we obtain

$$\sup_{B_{\frac{\rho}{2}}\setminus C_\theta} |u|^2 \le c \Big(\sup_{0>y_n>-\frac{\rho}{2}} |u(0,y_n)|^2 + \int\limits_{B_\rho} |\nabla(x)|^2 \, \frac{dx}{|x|^{n-2}} \Big).$$

Making use of (2.12.6), we arrive at

$$\sup_{B_{\frac{\rho}{2}}\setminus C_\theta} |u|^2 + \int\limits_{B_\rho} \Big(|\nabla_2 u|^2 + \frac{|\nabla u|^2}{|x|^2} \Big) \frac{dx}{|x|^{n-4}} \le c M_\rho(u).$$

Repeating the proof of Lemma 2.6.2, we find that for $\rho \in (0, R)$ and for small R, the inequality

$$\sup_{B_{\frac{\rho}{2}}\setminus C_\theta} |u|^2 + \int\limits_{B_\rho} \Big(|\nabla_2 u|^2 + \frac{|\nabla u|^2}{|x|^2} \Big) \frac{dx}{|x|^{n-4}} \le$$

$$\le c M_R(u) \exp\Big(- c \int\limits_\rho^R \mathrm{cap}_2(\overline{B}_\tau \setminus \Omega) \, \frac{d\tau}{\tau^{n-3}} \Big)$$

holds. The result follows. □

Chapter 3

Boundary Behavior of Solutions to the Polyharmonic Equations

The polyharmonic equation is, obviously, a particular case of general equations in Chapter 1. However, the results for this equation obtained previously can be made more explicit.

3.1 Weighted Positivity of $(-\Delta)^m$

Henceforth as above Ω is an open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ and O is a point of the closure $\overline{\Omega}$. In the sequel, c is a positive constant depending only on m and n, and ω_{n-1} is the $(n-1)$-dimensional measure of ∂B_1.

We shall deal with solution of the Dirichlet problem

$$(-\Delta)^m u = f, \quad u \in \overset{\circ}{H}{}^m(\Omega). \tag{3.1.1}$$

By Γ we denote the fundamental solution of the operator $(-\Delta)^m$,

$$\Gamma(x) = \begin{cases} \gamma |x|^{2m-n} & \text{for } 2m < n, \\ \gamma \log \dfrac{\mathscr{D}}{|x|} & \text{for } 2m = n, \end{cases}$$

where $\mathscr{D}$ is a positive constant and

$$\gamma^{-1} = 2^{m-1}(m-1)!(n-2)(n-4)\cdots(n-2m)\omega_{n-1}$$

for $n > 2m$, and

$$\gamma^{-1}\left[2^{m-1}(m-1)!\right]^2 \omega_{n-1}$$

for $n = 2m$.

Proposition 3.1.1. *Let $n \geq 2m$ and let*

$$\int_\Omega u(x)(-\Delta)^m u(x)\Gamma(x-p)\,dx \geq 0 \tag{3.1.2}$$

for all $u \in C_0^\infty(\Omega)$ *and for at least one point* $p \in \Omega$. *Then*

$$n = 2m,\ 2m+1,\ 2m+2 \ \textit{ for } \ m > 2$$

and

$$n = 4, 5, 6, 7 \ \textit{ for } \ m = 2.$$

Proof. Assume that $n \geq 2m+3$ for $m > 2$ and $n \geq 8$ for $m = 2$. Denote by (r, ω), $r > 0$, $\omega \in \partial B_1(p)$, the spherical coordinates with center p, and by G the image of Ω under the mapping $x \longmapsto (t, \omega)$, $t = -\log r$. Since

$$r^2 \Delta u = r^{2-n}(r\partial_r)\big(r^{n-2}(r\partial_r)u\big) + \delta_\omega u,$$

where δ_ω is the Beltrami operator on $\partial B_1(p)$, we have

$$\Delta = e^{2t}\big(\partial_t^2 - (n-2)\partial_t + \delta_\omega\big) = e^{2t}\Big\{\Big(\partial_t - \frac{n-2}{2}\Big)^2 - A\Big\},$$

where

$$A = -\delta_\omega + \frac{(n-2)^2}{4}\,. \tag{3.1.3}$$

Hence

$$r^{2m}\Delta^m = \prod_{j=0}^{m-1}\Big\{\Big(\partial_t - \frac{n-2}{2} + 2j\Big)^2 - A\Big\}. \tag{3.1.4}$$

Let u be a function in $C_0^\infty(\Omega)$ which depends only on $|x-p|$. We set $w(t) = u(x)$. Clearly,

$$\int\limits_\Omega (-\Delta)^m u(x) u(x) \Gamma(x-p)\, dx = \int\limits_{\mathbb{R}^1} w(t) \mathscr{P}\Big(\frac{d}{dt}\Big) w(t)\, dt, \tag{3.1.5}$$

where

$$\mathscr{P}(\lambda) = (-1)^m \gamma \omega_{n-1} \prod_{j=0}^{m-1} (\lambda + 2j)(\lambda - n + 2 + 2j) =$$

$$= (-1)^m \gamma \omega_{n-1} \lambda(\lambda - n + 2) \prod_{j=1}^{m-1} (\lambda + 2j)(\lambda - n - 2m + 2 + 2j).$$

Let

$$\mathscr{P}(\lambda) = (-1)^m \gamma \omega_{n-1} \lambda^{2m} + \sum_{k=1}^{2m-1} a_k \lambda^k.$$

We have

$$a_2 = \big(\lambda^{-1}\mathscr{P}(\lambda)\big)'\Big|_{\lambda=0} = \frac{1}{2-n} + \sum_{j=1}^{m-1}\Big(\frac{1}{2j} - \frac{1}{n-2-2m+2j}\Big).$$

Hence by $n \geq 2m+3$,

$$a_2 = \frac{1}{2} - \frac{1}{n-2} - \frac{1}{n-2m} + \sum_{j=2}^{m-1} \frac{n-2-m}{2j(n-2-2m+2j)} \geq$$
$$\geq \frac{1}{2} - \frac{1}{n-2} - \frac{1}{n-2m} > 0.$$

We choose a real-valued function $\eta \in C_0^\infty(1,2)$ normalized by

$$\int_{\mathbb{R}^1} |\eta'(\sigma)|^2 \, d\sigma = 1$$

and we set $u(x) = \eta(\varepsilon t)$, where ε is so small that $\operatorname{supp} u \subset \Omega$. The quadratic form on the right-hand side of (3.1.5) equals

$$\int_{\mathbb{R}^1} \Big(\varepsilon^{2m} \gamma \omega_{n-1} \big|\eta^{(m)}(\varepsilon t)\big|^2 + \sum_{k=1}^{m-1} a_{2k} (-1)^k \varepsilon^{2k} \big|\eta^{(k)}(\varepsilon t)\big|^2 \Big) \, dt =$$
$$= -a_2 \varepsilon + O(\varepsilon^3) < 0,$$

which contradicts the assumption (3.1.2). □

Now we prove the converse statement.

Proposition 3.1.2. *Let* $\Gamma_p(x) = \Gamma(x-p)$*, where* $p \in \Omega$*. If*

$$n = 2m,\ 2m+1,\ 2m+2 \ \textit{ for } \ m > 2,$$
$$n = 4, 5, 6, 7 \ \textit{ for } \ m = 2,$$
$$n = 2, 3, 4 \ \textit{ for } \ m = 1,$$

then for all $u \in C_0^\infty(\Omega)$*,*

$$\int_\Omega (-\Delta)^m u(x) \cdot u(x) \Gamma(x-\rho) \, dx \geq$$
$$\geq 2^{-1} u^2(p) + c \sum_{k=1}^{m} \int_\Omega \frac{|\nabla_k u(x)|^2}{|x-p|^{2(m-k)}} \Gamma(x-p) \, dx. \quad (3.1.6)$$

(*In the case* $n = 2m$*, the constant* $\mathscr{D}$ *in the definition of* Γ *is greater than* $|x-p|$ *for all* $x \in \operatorname{supp} u$*.)*

Proof. We preserve the notation introduced in the proof of Proposition 3.1.1. We note first that (3.1.6) becomes identity when $m = 1$. The subsequent proof will be divided into four parts.

(i) *The case* $n = 2m+2$.

By (3.1.4),

$$r^{-2m}\Delta^m = \prod_{j=0}^{m-1}(\partial_t - m + 2j - A^{\frac{1}{2}})\prod_{j=0}^{m-1}(\partial_t - m + 2j + A^{\frac{1}{2}}),$$

where $A = -\delta_\omega + m^2$, and $A^{\frac{1}{2}}$ is defined by using spherical harmonics. By setting $k = m - j$ in the second product, we rewrite the right-hand side as

$$\prod_{j=0}^{m-1}(\partial_t - m + 2j - A^{\frac{1}{2}})\prod_{k=1}^{m}(\partial_t + m - 2k + A^{\frac{1}{2}}).$$

This can be represented in the form

$$(\partial_t - m - A^{\frac{1}{2}})(\partial_t - m + A^{\frac{1}{2}})\prod_{j=1}^{m-1}(\partial_t^2 - \mathscr{B}_j^2),$$

where $\mathscr{B}_j = A^{\frac{1}{2}} + m - 2j$. Therefore,

$$\begin{aligned}2^m\Delta^m &= \Big(\partial_t^2 + \delta_\omega - 2m\,\frac{\partial}{\partial t}\Big)\prod_{j=1}^{m-1}(\partial_t^2 - \mathscr{B}_j^2) = \\ &= (\partial_t^2 + \delta_\omega)\prod_{j=1}^{m-1}(\partial_t^2 - \mathscr{B}_j^2) + \\ &\qquad + (-1)^m 2m\partial_t \sum_{\substack{0\le j\le m-1\\ k_1<\dots<k_j}}(-\partial_t^2)^{m-j-1}\mathscr{B}_{k_1}^2\cdots\mathscr{B}_{k_j}^2.\end{aligned}$$

We extend w by zero outside Ω and introduce the function w defined by $w(t,\omega) = u(x)$. We write the left-hand side of (3.1.6) in the form $\gamma(I_1 + I_2)$, where γ is the constant in the definition of Γ,

$$(2m^{-1}I_1) = \int_G \partial_t \sum_{\substack{0\le j\le m-1\\ k_1<\dots<k_j}}(-\partial_t^2)^{m-j-1}\mathscr{B}_{k_1}^2\cdots\mathscr{B}_{k_j}^2 w\cdot w\,dt\,d\omega,$$

and

$$I_2 = (-1)^m\int_G(\partial_t^2 + \delta_\omega)\prod_{j=1}^{m-1}(\partial_t^2 - \mathscr{B}_j^2)w\cdot w\,dt\,d\omega.$$

Since the operators $\mathscr{B}_j$ are symmetric, it follows that

$$\begin{aligned}m^{-1}I_1 &= \sum_{\substack{0\le j\le m-1\\ k_1<\dots<k_j}}\int_{\mathbb{R}^1}\partial_t\int_{\partial B_1}\left(\partial_t^{m-j-1}\mathscr{B}_{k_1}\cdots\mathscr{B}_{k_j}w\right)^2 d\omega\,dt = \\ &= \sum_{\substack{0\le j\le m-1\\ k_1<\dots<k_j}}\int_{\partial B_1}\left|\left(\partial_t^{m-j-1}\mathscr{B}_{k_1}\cdots\mathscr{B}_{k_j}w\right)(+\infty,\omega)\right|^2 d\omega.\end{aligned}$$

Since $u \in C_0^\infty(\Omega)$, we have $w(t,\omega) = u(p) + O(e^{-t})$ as $t \to +\infty$, and this can be differentiated. Therefore, all terms with $j < m-1$ are equal to zero, and we find

$$I_1 = m \int\limits_{\partial B_1} \big|(\mathscr{B}_1 \cdot \mathscr{B}_{m-1} w)(+\infty, \omega)\big|^2 \, d\omega =$$
$$= m u^2(p) \int\limits_{\partial B_1} |\mathscr{B}_1 \cdot \mathscr{B}_{m-1} 1|^2 \, d\omega.$$

By $\mathscr{B}_j = (-\delta_\omega + m^2)^{\frac{1}{2}} + m - 2j$, we have

$$I_1 = 4^{m-1} m \big[(m-1)!\big]^2 \omega_{2m+1} u^2(p).$$

Since in the case $n = 2m+2$,

$$\gamma^{-1} = 2^{2m-1} m \big[(m-1)!\big]^2 \omega_{2m+1},$$

we conclude that

$$I_1 = (2\gamma)^{-1} u^2(p). \tag{3.1.7}$$

We now wish to obtain the lower bound for I_2. Let $\widetilde{w}$ denote the Fourier transform of w with respect to t. Then

$$I_2 = \int\limits_{\partial B_1} \int\limits_{\mathbb{R}^1} (\lambda^2 - \delta_\omega) \prod_{j=1}^{m-1} (\lambda^2 + \mathscr{B}_j^2) \widetilde{w}(\lambda, \omega) \cdot \overline{\widetilde{w}(\lambda, \omega)} \, d\lambda \, d\omega.$$

Clearly,

$$\mathscr{B}_j \geq (m^2 - \delta_\omega)^{\frac{1}{2}} - m + 2 \geq 2m^{-1}(m^2 - \delta_\omega)^{\frac{1}{2}},$$

and

$$\lambda^2 + \mathscr{B}_j^2 \geq 4m^{-2}(\lambda^2 + 1 - \delta_\omega),$$

the operators being compared with respect to their quadratic forms. Thus

$$\Big(\frac{m}{2}\Big)^{2m-2} I_2 \geq$$
$$\geq \int\limits_{\partial B_1 \times \mathbb{R}^1} (\lambda^2 - \delta_\omega)(\lambda^2 + 1 - \delta_\omega)^{m-1} \widetilde{w}(\lambda, \omega) \cdot \overline{\widetilde{w}(\lambda, \omega)} \, d\lambda \, d\omega \geq$$
$$\geq c\Big(\|\partial_t w\|^2_{H^{m-1}(G)} + \|\nabla_\omega w\|^2_{H^{m-1}(G)}\Big),$$

where H^{m-1} is the Sobolev space. This is equivalent to the inequality

$$I_2 \geq c \int\limits_\Omega \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{|x-p|^{n-2k}} \, dx$$

which, along with (3.1.7), completes the proof for $n = 2m+2$.

(ii) *The case $n = 2m + 1$.*

We shall treat this case by descent from $n = 2m+2$ to $n = 2m+1$. Let $z = (x, s)$, where $x \in \Omega$, $s \in \mathbb{R}^1$, and let $q = (p, 0)$, where $p \in \Omega$, $0 \in \mathbb{R}^1$. We introduce a cut-off function $\eta \in C_0^\infty(-2, 2)$ which satisfies $\eta(s) = 1$ for $|s| \le 1$ and $0 \le \eta \le 1$ on $\mathbb{R}^1$. Let

$$U_\varepsilon(z) = u(x)\eta(\varepsilon s)$$

and let $\Gamma^{(n)}$ denote the fundamental solution of $(-\Delta)^m$ in $\mathbb{R}^n$.

By integrating

$$(-\Delta_z)^m \Gamma^{(n+1)}(z, q) = \delta(z - q),$$

with respect to $s \in \mathbb{R}^1$, we have

$$\Gamma^{(n)}(x, y) = \int\limits_{\mathbb{R}^1} \Gamma^{(k+1)}(z, q)\, ds. \tag{3.1.8}$$

From part (i) of the present proof we obtain

$$\int\limits_{\Omega\times\mathbb{R}^1} (-\Delta_z)^m U_\varepsilon(z) U_\varepsilon(z) \Gamma^{(n+1)}(z - q)\, dz \ge$$

$$\ge \frac{1}{2}\, U_\varepsilon^2(q) + c \int\limits_{\Omega\times\mathbb{R}^1} \sum_{k=1}^{m} \frac{|\nabla_k U_\varepsilon(z)|^2}{|z - q|^{2(m+1-k)}}\, dz.$$

By letting $\varepsilon \to 0$, we find

$$\int\limits_{\Omega\times\mathbb{R}^1} (-\Delta_z)^m u(x) \cdot u(x) \Gamma^{(n+1)}(z - q)\, ds\, dx \ge$$

$$\ge \frac{1}{2}\, u^2(p) + c \int\limits_{\Omega\times\mathbb{R}^1} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{|z - q|^{2(m+1-k)}}\, ds\, dx.$$

The result follows from (3.1.8).

(iii) *The case $m = 2$, $n = 7$.*

By (3.1.4),

$$30\omega_6 \int\limits_{\Omega} \Delta^2 u(x) \cdot u(x) \Gamma(x - p)\, dx =$$

$$= \int\limits_{\Omega} (w_{tt} - 5w_t + \delta_\varepsilon w)(w_{tt} + w_t - 6w + \delta_\omega w)\, dt\, d\omega.$$

Since $w(t,\omega) = u(p) + O(e^{-t})$ as $t \to +\infty$, the last integral equals

$$\int_\Omega \Big(w_{tt}^2 - 5w_t^2 - 6w_{tt}w + 2w_{tt}\delta_\omega w + (\delta_\omega w)^2 - 6w\delta_\omega w\Big)\, dt\, d\omega + 15\omega_6 u^2(p).$$

After integrating by parts, we rewrite this in the form

$$\int_\Omega \Big(w_{tt}^2(\delta_\omega w)^2 + 2(\nabla_\omega w_t)^2 + 6(\nabla_\omega w)^2 + w_t^2\Big)\, dt\, d\omega + 15\omega_6 u^2(p).$$

Using the variables (r,ω), we find that the left-hand side exceeds

$$c\int_\Omega \Big(\frac{(\Delta u(x))^2}{|x-p|^3} + \frac{|\Delta u(x)|^2}{|x-p|}\Big)\, dx + 15\omega_6 u^2(p).$$

Since

$$|\nabla_2 u|^2 - (\Delta u)^2 = \Delta\big((\nabla u)^2\big) - \frac{\partial^2}{\partial x_i \partial x_j}\Big(\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\Big),$$

it follows that

$$\int_\Omega \frac{(\nabla_2 u(x))^2}{|x-p|^3}\, dx \le \int_\Omega \frac{(\nabla u(x))^2}{|x-p|}\, dx + c\int_\Omega \frac{(\nabla u(x))^2}{|x-p|}\, dx,$$

which completes the proof.

(iv) *The case $n = 2m$.*

By (3.1.4),

$$\begin{aligned} r^{2m}\Delta^m &= \prod_{j=0}^{m-1}\Big\{(\partial_t - m + 1 + 2j)^2 - (m-1)^2 + \delta_\omega\Big\} = \\ &= \prod_{j=0}^{m-1}\big(\partial_t - m + 1 + 2j - \mathscr{E}^{\frac12}\big)\prod_{j=0}^{m-1}\big(\partial_t - m + 1 + 2j + \mathscr{E}^{\frac12}\big), \end{aligned}$$

where $\mathscr{E} = -\delta_\omega + (m-1)^2$. We introduce $k = m - 1 - j$ in the second product and obtain

$$r^{2m}\Delta^m = \prod_{j=0}^{m-1}(\partial_t^2 - \mathscr{F}_j^2),$$

where $\mathscr{F}_j = m - 1 - 2j + \mathscr{E}^{\frac12}$. Hence

$$\int_\Omega (-\Delta)^m u(x)\cdot u(x)\Gamma(x-p)\, dx =$$

$$= \gamma\int_G \prod_{j=0}^{m-1}(-\partial_t^2 + \mathscr{F}_j^2)w\cdot(\ell + t)w\; dt\, d\omega, \quad (3.1.9)$$

where $\ell = \log \mathscr{D}$. Since $w(t,\omega) = u(p) + O(e^{-t})$ and

$$\prod_{j=0}^{m-1}(-\partial_t^2 + \mathscr{F}_j^2) = \sum_{j=0}^{m}(-\partial_t^2)^{m-j} \sum_{k_1<\cdots<k_j} \mathscr{F}_{k_1}^2 \cdots \mathscr{F}_{k_j}^2,$$

the right-hand side in (3.1.9) can be rewritten as

$$\gamma \int_G \sum_{\substack{0\le j\le m-1\\ k_1<\cdots<k_j}} \partial_t^{m-j}\mathscr{F}_{k_1}\cdots\mathscr{F}_{k_j} w \partial_t^{m-j}\big((\ell+t)\mathscr{F}_{k_1}\cdots\mathscr{F}_{k_j}w\big)\,dt\,d\omega =$$

$$= \gamma \int_G \sum_{\substack{0\le j\le m-1\\ k_1<\cdots<k_j}} (\partial_t^{m-j}\mathscr{F}_{k_1}\cdots\mathscr{F}_{k_j}w)^2(\ell+t)\,dt\,d\omega +$$

$$+ \frac{\gamma}{2}\int_G \sum_{\substack{0\le j\le m-1\\ k_1<\cdots<k_j}} (m-j)\partial_t(\partial_t^{m-1-j}\mathscr{F}_{k_1}\cdots\mathscr{F}_{k_j}w)^2\,dt\,d\omega.$$

The second integral in the right-hand side equals

$$\lim_{t\to+\infty} \int_{\partial B_1(p)} \sum_{\substack{0\le j\le m-1\\ k_1<\cdots<k_j}} (m-j)\partial_t\big|\partial_t^{m-1-j}\mathscr{F}_{k_1}\cdots\mathscr{F}_{k_j}w\big|^2\,d\omega =$$

$$= \lim_{t\to+\infty}\int_{\partial B_1(p)} \sum_{k_1<\cdots<k_{m-1}} (\mathscr{F}_{k_1}\cdots\mathscr{F}_{k_{m-1}}e)^2\,d\omega$$

and since $(\mathscr{F}_{m-1}w)(t,\omega) = O(e^{-t})$, the last expression is equal to

$$\lim_{t\to+\infty}\int_{\partial B_1(p)} \big(\mathscr{F}_0\cdots\mathscr{F}_{m-2}w\big)^2\,d\omega = \big(2^{m-1}(m-1)!\big)^2\omega_{n-1}u^2(p).$$

Hence

$$\int_\Omega (-\Delta^m u(x)\cdot u(x)\Gamma(x-p)\,dx =$$

$$= \frac{1}{2}\,u^2(p) + \gamma\int_G (\ell+t) \sum_{\substack{0\le j\le m-1\\ k_1<\cdots<k_j}} \big(\partial_t^{m-1-j}\mathscr{F}_{k_1}\cdots\mathscr{F}_{k_j}w\big)^2\,dt\,d\omega.$$

Since $\mathscr{F}_{m-1} \ge c(-\delta)^{\frac{1}{2}}$ and $\mathscr{F}_k \ge c(-\delta+1)^{\frac{1}{2}}$ for $k < m-1$, the last integral majorizes

$$c\int_\Omega (\ell+t)\sum_{1\le\mu+\nu\le m-1}\big(\partial_t^\mu(-\delta)^{\frac{\nu}{2}}w\big)^2\,dt\,d\omega \ge$$

$$\ge c\int_\Omega \log\frac{\mathscr{D}}{|x-p|}\sum_{k=1}^m \frac{|\nabla_k u(x)|^2}{|x-p|^{2(m-k)}}\,dx,$$

which completes the proof. □

3.2 Local Estimates

We are in a position to obtain a growth estimate for the solution formulated in terms of a Wiener type m-capacitary integral. Before stating the result we note that the function $\gamma_m(\rho)$ is measurable not only for $n > 2m$ when it is monotonous, but also for $n = 2m$. In fact, one can easily show that the function

$$\Big(\frac{\rho}{2}, \infty\Big) \ni r \longmapsto \mathrm{cap}_m(S_\rho \setminus \Omega, B_{4r})$$

is continuous. Hence, being monotonous in ρ, the function of two variables $(\rho, r) \longmapsto \mathrm{cap}_m(S_\rho \setminus \Omega, B_{4r})$ satisfies the so-called Carathéodory conditions which imply the measurability of $\gamma_m(\rho)$ in the case $n = 2m$ (see [7], [68, p. 152]).

Theorem 3.2.1. *Let m and n be as in Proposition* 3.1.2 *and let the function $u \in \overset{\circ}{H}{}^m(\Omega)$ satisfy $\Delta^m u = 0$ on $\Omega \cap B_{2R}$. Then, for all $\rho \in (0, R)$,*

$$\mathrm{supp}\,\big\{|u(p)|^2 :\ p \in \Omega \cap B_\rho\big\} + \int\limits_{\Omega \cap B_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{|x|^{n-2k}}\, dx \le$$

$$\le c M_R(u) \exp\Big(- c \int\limits_\rho^R \gamma_m(\tau)\, \frac{d\tau}{\tau}\Big). \qquad (3.2.1)$$

Proof. For $n > 2m$, estimate (3.2.1) is contained in Lemma 2.6.2, Chapter 2. In the general case $n \ge 2m$, the proof is the same and is given here for readers convenience.

It is sufficient to assume that $2\rho \le R$, since in the opposite case the result follows from Corollary 2.6.1. Denote the first and the second terms on the left by φ_ρ and ψ_ρ, respectively. From Lemma 2.5.1 it follows that for $r \le R$,

$$\varphi_r + \psi_r \le \frac{c}{\gamma_m(r)}\,(\psi_{2r} - \psi_r) \le \frac{c}{\gamma_m(r)}(\psi_{2r} - \psi_r + \varphi_{2r} - \varphi_r).$$

This, along with the obvious inequality $\gamma_m(r) \le c$, implies

$$\varphi_r + \psi_r \le c e^{-c_0 \gamma_m(r)}(\varphi_{2r} + \psi_{2r}).$$

By setting $r = 2^{-j}R$, $j = 1, 2, \ldots$, we arrive at the estimate

$$\varphi_{2^{-\ell}R} + \psi_{2^{-\ell}R} \le c \exp\Big(- c \sum_{j=1}^{\ell} \gamma_m(2^{-j}R)\Big)(\varphi_R + \psi_R).$$

We choose ℓ so that

$$\ell < \log_2 \frac{R}{\rho} \leq \ell + 1$$

in order to obtain

$$\varphi_\rho + \psi_\rho \leq c \exp\Big(- c_0 \sum_{j=1}^{\ell} \gamma_m(2^{-j}R)\Big)(\varphi_R + \psi_R).$$

Now we notice that by Corollary 2.6.1,

$$\varphi_R + \psi_R \leq c M_R.$$

It remains to use the inequality

$$\sum_{j=1}^{\ell} \gamma_m(2^{-j}R) \geq c_1 \int_\rho^R \gamma_m(\tau)\, \frac{d\tau}{\tau} - c_2,$$

which follows from the subadditivity of the Riesz capacity. □

Now we obtain a positive estimate for a function, m-harmonic in $\Omega \setminus B_\rho$.

Theorem 3.2.2. *Let m and n be the same as in Proposition* 3.1.2 *and let $u \in \overset{\circ}{H}(\Omega)$ satisfy*

$$\Delta^m u = 0 \quad on \quad \Omega \setminus B_\rho.$$

Then for an arbitrary $p \in \Omega \setminus B_\rho$,

$$|u(p)| \leq c\big(M_\rho(u)\big)^{\frac{1}{2}} \Big(\frac{\rho}{|p|}\Big)^{n-2m} \exp\bigg(- c \int_\rho^{|p|} \gamma_m(\tau)\, \frac{d\tau}{\tau}\bigg). \tag{3.2.2}$$

Proof. Let w denote the Kelvin transform of u, i.e. the function

$$w(y) = |y|^{2m-n} u\Big(\frac{y}{|y|^2}\Big)$$

defined on the image $I\Omega$ of Ω under the inversion $x \longmapsto y = x|x|^{-2}$. It is well known that

$$\Delta_y^m \Big(|y|^{2m-n} u\Big(\frac{y}{|y|^2}\Big)\Big) = |y|^{-n-2m} (\Delta^m u)\Big(\frac{y}{|y|^2}\Big).$$

(A simple way to check this formula is to introduce the variables (t, ω), and to use (3.1.4).) Consequently,

$$\int_{I\Omega} w(y) \Delta_y^m w(y)\, dy = \int_\Omega u(x) \Delta_x^m u(x)\, dx \tag{3.2.3}$$

and therefore $w \in \overset{\circ}{H}{}^m(I\Omega)$ and $u \in \overset{\circ}{H}{}^m(\Omega)$ simultaneously.

By Corollary 2.6.1,

$$|w(q)| \le c\Big(\rho^n \int\limits_{B_{\frac{2}{\rho}} \setminus B_{\frac{1}{\rho}}} w^2(y)\,dy\Big)^{\frac{1}{2}} \exp\Big(-c\int\limits_{\frac{1}{|q|}}^{\frac{1}{\rho}} \gamma_m(\tau)\,\frac{d\tau}{\tau}\Big)$$

for all $q \in I\Omega \cap B_{\frac{1}{\rho}}$, which is equivalent to the inequality

$$|q|^{2m-n}\Big|u\Big(\frac{q}{|q|^2}\Big)\Big| \le$$
$$\le c\Big(\rho^n \int\limits_{B_{\frac{2}{\rho}} \setminus B_{\frac{1}{\rho}}} |y|^{2(2m-n)}u^2\Big(\frac{y}{|y|^2}\Big)\,dy\Big)^{\frac{1}{2}} \exp\Big(-c\int\limits_{|p|}^{\rho} \gamma_m(\tau)\,\frac{d\tau}{\tau}\Big).$$

By putting $p = 1|q|^{-2}$, $x = y|y|^{-2}$, we complete the proof. □

By (3.2.3) and Theorem 9.3.2.1 in [46] mentioned at the beginning of Section 3.2, one can find that $\mathrm{cap}_m(IK, B_{\frac{4}{\rho}})$ is equivalent to $\rho^{2(2m-n)} \times \mathrm{cap}_m(K, B_{\frac{4}{\rho}})$ for $K \subset S_\rho$. Hence the function

$$\gamma^*_m(\rho) = \rho^{2m-n}\,\mathrm{cap}_m(S_\rho \setminus I\Omega, B_{4\rho})$$

satisfies the equivalence relation

$$\gamma^*_m(\rho) \sim \rho^{n-2m}\,\mathrm{cap}_m(S_{\frac{1}{\rho}} \setminus \Omega, B_{\frac{4}{\rho}})$$

which, together with the easily checked property of the capacity

$$\mathrm{cap}_m(S_\rho \setminus \Omega, B_{4\rho}) \sim \mathrm{cap}_m(S_\rho \setminus \Omega),$$

valid for $n > 2m$ (see [46, Proposition 9.1.1.3]), implies

$$\int\limits_{\frac{1}{|p|}}^{\frac{1}{\rho}} \gamma^*_m(\tau) \sim \int\limits_{\rho}^{|p|} \gamma_m(\tau)\,\frac{d\tau}{\tau}\,.$$

Here $|p| > \rho$ and c_1, c_2 are positive constants depending on n and m. Furthermore, by the definition of w,

$$M_{\frac{1}{\rho}}(w) \sim \rho^{n-2m} M_\rho(u),$$

and the result follows from (3.2.1) applied to w.

By a standard argument, Theorems 3.2.1 and 3.2.2 yield the following variant of the Phragmén–Lindelöf principle.

Corollary 3.2.1. *Let m and n be the same as in Proposition* 3.1.2 *and let $\zeta u \in \overset{\circ}{H}{}^m(\Omega)$ for all $\zeta \in C^\infty(\mathbb{R}^n)$, $\zeta = 0$, near O. If*

$$\Delta^m u = 0 \quad on \ \ \Omega \cap B_1,$$

then either $u \in \overset{\circ}{H}(\Omega)$ and

$$\limsup_{\rho\to 0} \sup_{B_\rho\cap\Omega} |u(x)| \exp\Big(c \int_\rho^1 \gamma_m(\tau)\, \frac{d\tau}{\tau}\Big) < \infty \tag{3.2.4}$$

or

$$\liminf_{\rho\to 0} \rho^{n-2m} M_\rho(u) \exp\Big(-c \int_\rho^1 \gamma_m(\tau)\, \frac{d\tau}{\tau}\Big) > 0. \tag{3.2.5}$$

3.3 Estimates for the Green Function

Let G_m be the Green function of the Dirichlet problem for $(-\Delta)^m$, i.e. the solution of the equation

$$(-\Delta_x)^m G_m(x,y) = \delta(x-y), \quad y \in \Omega,$$

with zero Dirichlet data understood in the sense of the space $\overset{\circ}{H}{}^m$.

Theorem 3.3.1. *Let $n = 5, 6, 7$ for $m = 2$ and $n = 2m+1, 2m+2$ for $m > 2$. There exists a constant c which depends only on m, such that*

$$\begin{aligned} \big|G_m(x,y) - \gamma|x-y|^{2m-n}\big| &\le c d_y^{2m-n} \ \ for \ \ |x-y| \le d_y, \\ \big|G_m(x,y)\big| &\le c|x-y|^{2m-n} \ \ for \ \ |x-y| > d_y, \end{aligned}$$

where $d_y = \operatorname{dist}(y, \partial\Omega)$.

Proof. Let $\Omega_y = \{x \in \Omega : \ |x-y| < d_y\}$ and $a\Omega_y = \{x \in \Omega : \ |x-y| < a d_y\}$. We introduce the cut-off function $\eta \in C_0^\infty[0,1)$ equal to 1 on the segment $[0, \frac{1}{2}]$. Put

$$H(x,y) = G_m(x,y) - \eta\Big(\frac{|x-y|}{d_y}\Big)\Gamma(x-y).$$

Clearly, the function $x \longmapsto (-\Delta_x)^m H(x,y)$ is supported by $\Omega_y \setminus 2^{-1}\Omega_y$ and the inequality

$$\big|\Delta_x^m H(x,y)\big| \le c d_y^{-n}$$

holds.

By Corollary 2.6.1 applied to the function $x \longmapsto H(x,y)$, we have

$$H(p,y)^2 \le 2 \int_{\Omega_y} (-\Delta_x)^m H(x,y) \cdot H(x,y) \Gamma(x-p)\, dx.$$

Therefore,

$$\sup_{p \in \Omega_y} H(p,y)^2 \le 2 \sup_{p \in \Omega_y} |H(p,y)| \sup_{p \in 2\Omega_y} \int_{\Omega_y} \big|\Delta_x^m H(x,y)\big| \Gamma(x-p)\, dx, \quad (3.3.1)$$

and hence,

$$\sup_{p \in 2\Omega_y} |H(p,y)| \le c\, d_y^{-n} \sup_{p \in 2\Omega_y} \int_{\Omega_y} \Gamma(x-p)\, dx \le c\, d_y^{2m-n}. \quad (3.3.2)$$

Since $\Delta_p^m H(p,y) = 0$ for $p \notin \Omega_y$, we obtain from (3.3.2) and Corollary 3.2.1, where O is replaced by p, that for $p \notin 2\Omega_y$,

$$|H(p,y)| \le c \Big(\frac{d_y}{|p-y|} \Big)^{n-2m} \sup_{x \in 2\Omega_y} |H(x,y)| \le c|p-y|^{2m-n}.$$

The result follows. □

The just proven theorem, along with Corollary 2.6.1, yields

Corollary 3.3.1. *Let m and n be the same as in Theorem 3.3.1. The Green function G_m satisfies*

$$|G_m(x,y)| \le \frac{c}{|y|^{n-2m}} \exp \Big(-c \int_{|x|}^{|y|} \gamma_m(\tau) \frac{d\tau}{\tau} \Big)$$

for $2|x| < |y|$.

We conclude with the following analogue of Theorem 3.3.1 in the case $n = 2m$.

Theorem 3.3.2. *Let $n = 2m$ and let Ω be a domain of diameter D. Let also*

$$\gamma(x-y) = \gamma \log \frac{D}{|x-y|}.$$

Then

$$\big|G_m(x-y) - \gamma(x-y)\big| \le c_1 \log \frac{D}{d_y} + c_2 m \ \ \textit{if} \ \ |x-y| \le d_y,$$
$$|G_m(x,y)| \le c_3 \log \frac{D}{d_y} + c_4, \ \ \textit{if} \ \ |x-y| > d_y.$$

Proof. Proceeding in the dame way as in the proof of Theorem 3.3.1, we arrive at (3.3.1). Therefore,

$$\sup_{p\in 2\Omega_y} |H(p,y)| \le c\, d_y^{-2m} \sup_{p\in 2\Omega_y} \int\limits_{\Omega_y} \Gamma(x-p)\,dx \le c_1 \log\frac{D}{d_y} + c_2.$$

Hence by Corollary 2.6.1, we obtain

$$|H(p,y)| \le c \sup_{x\in 2\Omega_y} |H(p,y)| \le c\Big(c_1 \log\frac{D}{d_y} + c_2\Big)$$

for $p \notin \Omega_y$. Since $G_m(p,y) = H(p,y)$ for $p \notin 2\Omega_y$, the result follows. $\square$

Chapter 4

Wiener Type Regularity of a Boundary Point for the 3D Lamé System

4.1 Introduction

In the present chapter we consider the Dirichlet problem for the 3D Lamé system

$$Lu = -\Delta u - \alpha \operatorname{grad} \operatorname{div} u, \quad u = (u_1, u_2, u_3)^\top.$$

We derive sufficient conditions for its weighted positivity and show that some restrictions on the elastic constants are inevitable. We then prove that the divergence of the classical Wiener integral for a boundary point guarantees the continuity of solutions to the Lamé system at this point, assuming the weighted positivity.

We first give the following definition.

Definition 4.1.1. Let L be the 3D Lamé system

$$Lu = -\Delta u - \alpha \operatorname{grad} \operatorname{div} u = -D_{kk} u_i - \alpha D_{ki} u_k \quad (i = 1, 2, 3),$$

where as usual repeated indices indicate summation. The system L is said to be positive with weight $\Psi(x) = (\Psi_{ij}(x))_{i,j=1}^3$ if

$$\int\limits_{\mathbb{R}^3} (Lu)^T \Psi u \, dx = -\int\limits_{\mathbb{R}^3} \Big[D_{kk} u_i(x) + \alpha D_{ki} u_k(x) \Big] u_j(x) \Psi_{ij}(x) \, dx \geq 0 \quad (4.1.1)$$

for all real-valued, smooth, nonzero vector functions $u = (u_i)_{i=1}^3$, $u_i \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. As usual, D denotes the gradient $(D_1, D_2, D_3)^T$ and Du is the Jacobian matrix of u.

Remark 4.1.1. The 3D Lamé system satisfies the strong elliptic condition if and only if $\alpha > -1$, and we will make this assumption throughout this paper.

The fundamental matrix of the 3D Lamé system is given by $\Phi = (\Phi_{ij})_{i,j=1}^3$, where

$$\Phi_{ij} = c_\alpha T^{-1}\Big(\delta_{ij} + \frac{\alpha}{\alpha+2}\,\omega_i\omega_j\Big) \ \ (i,j=1,2,3), \tag{4.1.2}$$
$$c_\alpha = \frac{\alpha+2}{8\pi(\alpha+1)} > 0.$$

As usual, δ_{ij} is the Kronecker delta, $r = |x|$ and $\omega_i = \frac{x_i}{|x|}$.

The first result we shall prove is the following

Theorem 4.1.1. *The* 3D *Lamé system L is positive with weight Φ when $\alpha_- < \alpha < \alpha_+$, where $\alpha_- \approx -0.194$ and $\alpha_+ \approx 1.524$. It is not positive definite with weight Φ when $\alpha < \alpha_-^{(c)} \approx -0.902$, or $\alpha > \alpha_+^{(c)} \approx 39.450$.*

The proof of this theorem is given in Section 4.2.

Let Ω be an open set in $\mathbb{R}^3$ and consider the Dirichlet problem

$$Lu = f, \quad f_i \in C_0^\infty(\Omega), \quad u_i \in \overset{\circ}{H}{}^1(\Omega). \tag{4.1.3}$$

By $\overset{\circ}{H}{}^1(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ in the Sobolev norm:

$$\|f\|_{H^2(\Omega)} = \big[\|f\|^2_{L^2(\Omega)} + \|Df\|^2_{L^2(\Omega)}\big]^{\frac{1}{2}}.$$

Definition 4.1.2. The point $P \in \partial\Omega$ is regular with respect to L if for any $f = (f_i)_{i=1}^3$, $f_i \in C_0^\infty(\Omega)$, the solution of (4.1.3) satisfies

$$\lim_{\Omega\ni x\to P} u_i(x) = 0 \ \ (i=1,2,3). \tag{4.1.4}$$

Using Theorem 4.1.1, we will prove that the divergence of the classical Wiener integral for a boundary point P guarantees its regularity with respect to the Lamé system. To simplify notations we assume, without loss of generality, that $P = 0$ is the origin of the space.

Theorem 4.1.2. *Suppose the* 3D *Lamé system L is positive definite with weight Φ. Then $O \in \partial\Omega$ is regular with respect to L if*

$$\int_0^1 \operatorname{cap}(\overline{B}_\rho \setminus \Omega)\rho^{-2}\,d\rho = \infty. \tag{4.1.5}$$

As usual, B_ρ is the open ball centered at O with radius ρ, and cap (F) *is the compact set $F \subset \mathbb{R}^3$.*

The proof of this theorem is given in Section 4.3.

4.2 Proof of Theorem 4.1.1

We start the proof of Theorem 4.1.1 by rewriting the integral

$$\int\limits_{\mathbb{R}^3} (Lu)^T \Phi u \, dx = -\int\limits_{\mathbb{R}^3} (D_{kk}u_i + \alpha D_{ki}u_k)u_j \Phi_{ij} \, dx$$

in a more revealing form. In the following, we shall write $\int f\,dx$ instead of $\int\limits_{\mathbb{R}^3}$, and by u_{ii}^2 we always mean $\sum\limits_{i=1}^{3} u_{ii}^2$; to express $\big(\sum\limits_{i=1}^{3} u_{ii}\big)^2$ we will write $u_{ii}u_{jj}$ instead. Furthermore, we always assume $u_i \in C_0^\infty(\mathbb{R}^3)$, unless otherwise stated.

Lemma 4.2.1.

$$\int (Lu)^T \Phi u \, dx = \frac{1}{2}\,|u(0)|^2 = \mathscr{B}(u,u), \tag{4.2.1}$$

where

$$\mathscr{B}(u,u) = \frac{\alpha}{2}\int (u_j D_k u_k - u_k D_k u_j) D_i \Phi_{ij}\, dx + \\ + \int (D_k u_i D_k u_j + \alpha D_k u_k D_i u_j)\Phi_{ij}\, dx.$$

Proof. By definition,

$$\int (Lu)^T \Phi u \, dx = \\ = -\int D_{kk}u_i \cdot u_j \Phi_{ij}\, dx - \alpha \int D_{ki}u_k \cdot u_j \Phi_{ij}\, dx =: I_1 + I_2.$$

Since Φ is symmetric, we have $\Phi_{ij} = \Phi_{ji}$ and

$$\begin{aligned} I_1 &= -\int D_{kk}u_i \cdot u_j \Phi_{ij}\, dx = \\ &= -\frac{1}{2}\int \big[D_{kk}(u_i u_j) - 2 D_k u_i D_k u_j\big]\Phi_{ij}\, dx = \\ &= -\frac{1}{2}\int u_i u_j D_{kk}\Phi_{ij}\, dx + \int D_k u_i D_k u_j \cdot \Phi_{ij}\, dx. \end{aligned}$$

On the other hand, Φ is the fundamental matrix of L, so we have

$$-D_{kk}\Phi_{ij} - \alpha D_{ki}\Phi_{kj} = \delta_{ij}\delta(x),$$

and

$$
\begin{aligned}
-\frac{1}{2}\int u_i u_j D_{kk}\Phi_{ij}\,dx = \frac{1}{2}\int u_i u_j\big[\delta_{ij}\delta(x) + \alpha D_{ki}\Phi_{kj}\big]\,dx = \\
= \frac{1}{2}\,|u(0)|^2 - \frac{\alpha}{2}\int (D_i u_i \cdot u_j + u_i D_i u_j) D_k \Phi_{kj}\,dx = \\
= \frac{1}{2}\,|u(0)|^2 - \frac{\alpha}{2}\int (D_k u_k \cdot u_j + u_k D_k u_j) D_i \Phi_{ij}\,dx.
\end{aligned}
$$

Now I_2 can be written as

$$
I_2 = \alpha \int D_k u_k (D_i u_j \cdot \Phi_{ij} + u_j D_i \Phi_{ij})\,dx,
$$

and the lemma follows by adding up the results. □

Remark 4.2.1. With $\Phi(x)$ replaced by $\Phi_y(x) := \Phi(x-y)$, we have

$$
\begin{aligned}
\int (Lu)^T \Phi_y u\,dx = \int (Lu_y)^T \Phi u_y\,dx = \\
= \frac{1}{2}\,|u_y(0)|^2 + \mathscr{B}(u_y, u_y) =: \frac{1}{2}\,|u(y)|^2 + \mathscr{B}_y(u,u),
\end{aligned}
$$

where $u_y(x) = u(x+y)$ and

$$
\begin{aligned}
\mathscr{B}_y(u,u) = \frac{\alpha}{2}\int (u_j D_k u_k - u_k D_k u_j) D_i \Phi_{y,ij}\,dx + \\
+ \int (D_k u_i u_j u_j + \alpha D_k u_k D_i u_j)\Phi_{y,ij}\,dx.
\end{aligned}
$$

To proceed, we introduce the following decomposition for $C_0^\infty(\mathbb{R}^3)$ functions:

$$
f(x) = \overline{f}(x) + g(x), \quad \overline{f} \in C_0^\infty[0,\infty), \quad g \in C_0^\infty(\mathbb{R}^3),
$$

where

$$
\overline{f}(x) = \frac{1}{4\pi}\int_{S^2} f(r\omega)\,d\sigma.
$$

Note that

$$
\int_{S^2} g(r\omega)\,d\sigma = 0, \quad \forall\, r \geq 0,
$$

so we may think of $\overline{f}$ as the "0-th order harmonics" of the function f. We shall show below in Lemma 4.2.2 that all 0-th order harmonics in (4.2.1) are canceled, so it is possible to control u by Du.

Lemma 4.2.2. *With the decomposition*

$$u_i(x) = \overline{u}_i(r) + v_i(x) \quad (i = 1, 2, 3), \tag{4.2.2}$$

where

$$\begin{cases} \overline{u}_i(r) = \dfrac{1}{4\pi} \displaystyle\int\limits_{S^2} u_i(r\omega)\, d\sigma, \\ \displaystyle\int\limits_{S^2} v_i(r\omega)\, d\sigma = 0, \end{cases} \qquad \forall\, r \geq 0 \quad (i = 1, 2, 3),$$

we have

$$\int (Lu)^T \Phi u\, dx = \frac{1}{2}\, |u(0)|^2 + \mathscr{B}^*(u, u), \tag{4.2.3}$$

where

$$\begin{aligned} \mathscr{B}^*(u, u) = \frac{\alpha}{2} \int (v_j D_k v_k - v_k D_k v_j) D_i \Phi_{ij}\, dx + \\ + \int (D_k u_i D_k u_j + \alpha D_k u_k D_i u_j) \Phi_{ij}\, dx. \end{aligned} \tag{4.2.4}$$

Proof. By Lemma 4.2.1, it is enough to show

$$\int (u_j D_k u_k - u_k D_k u_j) D_i \Phi_{ij}\, dx = \int (v_j D_k v_k - v_k D_k v_j) D_i \Phi_{ij}\, dx.$$

Since

$$\begin{aligned} &\int (u_j D_k u_k - u_k D_k u_j) D_i \Phi_{ij}\, dx = \\ &= \int (\overline{u}_j D_k \overline{u}_k - \overline{u}_k D_k \overline{u}_j) D_i \Phi_{ij}\, dx + \int (\overline{u}_j D_k v_k - \overline{u}_k D_k v_j) D_i \Phi_{ij}\, dx + \\ &+ \int (v_j D_k \overline{u}_k - v_k D_k \overline{u}_j) D_i \Phi_{ij}\, dx + \int (v_j D_k v_k - v_k D_k v_j) D_i \Phi_{ij}\, dx =: \\ &=: I_1 + I_2 + I_3 + I_4, \end{aligned}$$

it suffices to show $I_1 = I_2 = I_3 = 0$. Now

$$\begin{aligned} D_i \Phi_{ij} &= D_i \Big[c_\alpha r^{-1} \Big(\delta_{ij} + \frac{\alpha}{\alpha + 2}\, \omega_i \omega_j \Big) \Big] = \\ &= -c_\alpha r^{-2} \omega_i \delta_{ij} + \\ &\quad + \frac{c_\alpha \alpha}{\alpha + 2}\, r^{-2} \Big[-\omega_i^2 \omega_j + (\delta_{ii} - \omega_i^2) \omega_j + (\delta_{ji} - \omega_j \omega_i) \omega_i \Big] = \\ &= -c_\alpha r^{-2} \omega_j + \frac{c_\alpha \alpha}{\alpha + 2}\, r^{-2} \omega_j =: d_\alpha r^{-2} \omega_j, \end{aligned} \tag{4.2.5}$$

where

$$d_\alpha = -\frac{2c_\alpha}{\alpha + 2} = -\frac{1}{4\pi(\alpha + 1)}\, .$$

Setting $D_r = \frac{\partial}{\partial r}$, we have

$$I_1 = d_\alpha \int r^{-2} \omega_j \big(\overline{u}_j D_r \overline{u}_k \cdot \omega_k - \overline{u}_k D_r \overline{u}_j \cdot \omega_k \big)\, dx =$$
$$= d_\alpha \int r^{-2} \big(\overline{u}_j D_r \overline{u}_k \cdot \omega_j \omega_k - \overline{u}_k D_r \overline{u}_j \cdot \omega_k \omega_j \big)\, dx = 0,$$
$$I_3 = d_\alpha \int r^{-2} \big(v_j D_r \overline{u}_k \cdot \omega_j \omega_k - v_k D_r \overline{u}_j \cdot \omega_k \omega_j \big)\, dx = 0.$$

As for I_2, we obtain

$$\begin{aligned} I_2 &= d_\alpha \int r^{-2} \big(\overline{u}_j D_k v_k \cdot \omega_j - \overline{u}_k D_k v_j \cdot \omega_j \big)\, dx = \\ &= d_\alpha \int r^{-2} \big(\overline{u}_j D_k v_k \cdot \omega_j - \overline{u}_j D_j v_k \cdot \omega_k \big)\, dx = \\ &= - \lim_{\varepsilon \to 0^+} d_\alpha \int\limits_{S^2} \Big[\overline{u}_k(\varepsilon) v_k(\varepsilon\omega) \omega_j \omega_k - \overline{u}_j(\varepsilon) v_k(\varepsilon\omega) \omega_j \omega_k \Big]\, d\sigma - \\ &\quad - \lim_{\varepsilon \to 0^+} d_\alpha \int\limits_{\mathbb{R}^3 \setminus B_\varepsilon} \Big\{ v_k r^{-3} \Big[- 2\overline{u}_j \cdot \omega_j \omega_k + \\ &\qquad + r D_r \overline{u}_j \cdot \omega_j \omega_k + \overline{u}_j \cdot (\delta_{jk} - \omega_j \omega_k) \Big] - \\ &\quad - v_k r^{-3} \Big[- 2\overline{u}_j \cdot \omega_j \omega_k + r D_r \overline{u}_j \cdot \omega_j \omega_k + \overline{u}_j \cdot (\delta_{kj} - \omega_k \omega_j) \Big] \Big\} = 0. \end{aligned}$$

The result follows. □

Remark 4.2.2. With $\Phi(x)$ replaced by $\Phi_y(x) := \Phi(x-y)$ and (4.2.2) replaced by

$$u_i(x) = \overline{u}_j(r-y) + v_i(x) \ \ (i = 1,2,3),$$

where $r_y = |x-y|$ and

$$\begin{cases} \overline{u}_j(r_y) = \dfrac{1}{4\pi} \displaystyle\int\limits_{S^2} u_i(y + r_y\omega)\, d\sigma, \\ \displaystyle\int\limits_{S^2} v_i(y + r_y\omega)\, d\sigma = 0, \end{cases} \qquad \forall\, r_y \geq 0 \ \ (i = 1,2,3),$$

we have

$$\int (Lu)^T \Phi_y u\, dx = \frac{1}{2} |u(y)|^2 + \mathscr{B}^*_y(u,u),$$

where

$$\mathscr{B}^*_y(u,u) = \frac{\alpha}{2} \int \big(v_j D_k v_k - v_k D_k v_j \big) D_i \Phi_{y,ij}\, dx +$$
$$+ \int \big(D_k u_i D_k u_j + \alpha D_k u_k D_i u_j \big) \Phi_{y,ij}\, dx.$$

In the following lemma, we use the definition of Φ and derive an explicit expression for the bilinear form $\mathscr{B}^*(u,u)$ defined in (4.2.4).

Lemma 4.2.3. *We have*

$$\mathscr{B}^*(u,u) = c_\alpha \int \Big\{ \frac{\alpha}{\alpha+2}\, r^{-2}\Big[v_k(D_k v)\cdot\omega - (\operatorname{div} v)(v\cdot\omega)\Big] + \\ + r^{-1}\Big[\big|D_r\overline{u}\big|^2 + \alpha\,\frac{2\alpha+3}{\alpha+2}\,(D_r\overline{u}_i)^2\omega_i^2 + |Dv|^2 + \alpha(\operatorname{div} v)^2 + \\ + \frac{\alpha}{\alpha+2}\,\big|(D_k v)\cdot\omega\big|^2 + \frac{\alpha^2}{\alpha+2}\,(\operatorname{div} v)\big[\omega_i(D_i v)\cdot\omega\big] + \\ + \alpha\,\frac{3\alpha+4}{\alpha+2}\,(D_r\overline{u}\cdot\omega)(\operatorname{div} v) + \alpha(D_r\overline{u}\cdot\omega)\big[\omega_i(D_i v)\cdot\omega\big]\Big]\Big\}\,dx. \quad (4.2.6)$$

Before proving this lemma, we need a simple yet important observation that will be useful in the following computation.

Lemma 4.2.4. *Let $g \in C_0^\infty(\mathbb{R}^3)$ be such that*

$$\int_{S^2} g(r\omega)\,d\sigma = 0, \quad \forall\, r \geq 0.$$

Then

$$\begin{cases} \displaystyle\int f(r)g(x)\,dx = 0, \\ \displaystyle\int r^{-1}Df(x)\cdot Dg(x)\,dx = 0, \end{cases} \qquad \forall\, f \in C_0^\infty[0,\infty).$$

Proof. By switching to the spherical coordinates, we easily see that

$$\int f(r)g(x)\,dx = \int_0^\infty r^2 f(r) \int_{S^2} g(r\omega)\,d\sigma = 0.$$

On the other hand,

$$\int r^{-1}Df(r)\cdot Dg(x)\,dx = \int r^{-1}D_r f D_i g\cdot\omega_i\,dx = \\ = -\int g\Big[-r^{-2}(D_r f)\omega_i^2|r^{-1}(D_{rr}f)\omega_i^2 + r^{-2}D_r f(\delta_{ii}-\omega_i^2)\Big]\,dx = \\ = -\int g\big(-r^{-2}D_r f + r^{-1}D_{rr}f\big)\,dx = 0,$$

where the last equality follows by switching to polar coordinates. $\square$

Proof of Lemma 4.2.3. By definition,

$$\mathscr{B}^*(u,u) = \frac{\alpha}{2}\int \big(v_j D_k v_k - v_k D_k v_j\big) D_i \Phi_{ij}\, dx + \\ + \int \big(D_k u_i D_k u_j + \alpha D_k u_k D_i u_j\big)\Phi_{ij}\, dx =: I_1 + I_2.$$

We have shown in Lemma 4.2.2 that (see (4.2.5))

$$I_1 = 2^{-1}\alpha d_\alpha \int r^{-2}\omega_j \big(v_j D_k v_k - v_k D_k v_j\big)\, dx = \\ = \frac{c_\alpha \alpha}{\alpha+2}\int r^{-2}\Big[v_k (D_k v)\cdot\omega - (\operatorname{div} v)(v\cdot\omega)\Big]\, dx.$$

On the other hand,

$$I_2 = c_\alpha \int r^{-1} D_k u_i D_k u_i\, dx + \frac{c_\alpha \alpha}{\alpha+2}\int r^{-1} D_k u_i D_k u_j \cdot \omega_i\omega_j\, dx + \\ + c_\alpha \alpha \int r^{-1} D_k u_k D_i u_i\, dx + \frac{c_\alpha \alpha^2}{\alpha+2}\int r^{-1} D_k u_k D_i u_j \cdot \omega_i \omega_j\, dx =: \\ =: I_3 + I_4 + I_5 + I_6.$$

Substituting $u_i = \overline{u}_i + v_i$ into I_3 and using Lemma 4.2.4, we get

$$I_3 = c_\alpha \int r^{-1}\big(D_r \overline{u}_i D_r \overline{u}_i \cdot \omega_k^2 + D_k v_i D_k v_i\big)\, dx + \\ + 2c_\alpha \int r^{-1} D_k \overline{u}_i D_k v_i\, dx = \\ = c_\alpha \int r^{-1}\big(|D_r \overline{u}|^2 + |Dv|^2\big)\, dx. \tag{4.2.7}$$

Next,

$$I_5 = c_\alpha \alpha \int r^{-1}\Big(D_r \overline{u}_k D_r \overline{u}_i \cdot \omega_k \omega_i + 2 D_i v_i D_r \overline{u}_k \cdot \omega_k + D_k v_k D_i v_i\Big)\, dx.$$

Note that for $k \neq i$,

$$\int r^{-1} D_r \overline{u}_k D_r \overline{u}_i \cdot \omega_k \omega_i\, dx = \int_0^\infty r D_r \overline{u}_k D_r \overline{u}_i\, dr \int_{S^2} \omega_k \omega_i \sigma = 0,$$

and therefore,

$$I_5 = c_\alpha \alpha \int r^{-1}\Big[(D_r \overline{u}_i)^2 \omega_i^2 + 2(\operatorname{div} v)(D_r \overline{u}\cdot\omega) + (\operatorname{div} v)^2\Big]\, dx.$$

As for I_4, we obtain

$$
\begin{aligned}
I_4 &= \frac{c_\alpha \alpha}{\alpha+2} \int r^{-1} D_k(\overline{u}_i + v_i) D_k(\overline{u}_j + v_j) \cdot \omega_i \omega_j \, dx = \\
&= \frac{c_\alpha \alpha}{\alpha+2} \int r^{-1} \Big(D_r \overline{u}_i D_r \overline{u}_j \cdot \omega_i \omega_j \omega_k^2 + D_r \overline{u}_i D_k v_j \cdot \omega_i \omega_j \omega_k + \\
&\qquad + D_k v_i D_r \overline{u}_j \cdot \omega_i \omega_j \omega_k + D_k v_i D_k v_j \cdot \omega_i \omega_j \Big) \, dx = \\
&= \frac{c_\alpha \alpha}{\alpha+2} \int r^{-1} \Big[(D_r \overline{u}_i)^2 \omega_i^2 + \\
&\qquad + 2 (D_r \overline{u} \cdot \omega) \big[\omega_k (D_k v) \cdot \omega \big] + |D_k v \cdot \omega|^2 \Big] \, dx.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_6 &= \frac{c_\alpha \alpha^2}{\alpha+2} \int r^{-1} D_k(\overline{u}_k + v_k) D_i(\overline{u}_j + v_j) \cdot \omega_i \omega_j \, dx = \\
&= \frac{c_\alpha \alpha^2}{\alpha+2} \int r^{-1} \Big(D_r \overline{u}_k D_r \overline{u}_j \cdot \omega_i^2 \omega_j \omega_k + D_r \overline{u}_k D_i v_j \cdot \omega_i \omega_j \omega_k + \\
&\qquad + D_r \overline{u}_j D_k v_k \cdot \omega_i^2 \omega_j + D_k v_k D_i v_j \cdot \omega_i \omega_j \Big) \, dx = \\
&= \frac{c_\alpha \alpha^2}{\alpha+2} \int r^{-1} \Big[(D_r \overline{u}_j)^2 \omega_j^2 + 2 (D_r \overline{u} \cdot \omega) \big[\omega_i (D_i v) \cdot \omega \big] + \\
&\qquad + (D_r \overline{u} \cdot \omega)(\operatorname{div} v) + (\operatorname{div} v) \big[\omega_i (D_i v) \cdot \omega \big] \Big] \, dx.
\end{aligned}
$$

The lemma follows by adding up all these integrals. □

With the help of Lemma 4.2.3, we now complete the proof of Theorem 4.1.1.

Proof of Theorem 4.1.1. By Lemmas 4.2.2 and 4.2.3

$$
-c_\alpha^{-1} \int (Lu)^T \Phi u \, dx = \frac{1}{2} \, c_\alpha^{-1} |u(0)|^2 + I_1 + I_2 + I_3,
$$

where

$$
\begin{aligned}
I_1 &= \int r^{-1} \Big[|D_r \overline{u}|^2 + \alpha \, \frac{2\alpha+3}{\alpha+2} \, (D_r \overline{u}_i)^2 \omega_i^2 + \\
&\qquad + |Dv|^2 + \alpha (\operatorname{div} v)^2 + \frac{\alpha}{\alpha+2} \, \big| (D_k v) \cdot \omega \big|^2 \Big] \, dx, \\
I_2 &= \int r^{-1} \Big[\frac{\alpha^2}{\alpha+2} \, (\operatorname{div} v) \big[\omega_i (D_i v) \cdot \omega \big] + \alpha \, \frac{3\alpha+4}{\alpha+2} \, (D_r \overline{u} \cdot \omega)(\operatorname{div} v) + \\
&\qquad + \alpha (D_r \overline{u} \cdot \omega) \big[\omega_i (D_i v) \cdot \omega \big] \Big] \, dx, \\
I_3 &= \int \frac{\alpha}{\alpha+2} \, r^{-2} \Big[v_k (D_k v) \cdot \omega - (\operatorname{div} v)(v \cdot \omega) \Big] \, dx.
\end{aligned}
$$

Consider first the case $\alpha \geq 0$. By switching to the polar coordinates, we have

$$I_1 \geq \int r^{-1}\Big[|D_r\overline{u}|^2 + \alpha\,\frac{2\alpha+3}{\alpha+2}\,(D_r\overline{u}_i)^2\omega_i^2 + |Dv|^2 + \alpha(\operatorname{div} v)^2\Big]\,dx =$$
$$= \int_0^\infty r\Big[\Big(1+\frac{\alpha}{3}\cdot\frac{2\alpha+3}{\alpha+2}\Big)\|D_r\overline{u}\|_\omega^2 + \|Dv\|_\omega^2 + \alpha\|\operatorname{div} v\|_\omega^2\Big]\,dr,$$

where we have written $\|\cdot\|_\omega$ for $\|\cdot\|_{L^2(S^2)}$ and used the fact that

$$\int_{S^2}(D_r\overline{u}_i)^2\omega_i^2\,d\sigma = \frac{4\pi}{3}\sum_{i=1}^3(D_r\overline{u}_i)^2 = \frac{1}{3}\int_{S^2}|D_r\overline{u}|^2\,d\sigma = \frac{1}{3}\,\|D_r\overline{u}\|_\omega^2.$$

Next,

$$|I_2| \leq \int r^{-1}\Big[\frac{\alpha^2}{\alpha+2}\,|\operatorname{div} v|\,|Dv| +$$
$$+ \alpha\,\frac{3\alpha+4}{\alpha+2}\,|D_r\overline{u}\cdot\omega|\,|\operatorname{div} v| + \alpha|D_r\overline{u}\cdot\omega|\,|Dv|\Big]\,dx \leq$$
$$\leq \int_0^\infty r\Big[\frac{\alpha^2}{\alpha+2}\,\|\operatorname{div} v\|_\omega\|Dv\|_\omega +$$
$$+ \frac{\alpha}{\sqrt{3}}\cdot\frac{3\alpha+4}{\alpha+2}\,\|D_r\overline{u}\|_\omega\|\operatorname{div} v\|_\omega + \frac{\alpha}{\sqrt{3}}\,\|D_r\overline{u}\|_\omega\|Dv\|_\omega\Big]\,d\tau,$$

where we have used

$$\|D_r\overline{u}\cdot\omega\|_\omega^2 = \int_{S^2} D_r\overline{u}_i D_r\overline{u}_j\cdot\omega_i\omega_j\,d\sigma =$$
$$= D_r\overline{u}_i D_r\overline{u}_j\cdot\frac{4\pi}{3}\,\delta_{ij} = \frac{4\pi}{3}\sum_{i=1}^3(D_r\overline{u}_i)^2 = \frac{1}{3}\,\|D_r\overline{u}\|_\omega^2.$$

As for I_3, we note that

$$|I_3| \leq \frac{\alpha}{\alpha+2}\int r^{-2}\big(|v|\,|D_v| + |v|\,|\operatorname{div} v|\big)\,dx \leq$$
$$\leq \frac{\alpha}{\alpha+2}\int_0^\infty \|v\|_\omega\big(\|D_v\|_\omega + \|\operatorname{div} v\|_\omega\big)\,dr.$$

Since 2 is the first non-trivial eigenvalue of the Laplace–Beltrami operator on S^2, we have

$$\|v\|_\omega^2 = \int\limits_{S^2} \big|v(r\omega)\big|^2\, d\sigma \le \frac{1}{2} \int\limits_{S^2} \big|D_\omega[v(r\omega)]\big|^2\, d\sigma =$$
$$= \frac{r^2}{2} \int\limits_{S^2} \big|(D_\omega v)(r\omega)\big|^2\, d\sigma \le \frac{r^2}{2}\, \|Dv\|_\omega^2, \tag{4.2.8}$$

where D_ω is the gradient operator on S^2. Thus

$$|I_3| \le \frac{1}{\sqrt{2}} \cdot \frac{\alpha}{\alpha+2} \int\limits_0^\infty r\Big[\|Dv\|_\omega^2 + \|Dv\|_\omega \|\operatorname{div} v\|_\omega\Big]\, dr,$$

and by putting all pieces together, we obtain

$$I_1 + I_2 + I_3 \ge \int\limits_0^\infty r(w^T B_+ w)\, dr, \tag{4.2.9}$$

where

$$w = \big(\|D_r \overline{u}\|_\omega, \|Dv\|_\omega, \|\operatorname{div} v\|_\omega\big)^T,$$
$$B_+ = \begin{bmatrix} 1 + \frac{\alpha}{3} \cdot \frac{2\alpha+3}{\alpha+2} & -\frac{\alpha}{2\sqrt{3}} & -\frac{\alpha}{2\sqrt{3}} \cdot \frac{3\alpha+4}{\alpha+2} \\ -\frac{\alpha}{2\sqrt{3}} & 1 - \frac{1}{\sqrt{2}} \cdot \frac{\alpha}{\alpha+2} & -\frac{\alpha}{2} \cdot \frac{\alpha + 2^{-\frac{1}{2}}}{\alpha+2} \\ -\frac{\alpha}{2\sqrt{3}} \cdot \frac{3\alpha+4}{\alpha+2} & -\frac{\alpha}{2} \cdot \frac{\alpha + 2^{-\frac{1}{2}}}{\alpha+2} & \alpha \end{bmatrix}.$$

Clearly, the weighted positivity of L follows from the weighted positivity of B_+, because the latter implies, for some $c > 0$, that

$$\int\limits_0^\infty r(w^T B_+ w)\, dr \ge c \int\limits_0^\infty r|w|^2\, dr \ge$$
$$\ge c \int\limits_0^\infty r\big(\|D_r \overline{u}\|_\omega^2 + \|Dv\|_\omega^2\big)\, dr = c \int r^{-1} |Du|^2\, dx.$$

The weighed positivity of B_+, on the other hand, is equivalent to the positivity of the determinants of all leading principal minors of B_+:

$$p_{+,1}(\alpha) = \frac{2\alpha^2+6\alpha+6}{3(\alpha+2)} > 0, \tag{3.2.10a}$$

$$p_{+,2}(\alpha) = -\frac{1}{12(\alpha+2)^2}\Big[\alpha^4 - 4(1-\sqrt{2})\alpha^3 - 12(3-\sqrt{2})\alpha^2 - - 12(6-\sqrt{2})\alpha - 48\Big] > 0, \tag{3.2.10b}$$

$$p_{+,3}(\alpha) = -\frac{\alpha}{12(\alpha+2)^3}\Big[6\alpha^5 + (23+3\sqrt{2})\alpha^4 + (13+19\sqrt{2})\alpha^3 - - (77-38\sqrt{2})\alpha^2 - (157-24\sqrt{2})\alpha - 96\Big] > 0. \tag{3.2.10c}$$

With the help of computer algebra packages, we find that (3.2.10c) holds for $0 \le \alpha < \alpha_+$, where $\alpha_+ \approx 1.524$ is the largest real root of $p_{+,3}$.

The estimates of I_1, I_2 and I_3 are slightly different when $\alpha < 0$, since now the quadratic term $\alpha\|\operatorname{div} v\|^2_\omega$ in I_1 is negative. This means that it is no longer possible to control the $\|\operatorname{div} v\|_\omega$ terms in I_2, I_3 by $\alpha\|\operatorname{div} v\|^2_\omega$, and in order to obtain positivity, we need to bound $\|\operatorname{div} v\|_\omega$ by $\|Dv\|_\omega$ as follows:

$$\|\operatorname{div} v\|^2_\omega \le 3\|Dv\|^2_\omega.$$

This leads to the revised estimates:

$$I_1 \ge \int_0^\infty r\Big[\Big(1 + \frac{\alpha}{3}\cdot\frac{2\alpha+3}{\alpha+2}\|D_r\overline{u}\|^2_\omega + + \|Dv\|^2_\omega + 3\alpha\|Dv\|^2_\omega + \frac{\alpha}{\alpha+2}\,\|Dv\|^2_\omega\Big]\,dr,$$

$$|I_2| \le \int_0^\infty r\Big[\frac{\sqrt{3}\,\alpha^2}{\alpha+2}\,\|Dv\|^2_\omega - \alpha\,\frac{3\alpha+4}{\alpha+2}\,\|D_r\overline{v}\|\|_\omega\|Dv\|_\omega - - \frac{\alpha}{\sqrt{3}}\,\|D_r\overline{u}\|_\omega\|Dv\|_\omega\Big]\,dr,$$

$$|I_3| \le -\frac{1}{\sqrt{2}}\cdot\frac{\alpha}{\alpha+2}\int_0^\infty r\Big[\|Dv\|^2_\omega + \sqrt{3}\|Dv\|^2_\omega\Big]\,dr.$$

Hence

$$I_1 + I_2 + I_3 \ge \int_0^\infty r(w^T B_- w)\,dr, \tag{4.2.10}$$

where

$$w = \big(\|D_r\overline{u}\|_\omega, \|Dv\|_\omega\big)^T,$$

$$B_- = \begin{bmatrix} 1+\dfrac{\alpha}{3}\cdot\dfrac{2\alpha+3}{\alpha+2} & \dfrac{\alpha}{2}\cdot\dfrac{3\alpha+4}{\alpha+2}+\dfrac{\alpha}{2\sqrt{3}} \\ \dfrac{\alpha}{2}\cdot\dfrac{3\alpha+4}{\alpha+2}+\dfrac{\alpha}{2\sqrt{3}} & 1+3\alpha+\dfrac{\alpha}{\alpha+2}\Big(1+\dfrac{1+\sqrt{3}}{\sqrt{2}}-\sqrt{3}\,\alpha\Big) \end{bmatrix}.$$

The positive definiteness of B_- is equivalent to

$$p_{-,1}(\alpha) = \frac{2\alpha^2+6\alpha+6}{3(\alpha+2)} > 0, \tag{3.2.12a}$$

$$\begin{aligned} p_{-,2}(\alpha) = \frac{1}{6(\alpha+2)^2}\Big[&-(2+7\sqrt{3})\alpha^4 + 2\big(15+\sqrt{2}-11\sqrt{3}+\sqrt{6}\big)\alpha^3 + \\ &+ 2\big(57+3\sqrt{2}-10\sqrt{3}+3\sqrt{6}\big)\alpha^2 + \\ &+ 6\big(20+\sqrt{2}\sqrt{6}\big)\alpha + 24\Big] > 0, \end{aligned} \tag{3.2.12b}$$

and (3.2.12b) holds for $\alpha_- < \alpha < 0$, where $\alpha_- \approx -0.194$ is the smallest real root of $p_{-,2}$.

Now we show that the 3D Lamé system is not positive with weight Φ when α is either too close to -1, or too large. By Proposition 3.11 in [8], the 3D Lamé system is positive with weight Φ only if

$$\sum_{i,\beta,\gamma} A_{ip}^{\beta\gamma}\xi_\beta\xi_\gamma\Phi_{ip}(\omega) \geq 0, \quad \forall\,\xi\in\mathbb{R}^3, \quad \forall\,\omega\in S^2 \quad (i=1,2,3),$$

where

$$A_{ij}^{\beta\gamma} = \delta_{ij}\delta_{\beta\gamma} + \frac{\alpha}{2}\left(\delta_{i\beta}\delta_{j\gamma}+\delta_{i\gamma}\delta_{j\beta}\right)$$

and (see equation (4.1.2))

$$\Phi_{ij}(\omega) = c_\alpha r^{-1}\Big(\delta_{ij}+\frac{\alpha}{\alpha+2}\,\omega_i\omega_j\Big) \quad (i,j=1,2,3).$$

This means, in particular, that the matrix

$$\begin{aligned} A(\omega;\alpha) &:= \Big(\sum_{i=1}^{3} A_{i1}^{\beta\gamma}\Phi_{i1}(\omega)\Big)_{\beta,\gamma=1}^{3} = \\ &= \frac{c_\alpha r^{-1}}{2(\alpha+2)}\begin{bmatrix} 2(\alpha+1)(\alpha+2+\alpha\omega_1^2) & \alpha^2\omega_1\omega_2 & \alpha^2\omega_1\omega_3 \\ \alpha^2\omega_1\omega_2 & 2(\alpha+2+\alpha\omega_1^2) & 0 \\ \alpha^2\omega_1\omega_3 & 0 & 2(\alpha+2+\alpha\omega_1^2) \end{bmatrix} \end{aligned}$$

is semi-positive definite for any $\omega\in S^2$ if the 3D Lamé system is positive with weight Φ. But $A(\omega;\alpha)$ is semi-positive definite only if the determinant of the leading principal minor

$$d_2(\omega;\alpha) := \det\begin{bmatrix} 2(\alpha+1)(\alpha+2+\alpha\omega_1^2) & \alpha^2\omega_1\omega_2 \\ \alpha^2\omega_1\omega_2 & 2(\alpha+2+\alpha\omega_1^2) \end{bmatrix} =$$

$$= 4(\alpha+1)(\alpha+2+\alpha\omega_1^2)^2 - \alpha^4\omega_1^2\omega_2^2$$

is nonnegative, and elementary estimate shows that

$$\min_{\omega\in S^2} d_2(\omega;\alpha) \le d_2\big[(2^{-\frac{1}{2}}, 2^{-\frac{1}{2}}, 0);\alpha\big] = (\alpha+1)(3\alpha+4)^2 - \frac{\alpha^4}{4} =: q(\alpha).$$

It follows that the 3D Lamé system is not positive with weight Φ when $q(\alpha) < 0$, which holds for $\alpha < \alpha_-^{(c)} \approx -0.902$ or $\alpha > \alpha_+^{(c)} \approx 39.450$. $\square$

Remark 4.2.3. We have in fact shown that for $\alpha_- < \alpha < \alpha_+$ and some $c > 0$ depending on α,

$$\int (Lu)^T \Phi u\, dx \ge \frac{1}{2}\,|u(0)|^2 + c\int |Du(x)|^2\,\frac{dx}{|x|}\,.$$

If we replace $\Phi(x)$ by $\Phi_y(x) := \Phi(x-y)$, then

$$\begin{aligned}\int (Lu)^T \Phi_y u\, dx &= \int \big[Lu(x+y)\big]^T \Phi u(x+y)\, dx \ge \\ &\ge \frac{1}{2}\,|u(y)|^2 + c\int \big|Du(x+y)\big|^2\,\frac{dx}{|x|} \ge \\ &\ge \frac{1}{2}\,|u(y)|^2 + c\int \frac{|Du(x)|^2}{|x-y|}\,dx. \qquad (4.2.11)\end{aligned}$$

4.3 Proof of Theorem 4.1.2

In the next lemma and henceforth, we use the notation $S_\rho = \{x :\ \rho < |x| < 2\rho\}$ and

$$m_\rho(u) = \rho^{-3}\int\limits_{\Omega\cap S_\rho} |u(x)|^2\,dx,$$

$$M_\rho(u) = \rho^{-3}\int\limits_{\Omega\cap B_\rho} |u(x)|^2\,dx.$$

Lemma 4.3.1. *Suppose L is positive with weight Φ, and let $u = (u_i)_{i=1}^3$, $u_i \in \overset{\circ}{H}{}^1(\Omega)$ be a solution of*

$$Lu = 0 \ \ on\ \ \Omega\cap B_{2\rho}.$$

Then

$$\int\limits_\Omega \big[L(u\eta_\rho)\big]^T \Phi_y u\eta_\rho\, dx \le c\, m_\rho(u), \ \ \forall\, y\in B_\rho,$$

where $\eta_\rho(x) = \eta(\frac{x}{\rho})$, $\eta \in C_0^\infty(B_{\frac{5}{3}})$, $\eta = 1$ on $B_{\frac{4}{3}}$, and $\Phi_y(x) = \Phi(x-y)$.

Proof. By definition of u,

$$\int\limits_{\Omega} \left[L(u\eta_\rho)\right]^T \Phi_y u\eta_\rho \, dx =$$

$$= \int\limits_{\Omega} \left[L(u\eta_\rho)\right]^T \Phi_y u\eta_\rho \, dx - \int\limits_{\Omega} (Lu)^T \Phi_y u\eta_\rho^2 \, dx,$$

where the second integral on the right=hand side vanishes and the first one equals

$$- \int\limits_{\Omega} \Big[2D_k u_k D_k \eta_r ho + u_i D_{kk}\eta_\rho +$$

$$+ \alpha\big(D_i u_k D_k \eta\rho + D_k u_k D_i \eta_\rho + u_k D_{ki}\eta_\rho\big)\Big] u_j \eta_\rho (\Phi_y)_{ij} \, dx.$$

Note that $D\eta_\rho$, $D^2\eta_\rho$ have compact supports in $R := B_{\frac{5\rho}{3}} \backslash B_{\frac{4\rho}{3}}$ and $|D^k\eta_\rho| \le c\rho^{-k}$. Besides,

$$|\Phi_{y,ij}(x)| \le \frac{c}{|x-y|} \le c\rho^{-1}, \ \ \forall\, x \in R, \ \ \forall\, y \in B_\rho.$$

Thus

$$\int\limits_{\Omega} \left[L(u\eta_\rho)\right]^T \Phi_y u\eta_\rho \, dx \le$$

$$\le \int\limits_{\Omega\cap R} \rho^{-2}|u|\,|Du| \, dx + c \int\limits_{\Omega\cap R} \rho^{-3}|u|^2 \, dx \le$$

$$\le c\Bigg[\rho^{-3} \int\limits_{\Omega\cap S_\rho} |u|^2 \, dx\Bigg]^{\frac{1}{2}} \Bigg[\rho^{-1} \int\limits_{\Omega\cap R} |Du|^2 \, dx\Bigg]^{\frac{1}{2}} + c\rho^{-3} \int\limits_{\Omega\cap S_\rho} |u|^2 \, dx.$$

The lemma then follows from the well known local energy estimate [49]

$$\rho^{-1} \int\limits_{\Omega\cap R} |Du|^2 \, dx \le \rho^{-3} \int\limits_{\Omega\cap S_\rho} |u|^2 \, dx. \qquad \square$$

Combining (4.2.11) (with u replaced by $u\eta_\rho$) and Lemma 4.3.1, we arrive at the following local estimate.

Corollary 4.3.1. *Let the conditions of Lemma* 4.3.1 *be satisfied. Then*

$$|u(y)|^2 + \int\limits_{\Omega\cap B_\rho} \frac{|Du(x)|^2}{|x-y|} \, dx \le c\, m_\rho(u), \ \ \forall\, y \in \Omega \cap B_\rho.$$

To proceed, we need the following Poincaré-type inequality (see Proposition 2.5.1).

Lemma 4.3.2. *Let $u = (u_i)_{i=1}^3$ be any vector function with $u_i \in \overset{\circ}{H}{}^1(\Omega)$. Then for any $\rho > 0$,*

$$m_\rho(u) \le \frac{c}{\operatorname{cap}(\overline{S}_\rho \setminus \Omega)} \int\limits_{\Omega \cap S_\rho} |Du|^2 \, dx,$$

where c is independent of ρ.

The next corollary is a direct consequence of Corollary 4.3.1 and Lemma 4.3.2.

Corollary 4.3.2. *Let the conditions of Lemma* 4.3.1 *be satisfied. Then*

$$|u(y)|^2 + \int\limits_{\Omega \cap B_\rho} \frac{|Du(x)|^2}{|x-y|} \, dx \le \frac{c}{\operatorname{cap}(\overline{S}_\rho \setminus \Omega)} \int\limits_{\Omega \cap S_\rho} |Du|^2 \, dx, \quad \forall\, y \in \Omega \cap B_\rho.$$

We are now in a position to prove the following lemma which is the key ingredient in the proof of Theorem 4.1.2.

Lemma 4.3.3. *Suppose L is positive with weight Φ, and let $u = (u_i)_{i=1}^3$, $u_i \in \overset{\circ}{H}{}^1(\Omega)$ be a solution of $Lu = 0$ on $\Omega \cap B_{2R}$. Then, for all $\rho \in (0, R)$,*

$$\sup_{x \in \Omega \cap B_\rho} |u(x)|^2 + \int\limits_{\Omega \cap B_\rho} \big|Du(x)\big|^2 \frac{dx}{|x|} \le$$

$$\le c_1 M_{2R}(u) \exp\Big[- c_2 \int\limits_\rho^R \operatorname{cap}(\overline{B}_r \setminus \Omega) r^{-2} \, dr \Big], \qquad (4.3.1)$$

where c_1, c_2 are independent of ρ.

Proof. Define

$$\gamma(r) := r^{-1} \operatorname{cap}(\overline{S}_r \setminus \Omega).$$

We first claim that $\gamma(r)$ is bounded from above by some absolute constant A. Indeed, the monotonicity of capacity implies that

$$\operatorname{cap}(\overline{S}_r \setminus \Omega) \le \operatorname{cap}(\overline{B}_r).$$

By choosing smooth test functions $\eta_r(x) = \eta(\frac{x}{r})$ with $\eta \in C_0^\infty(B_2)$ and $\eta = 1$ on $B_{\frac{3}{2}}$, we also have

$$\operatorname{cap}(\overline{B}_r) \le \int\limits_{\mathbb{R}^3} |D\eta_r|^2 \, dx \le$$

$$\le \sup_{x \in \mathbb{R}^3} |D\eta(x)|^2 \int\limits_{B_{2r}} r^{-2} \, dx = \Big[\frac{32}{3} \pi \sup_{x \in \mathbb{R}^3} |D\eta(x)|^2 \Big] r.$$

Hence the claim follows.

We next consider the case $\rho \in (0, \frac{R}{2}]$. Denote the first and the second terms on the left-hand side of (4.3.1) by φ_ρ and ψ_ρ, respectively. From Corollary 4.3.2, it follows that for $r \le R$,

$$\varphi_r + \psi_r \le \frac{c}{\gamma(r)}\,(\psi_{2r} - \psi_r) \le \frac{c}{\gamma(r)}\,(\psi_{2r} - \psi_r + \varphi_{2r} - \varphi_r),$$

which implies that

$$\begin{aligned}\varphi_r + \psi_r \le \frac{c}{c+\gamma(r)}\,(\varphi_{2r} + \psi_{2r}) &= \\ &= \frac{ce^{c_0\gamma(r)}}{c+\gamma(r)}\,\big[e^{-c_0\gamma(r)}(\varphi_{2r} + \psi_{2r})\big], \quad \forall\, c_0 > 0.\end{aligned}$$

Since $\gamma(r) \le A$ and

$$\sup_{s\in[0,A]} \frac{ce^{c_0 s}}{c+s} \le \max\Big\{1, \frac{ce^{c_0 A}}{c+A}\,, cc_0 e^{1-cc_0}\Big\},$$

it is possible to choose $c_0 > 0$ sufficiently small so that

$$\sup_{r>0} \frac{ce^{c_0\gamma(r)}}{c+\gamma(r)} \le 1.$$

It follows, for c_0 chosen this way, that

$$\varphi_r + \psi_r \le e^{-c_0\gamma(r)}(\varphi_{2r} + \psi_{2r}). \tag{4.3.2}$$

By setting $r = 2^{-l}R$, $l \in \mathbb{N}$, and repeatedly applying (4.3.2), we obtain

$$\varphi_{2^{-l}R} + \psi_{2^{-l}R} \le \exp\Big[-c_0\sum_{j=1}^{l}\gamma(2^{-j}R)\Big](\varphi_R + \psi_R).$$

If l is such that $l \le \log_2(\frac{R}{\rho}) < l+1$, then $\rho \le 2^{-l}R < 2\rho$ and

$$\varphi_\rho + \psi_\rho \le \varphi_{2^{-l}R} + \psi_{2^{-l}R} \le \exp\Big[-c_0\sum_{j=1}^{l}\gamma(2^{-j}R)\Big](\varphi_R + \psi_R).$$

Note that by Corollary 4.3.1,

$$\varphi_R + \psi_R \le c\, m_R(u) \le c\, M_{2R}(u).$$

In addition, the subadditivity of the harmonic capacity implies that

$$\sum_{j=1}^{l}\gamma(2^{-j}R)\geq\sum_{j=1}^{l}\frac{\operatorname{cap}(\overline{B}_{2^{1-j}R}\setminus\Omega)-\operatorname{cap}(\overline{B}_{2^{-j}R}\setminus\Omega)}{2^{-j}R}=$$

$$=\frac{\operatorname{cap}(\overline{B}_{R}\setminus\Omega)}{2^{-1}R}-\frac{\operatorname{cap}(\overline{B}_{2^{-l}R}\setminus\Omega)}{2^{-l}R}+\sum_{j=1}^{l-1}\frac{\operatorname{cap}(\overline{B}_{2^{-j}R}\setminus\Omega)}{2^{-j}R}=$$

$$=\frac{1}{2}\cdot\frac{\operatorname{cap}(\overline{B}_{R}\setminus\Omega)}{R}-2\,\frac{\operatorname{cap}(\overline{B}_{2^{-l}R}\setminus\Omega)}{2^{-l}R}+\sum_{j=1}^{l}\frac{\operatorname{cap}(\overline{B}_{2^{-j}R}\setminus\Omega)}{2^{-j}R}\geq$$

$$\geq-2\,\frac{\operatorname{cap}(\overline{B}_{2^{-l}R}\setminus\Omega)}{2^{-l}R}+\frac{1}{2}\sum_{j=0}^{l}\frac{\operatorname{cap}(\overline{B}_{2^{-j}R}\setminus\Omega)}{2^{-j}R}\,.$$

Since

$$\frac{\operatorname{cap}(\overline{B}_{2^{-l}R}\setminus\Omega)}{2^{-l}R}\leq A,$$

$$\sum_{j=0}^{l}\frac{\operatorname{cap}(\overline{B}_{2^{-j}R}\setminus\Omega)}{2^{-j}R}\geq\frac{1}{2}\sum_{j=1}^{l+1}\frac{\operatorname{cap}(\overline{B}_{2^{1-j}R}\setminus\Omega)}{(2^{-j}R)^{2}}\cdot 2^{-j}R\geq$$

$$\geq\frac{1}{2}\sum_{j=1}^{l+1}\int_{2^{-j}R}^{2^{1-j}R}\operatorname{cap}(\overline{B}_{r}\setminus\Omega)r^{-2}\,dr\geq\frac{1}{2}\int_{\rho}^{R}\operatorname{cap}(\overline{B}_{r}\setminus\Omega)r^{-2}\,dr,$$

we have

$$\exp\Big[-c_0\sum_{j=1}^{l}\gamma(2^{-j}R)\Big]\leq\exp\Big[-\frac{c_0}{4}\int_{\rho}^{R}\operatorname{cap}(\overline{B}_{r}\setminus\Omega)r^{-2}\,dr+2c_0A\Big].$$

Hence (4.3.1) follows with $c_1=ce^{2c_0A}$ and $c_2=\frac{c_0}{4}$.

Finally, we consider the case $\rho\in(\frac{R}{2}\,,R)$. By Corollary 4.3.1,

$$|u(y)|^{2}+\int_{\Omega\cap B_{\rho}}\frac{|Du(x)|^{2}}{|x-y|}\,dx\leq c\,m_{\rho}(u),\ \ \forall\,y\in\Omega\cap B_{\rho},$$

which implies that

$$\sup_{y\in\Omega\cap B_{\rho}}|u(y)|^{2}+\int_{\Omega\cap B_{\rho}}|Du(x)|^{2}\,\frac{dx}{|x|}\leq c\,M_{2R}(u).$$

In addition,

$$\int_{\rho}^{R}\operatorname{cap}(\overline{B}_{r}\setminus\Omega)r^{-2}\,dr\leq A\int_{\frac{R}{2}}^{R}r^{-1}\,dr=A\log 2,$$

so

$$\Big[\sup_{y\in\Omega\cap B_\rho}|u(y)|^2+\int\limits_{\Omega\cap B_\rho}|Du(x)|^2\,\frac{dx}{|x|}\Big]\times$$

$$\times\exp\Big[c_2\int\limits_\rho^R \operatorname{cap}(\overline{B}_r\setminus\Omega)r^{-2}\,dr\Big]\le c_1M_{2R}(u),$$

provided that $c_1\ge ce^{c_2A\log 2}$. □

Proof of Theorem 4.1.2. Consider the Dirichlet problem (4.1.3)

$$Lu=f,\qquad f_i\in C_0^\infty(\Omega),\ \ u_i\in \overset{\circ}{H}{}^1(\Omega).$$

Since f vanishes near the boundary, there exists $R>0$ such that $f=0$ in $\Omega\cap B_{2R}$. By Lemma 4.3.3,

$$\sup_{y\in\Omega\cap B_\rho}|u(x)|^2\le c_1M_{2R}(u)\exp\Big[-c_2\int\limits_\rho^R \operatorname{cap}(\overline{B}_r\setminus\Omega)r^{-2}\,dr\Big],$$

and in particular,

$$\limsup_{x\to 0}|u(x)|^2\le c_1M_{2R}(u)\exp\Big[-c_2\int\limits_0^R \operatorname{cap}(\overline{B}_r\setminus\Omega)r^{-2}\,dr\Big]=0,$$

where the last equality follows from the divergence of the Wiener integral

$$\int\limits_0^1 \operatorname{cap}(\overline{B}_r\setminus\Omega)r^{-2}\,dr=\infty.$$

Thus O is regular with respect to L. □

Remark 4.3.1. In the paper by Guo Luo and Maz'ya [33] we studied weighted integral inequalities of

$$\int\limits_\Omega Lu\cdot\Psi u\,dx\ge 0 \tag{4.3.3}$$

for general second order elliptic systems L in $\mathbb{R}^n$ ($n\ge 3$). For weights that are smooth and positive homogeneous of order $2-n$, we have shown that$L$ is positive in the sense of (4.3.3) only if the weight is the fundamental matrix of L, possibly multiplies by a semi-positive definite constant matrix.

Chapter 5

An Analogue of the Wiener Criterion for the Zaremba Problem in a Cylindrical Domain

In this chapter asymptotic behavior at infinity of solutions to the Zaremba problem for the Laplace operator in a half-cylinder is studied. Pointwise estimates for solutions, the Green function and the harmonic measure are obtained in terms of the Wiener capacity. The main result is a necessary and sufficient condition for regularity of a point at infinity.

5.1 Formulation of the Zaremba Problem

Let G be the semicylinder $\{x = (x', x_n) :\ x_n > 0,\ x' \in \omega\}$, where ω is a domain in $\mathbb{R}^{n-1}$ with compact closure and smooth boundary. Suppose that a closed subset F is selected on $\partial\sigma$ with limit points at infinity. Further, let

$$G_\tau = \bigl\{x \in G :\ x_n > \tau\bigr\}, \quad S_\tau = \bigl\{x \in G :\ x_n = \tau\bigr\},$$
$$F_\tau = \bigl\{x \in F :\ x_n > \tau\bigr\}.$$

By $k, k_0, k_1, \dots$ we mean positive constants depending on n and the domain ω. In the case $n > 2$, by $\operatorname{cap}(e)$ we denote the harmonic capacity of a Borel set $e \subset \mathbb{R}^n$. For $n = 2$ we use the same notation for the capacity generated by the operator $-\Delta + 1$. By "quasi-everywhere" we mean "outside of a set of zero capacity".

We introduce the space $\overset{\circ}{L}{}^1_2(G; F)$ of functions given on G having the finite norm

$$\|u\|_{\overset{\circ}{L}{}^1_2(G;F)} = \Bigl(\int\limits_G (\operatorname{grad} u)^2\, dx + \int\limits_{G\setminus G_1} u^2\, dx\Bigr)^{\frac{1}{2}}, \tag{5.1.1}$$

and vanishing quasi-everywhere on F. By Hardy's inequality the above norm is equivalent to

$$\Big(\int_G \Big[(\operatorname{grad} u)^2 + (x_n+1)^{-2}u^2\Big]\,dx\Big)^{\frac{1}{2}}. \tag{5.1.2}$$

This implies that the set of functions in $\overset{\circ}{L}{}^1_2(G;F)$ with compact support in G is dense in $\overset{\circ}{L}{}^1_2(G;F)$. Since any function from that set can be approximated in $W^1_2(G)$ by a sequence of smooth functions vanishing near F, it follows that the space $C_0^\infty(\overline{G}\setminus F)$ is dense in $\overset{\circ}{L}{}^1_2(G;F)$ (cf. [18]).

Let $L_2^{-1}(G;F)$ stand for the space of linear functionals on $\overset{\circ}{L}{}^1_2(G;F)$. Any functional $f \in L_2^{-1}(G;F)$ can be represented in the form

$$f(v) = \int_G \Big(\sum_{i=1}^n f_i\,\frac{\partial v}{\partial x_i} + f_0 v\Big), \quad v \in \overset{\circ}{L}{}^1_2(G;F), \tag{5.1.3}$$

where f_i and $(x_n+1)f_0$ belong to $L_2(G)$ (see [25]). Note that for any $\tau \in (0,\infty)$ the inequality

$$\|u\|^2_{L_2(G\setminus G_\tau)} \le \frac{k(\tau)}{\operatorname{cap}(F\setminus F_\tau)}\,\|\nabla u\|^2_{L_2(G\setminus G_\tau)},$$

holds (see [25, Chapter 10]). Hence, given a set F of positive capacity which is always assumed in what follows, we see that the norm (5.1.1) is equivalent to $\|\nabla u\|_{L_2(G)}$.

Consider the integral identity

$$\int_G \nabla u \nabla v\,dx = f(v), \tag{5.1.4}$$

where $f \in L_2^{-1}(G;F)$, $v \in C_0^\infty(\overline{G}\setminus F)$, u belongs to $W^1_2(G\setminus G_\tau)$ for any τ, and u vanishes quasi-everywhere on F. Assuming additionally that $f_i \in W^1_2(G)$ in (5.1.3), we obtain, as is well known (cf. [17, Section 15]), that $u \in W^2_2$ in a small neighborhood of any point in $\overline{G}\setminus F$, and the equality (5.1.4) can be understood in the strong sense:

$$-\Delta u = f_0 - \operatorname{div} f \ \text{ in } \ G, \quad \frac{\partial u}{\partial \nu} = f\cdot\nu \ \text{ on } \ \partial G\setminus F,$$
$$u = 0 \ \text{ quasi-everywhere on } \ F,$$

where $f = (f_1,\dots,f_n)$ and ν is the outward normal to ∂G.

Therefore, it is natural to call u as the (generalized) solution of the Zaremba problem. If $u \in \overset{\circ}{L}{}^1_2(G;F)$, we call u a solution with the finite Dirichlet integral. In this case, one can take v in (5.1.4) as an arbitrary function in $\overset{\circ}{L}{}^1_2(G;F)$. Since the left-hand side of (5.1.4) is the scalar product in $\overset{\circ}{L}{}^1_2(G;F)$ and the right-hand side of (5.1.4) is a linear functional on $\overset{\circ}{L}{}^1_2(G;F)$, it follows that the solution with the finite Dirichlet integral exists and is unique.

5.2 Auxiliary Assertions

In this section we prove two auxiliary assertions and provide information on solutions of a certain ordinary differential equation.

Lemma 5.2.1. *Let u have the finite norm* (5.1.2) *and satisfy the inequalities*

$$\Delta u \le 0 \;\; on \;\; G, \qquad \frac{\partial u}{\partial \nu} \ge 0 \;\; on \;\; \partial G \setminus F$$

in the sense that

$$\int_G \nabla u \nabla v \, dx \ge 0 \;\; for \;\; 0 \le v \in \overset{\circ}{L}{}^1_2(G;F). \tag{5.2.1}$$

Besides, let $u \ge 0$ quasi-everywhere on F. Then $u \ge 0$ on $\overline{G} \setminus F$.

Proof. Since $u_- = \frac{|u|-u}{2} = 0$ quasi-everywhere on F, we have $u_- \in \overset{\circ}{L}{}^1_2(G;F)$ and can put $v = u_-$ in (5.2.1). Then $\|\nabla u_-\|_{L_2(G)} = 0$ and hence $u_- = const$. This constant is zero, because $u_- = 0$ on a set of positive capacity. □

Lemma 5.2.2. *Let $f = 0$ on G_τ and u be the solution of the Zaremba problem with the finite Dirichlet integral. Then*

$$\sup_{S_{\lambda+1}} |u| \le k \|u\|_{L_2(G_\lambda \setminus G_{\lambda+2})} \;\; for \;\; \lambda > \tau. \tag{5.2.2}$$

For elliptic equations of the second order in divergence form with measurable bounded coefficients, estimate (5.2.2) was proved by Moser [57]. To be more precise, [57] contains an interior local estimate of the type (5.2.2). However, its proof can be easily extended to the case under consideration.

Consider now the ordinary differential equation

$$\xi''(\sigma) - p(\sigma)\xi(\sigma) = 0 \tag{5.2.3}$$

on the half-axis $(0, \infty)$ with a nonnegative measurable function p, not vanishing identically. By Z we denote a solution of (5.2.3) satisfying the initial conditions

$$Z(0) = 1, \quad Z'(0) = 0.$$

Clearly, Z is a convex nondecreasing function obeying the inequalities

$$Z(\sigma) \ge 1 \ \text{ and } \ \lim_{\sigma\to\infty} \frac{Z(\sigma)}{\sigma} > 0.$$

Let

$$z(\sigma) = Z(\sigma) \int_\sigma^\infty \frac{d\tau}{(Z(\tau))^2}$$

be another solution of (5.2.3), positive for $\sigma > 0$. We have $zZ' - z'Z = 1$ and $z'(0) = -1$. The function z is nonincreasing because

$$z'(\sigma) = Z'(\sigma) \int_\sigma^\infty \frac{d\tau}{Z(\sigma)^2} - \frac{1}{Z(\sigma)} \le \int_\sigma^\infty \frac{Z'(\sigma)}{Z(\sigma)^2}\, d\sigma - \frac{1}{Z(\sigma)} = 0$$

By (5.2.3), the function Z' is nondecreasing and tends to zero at infinity. Therefore, for any $a > 0$,

$$\int_a^\infty \left[z'(\sigma)^2 + p(\sigma) z(\sigma)^2\right] d\sigma = z(\sigma) z'(\sigma)\Big|_a^\infty = -z(a)z'(a).$$

In view of this identity, the function $\sigma \to A\,\frac{z(\sigma)}{z(a)}$ provides the minimum of the functional

$$\xi \longrightarrow \int_a^\infty \left[\xi'(\sigma)^2 + p(\sigma)\xi(\sigma)^2\right] d\sigma$$

on the set of absolutely continuous functions satisfying the condition $\xi(a) = A$, and the value of this minimum is equal to $-A^2 \frac{z'(a)}{Z(a)}$. Note also that $A\,\frac{Z(\sigma)}{Z(a)}$ provides the minimum value $A^2\,\frac{z'(a)}{Z(a)}$ to the functional

$$\xi \longrightarrow \int_0^a \left[\xi'(\sigma)^2 + p(\sigma)\xi(\sigma)^2\right] d\sigma.$$

Here $a \in (0, \infty)$ and ξ is an arbitrary absolutely continuous function satisfying the condition $\xi(a) = A$. Information on minimum values of these functionals implies that both $\frac{|z'|}{z}$ and $\frac{Z'}{Z}$ do not decrease a p grows. This enables one to obtain estimates for solutions z and Z under additional assumptions on p. For example, if $p(\sigma) \le \varkappa = const$, which will take place in what follows, then, combining (5.2.3) with the equation $\xi'' - \varkappa\xi = 0$, we obtain

$$0 \le \frac{Z'(\sigma)}{Z(\sigma)} \le \varkappa^{\frac12} \operatorname{th}(\varkappa^{\frac12}\sigma) \le \varkappa^{\frac12} \ \text{ and } \ -\varkappa^{\frac12} \le \frac{z'(\sigma)}{z(\sigma)} \le 0.$$

Therefore, for any positive a and σ,

$$Z(\sigma) \le Z(\sigma + a) \le Z(\sigma) e^{\varkappa^{\frac12} a} \ \text{ and } \ z(\sigma) e^{-\varkappa^{\frac12} a} \le z(\sigma + a) \le z(\sigma).$$

5.3 Estimates for Solutions of the Zaremba Problem

We set $\mathfrak{C}(\sigma) = \mathrm{cap}\,(F_\sigma \setminus F_{\sigma+1})$ and consider the ordinary differential equation

$$\xi''(\sigma) - k\mathfrak{C}(\sigma)\xi(\sigma) = 0, \quad \sigma > 0. \tag{5.3.1}$$

This means that we put $p(\sigma) = k\mathfrak{C}(\sigma)$ in (5.2.3). In the same way as in Section 5.2, by z and Z we denote the nonincreasing and nondecreasing solutions of (5.2.3)with that choice of p.

Given a compact set $F \subset \mathbb{R}^n$, denote $\Pi_a^b = ([a,b] \times \mathbb{R}^{n-1})$ and $\Phi(a,b) = \mathrm{cap}\,(\Pi_a^b \cap F)$.

Lemma 5.3.1. *For any compact set $F \subset \mathbb{R}^n$, the function $\Phi(a, a+1)$ is Lebesgue measurable on $\mathbb{R}$ as a function of variable a.*

Proof. For any compact set F the function $\Phi(a,b)$ is increasing in variable b and decreasing in variable a. Therefore, this function is Lebesgue measurable on $\mathbb{R}^2$.

By the Fubini theorem, $\Phi(a, a+\lambda)$ is a measurable function of a for almost all $\lambda \in \mathbb{R}$.

Consider now a δ-neighbourhood of F with $\delta > 0$. For this domain we use the notation F_δ. Letting

$$\Phi_\delta(a,b) = \mathrm{cap}\,(\Pi_a^b \cap F_\delta),$$

in the same way as above we obtain that $\Phi_\delta(a, a+\lambda)$ is a measurable function of a for almost all $\lambda \in \mathbb{R}$.

Obviously, there exists $\lambda_0(\delta) > 1$ such that for all $\lambda \in (1, \lambda_0(\delta))$ we have

$$F \subset \lambda^{-1} F_\delta \subset F_{2\delta}.$$

Choosing now $\lambda \in (1, \lambda_0(\delta))$ so that $\Phi_\delta(a, a+\lambda)$ is measurable, by the scaling arguments we deduce that $\mathrm{cap}\,(\Pi_{\lambda^{-1}a}^{\lambda^{-1}a+1} \cap \lambda^{-1} F_\delta)$ is a measurable function of a. Therefore, $\mathrm{cap}\,(\Pi_a^{a+1} \cap \lambda^{-1} F_\delta)$ is also a measurable function of a.

It remains to send $\delta \to 0$. Since $\mathrm{cap}\,(\Pi_a^{a+1} \cap \lambda^{-1} F_\delta)$ converges, as $\delta \to 0$, to $\mathrm{cap}\,(\Pi_a^{a+1} \cap F)$ for all a, we obtain the desired measurability. $\square$

The following lemma, similar to analogous assertions related to the Dirichlet problem in [69] and [37], is the key one.

Lemma 5.3.2. *Let $f = 0$ on G_τ and let G_τ be the solution of the Zaremba problem with the finite Dirichlet integral. Then*

$$\int\limits_\omega u(x)^2\, dx' \le \frac{z(x_n)}{z(y_n)} \int\limits_\omega u(y)^2\, dy' \quad \textit{for } x_n > y_n > \tau. \tag{5.3.2}$$

Proof. Let $s \to \eta(s)$ be a piecewise linear function given on $\mathbb{R}^1$, vanishing for $s < 0$ and equal to unity for $s > 1$. Setting the function

$$x \longrightarrow v(x) = \eta\big(\xi^{-1}(t-\sigma)\big)u(x), \quad x \in G,$$

with $t > \sigma > \tau$, into (5.1.4), we obtain

$$\int_G \eta\Big(\frac{t-\sigma}{\varepsilon}\Big)(\nabla u)^2\,dx + \frac{1}{\varepsilon}\int_G \eta'\Big(\frac{t-\sigma}{\varepsilon}\Big)u(x)\,\frac{\partial u}{\partial x_n}\,(x)\,dx = 0$$

which is equivalent to

$$\int_G \eta\Big(\frac{t-\sigma}{\varepsilon}\Big)(\nabla u)^2\,dx + \frac{\mathfrak{F}(\sigma+\varepsilon)-\mathfrak{F}(\sigma)}{2\varepsilon} = 0,$$

where $\mathfrak{F}(\sigma) = \|u\|^2_{L_2(S_\sigma)}$. Passing to the limit as $\varepsilon \to 0$, we find

$$\int_{G_\sigma} (\nabla u)^2\,dx = -\frac{\mathfrak{F}'(\sigma)}{2}\,. \tag{5.3.3}$$

Hence, for any $\varepsilon \in (0,1)$,

$$\int_{G_\sigma} \Big(\frac{\partial u}{\partial t}\Big)^2\,dx + \int_\sigma^\infty \int_{G_t\setminus G_{t+1}} (\nabla u)^2\;dx\,d\sigma \le -\mathfrak{F}'(\sigma). \tag{5.3.4}$$

Combining the inequality

$$\int_{G_t\setminus G_{t+1}} (\nabla u)^2\,dx \ge k\mathfrak{E}(t) \int_{G_t\setminus G_{t+1}} u^2\,dx$$

with the known estimate

$$\int_{S_t} u^2\,dx' \le k\int_\sigma^\infty \int_{G_t\setminus G_{t+1}} \big[(\nabla u)^2 + u^2\big]\,dx,$$

we have

$$\int_{G_t\setminus G_{t+1}} (\nabla u)^2\,dx \ge k\mathfrak{E}(t)\mathfrak{F}(t).$$

Substituting it in (5.3.4), we obtain

$$\int_\omega dx \int_G^\infty \big[u_t^2 + k\mathfrak{E}(t)u^2\big]\,dt \le -\mathfrak{F}'(\sigma), \tag{5.3.5}$$

where

$$u_t(x',t) = \frac{\partial u(x',t)}{\partial t}\,.$$

The functional

$$\xi \longrightarrow \int\limits_0^\infty \Big[\Big(\frac{d\xi}{dt}\Big)^2 + k\mathfrak{C}(t)\xi^2\Big]\,dt$$

defined on the functions obeying the condition $\xi(\sigma) = u(x',\sigma)$, attains its minimum value at the solution

$$t \longrightarrow u(x',\sigma)\,\frac{z(t)}{z(\sigma)}$$

of equation (5.3.1), and the value of that minimum is $-u^2(x',\sigma)\,\frac{z'(\sigma)}{z(\sigma)}$ (cf. Section 5.2). Hence (5.3.5) implies the differential inequality

$$\Big(\frac{z'(\sigma)}{z(\sigma)}\Big)\mathfrak{F}(\sigma) \geq \mathfrak{F}'(\sigma)$$

which results in (5.3.2). □

Corollary 5.3.1. *Let $f = 0$ on G_τ and let u be the solution of the Zaremba problem with the finite Dirichlet integral. Then, with $y = (y', y_n)$,*

$$\sup_{x'\in\omega} u(x)^2 \leq k\,\frac{z(x_n)}{z(y_n)} \int\limits_\omega u(y)^2\,dy' \quad \textit{for } \; x_n - 1 > y_n > \tau. \tag{5.3.6}$$

Proof. Using (5.2.2) and the monotonicity of the function $\mathfrak{F}$ (cf. (5.3.3)), we have

$$\sup_{S_t} u^2 \leq k\|u\|_{L_2(G_{t-1}\setminus G_{t+1})} \leq k\big[2\mathfrak{F}(t-a)\big]^{\frac{1}{/2}} \;\; \text{for} \;\; t-1 > \tau,$$

which together with (5.3.2) implies the estimate

$$\sup_{S_t} u^2 \leq k\Big[2\,\frac{z(t-1)}{z(\sigma)}\,\mathfrak{F}(\sigma)\Big]^{\frac{1}{2}} \;\; t-1 > \sigma > \tau.$$

It remains to use the inequality

$$z(\sigma)e^{-\varkappa^{\frac{1}{2}}a} \leq z(\sigma + a)$$

(see the end of Section 5.2). □

Remark 5.3.1. If the function $\mathfrak{E}$ is sufficiently regular at infinity or has a regular minorant, then, using the known asymptotic formulas or estimates for solutions of (5.3.1), one can obtain more precise information on solutions of the Zaremba problem by (5.3.2) and (5.3.6). Roughly speaking, there exist three alternatives:

$$\frac{z(t)}{z(\sigma)} = \begin{cases} O\Big(\exp\Big(-k\int\limits_{\sigma}^{t}\sqrt{\mathfrak{E}(s)}\,ds\Big)\Big), & \text{if } \mathfrak{E}(s) \gg s^{-2}, \\ O\Big(\Big(\frac{\sigma}{t}\Big)^{k}\Big), & \text{if } \mathfrak{E}(s) \sim s^{-2}, \\ O\Big(\exp\Big(-k\int\limits_{\sigma}^{t} s\,\mathfrak{E}(s)\,ds\Big)\Big), & \text{if } \mathfrak{E}(s) \ll s^{-2} \end{cases} \tag{5.3.7}$$

(cf. [17, Chapter II] and [15]). In order to check this, it suffices to reduce (5.3.1) to the Riccati equation

$$Y'(\sigma) = Y^2(\sigma) = k\,\mathfrak{E}(\sigma),$$

where $Y(\sigma) = \frac{\xi'(\sigma)}{\xi(\sigma)}$, and to note that the above estimates for z are valid for $Y' \ll Y^2$, $Y' \approx Y^2$ and $Y' \gg Y^2$ at infinity. Similar estimates hold for the increasing solution Z.

5.4 Regularity Criterion for a Point at Infinity

We say that a point at infinity is regular for the Zaremba problem if for all $f \in L_2^{-1}(G;F)$ with a bounded support, the solution with the finite Dirichlet integral tends to zero as $x_n \to \infty$ and $x \in G$. Here is the main result.

Theorem 5.4.1. *A point at infinity is regular for the Zaremba problem if and only if the function* $t\mathfrak{E}(t)$ *is not integrable on* $(0,\infty)$*, or equivalently,*

$$\sum_{j=1}^{\infty} j\,\mathrm{cap}\,(F_j \setminus F_{j+1}) = \infty. \tag{5.4.1}$$

Consider an example of a set F for which the above regularity criterion can be expressed explicitly. Let p be a point at $\partial\omega$ and let ψ denote a decreasing positive continuous function given on $[0,\infty]$ and such that $\psi(0)=1$.

Let

$$F = \Big\{x \in \partial G : \frac{x'-p}{\psi(x_n)} \in \delta,\ x_n \geq 0\Big\},$$

where δ is a domain on $\partial\omega$. The well known estimates for the capacity of a parallelepiped (cf. [2]) imply the inequalities

$$\left.\begin{matrix} \dfrac{k_1}{\log\frac{k_2}{\psi(j+1)}} \\ k_5\psi(j+1)^{n-3} \end{matrix}\right\} \leq \mathrm{cap}\,(F_j \setminus F_{j+1}) \leq \left\{\begin{matrix} \dfrac{k_3}{\log\frac{k_4}{\psi(j)}} & (n=3), \\ k_6\psi(j)^{n-3} & (n>3). \end{matrix}\right.$$

Hence (5.4.1) holds, if and only if

$$\int^{\infty} \frac{s}{|\log\psi(s)|}\, ds = \infty \ \text{ for } \ n = 3, \qquad \int^{\infty} \psi(s)^{n-3} s\, ds = \infty \ \text{ for } \ n > 3.$$

Proof of Theorem 5.4.1. *Sufficiency.* Let u and z be the same as in Lemma 5.3.2. By Corollary 5.3.1, for $x \in G$,

$$u(x) \to 0 \ \text{ as } \ x \to \infty, \ \text{ if } \ z(\sigma) \to 0 \ \text{ as } \ \sigma \to \infty.$$

Suppose that the limit $z(\infty)$ is positive. Since $z'(\infty) = 0$ and $z(\sigma) > \frac{z(\infty)}{2}$ for large σ, we have after integrating (5.3.1) from σ to ∞ that

$$-z'(\sigma) \geq \frac{k}{2}\, z(\infty) \int_{\sigma}^{\infty} \operatorname{cap}(F_\mu \setminus F_{\mu+1})\, d\mu.$$

This implies the estimate

$$z(t) - z(\sigma) \geq \frac{k}{2}\, z(\sigma) \int_{t}^{\infty} d\sigma \int_{\sigma}^{\infty} \operatorname{cap}(F_\mu \setminus F_{\mu+1})\, d\mu =$$

$$= \frac{k_0}{2}\, z(\infty) \int_{t}^{\infty} (\mu - t) \operatorname{cap}(F_\mu \setminus F_{\mu+1})\, d\mu.$$

Hence

$$\int^{\infty} \mu \operatorname{cap}(F_\mu \setminus F_{\mu+1})\, d\mu < \infty$$

which is equivalent to (5.4.1).

Necessity will be proved with the help of the following lemma on estimates of the Neumann function $N(x, y)$ in a cylinder. $\square$

Lemma 5.4.1. *Given $y \in \overline{G}$, let $N(x, y)$ stand for the solution of the problem*

$$-\Delta_x N(x, y) = \delta(x - y) - \lambda(x) \ \text{ in } \ G,$$

$$\frac{\partial N(x, y)}{\partial \nu_x} = 0 \ \text{ on } \ \partial G \setminus \{y\},$$

vanishing for $x \to \infty$ and any fixed y. Here

$$\lambda \in C_0^\infty(\overline{G}) \ \text{ and } \int_G \lambda(x)\, dx = 1.$$

Further, let γ_1^2 be the first positive eigenvalue of the Laplace operator in ω with zero Neumann condition on $\partial\omega$ and let $|\omega|$ stand for the $(n-1)$-dimensional Lebesgue measure of ω.

There exist positive constants $\varkappa$ and k depending on n, ω and λ such that

(i) $|N(x,y)| \le ke^{-\gamma_1(x_n-y_n)}$ *for* $x_n - y_n > \varkappa$,

(ii) $|N(x,y) + \frac{x_n-y_n}{|\omega|}| \le ke^{\gamma_1(x_n-y_n)}$ *for* $y_n - x_n > \varkappa$,

(iii) *the ratio of* N *to the fundamental solution of the Laplace operator in* $\mathbb{R}^n$ *is bounded from above and is separated from zero from below by positive constants in the zone* $|x-y| < \varkappa$.

The proof of this lemma will be given at the end of this section, while we turn to the necessity of the condition (5.4.1).

Let

$$\sum_{j=0}^{\infty}(j+1)\,\mathrm{cap}\,(F_j \setminus F_{j+1}) = \infty. \tag{5.4.2}$$

Suppose that a point at infinity is regular. Since the solution of the Zaremba problem in G_t multiplied by a smooth function in $\overline{G}$, supported in $\overline{G}_t$ and equal to unity in a neighborhood of infinity, becomes the solution of a similar problem in G, it follows that a point at infinity is regular for the cylinder G_t with any $t > 0$. Hence, from the very beginning, one may assume the sum in (5.4.2) to be sufficiently small. Let

$$F_j \setminus F_{j+1} = \bigcup_{k=1}^{L} F_j^{(k)} \quad \text{with} \quad \mathrm{diam}\, F_j^{(k)} < \frac{\varkappa}{4}\,,$$
$$F_j^{(k_1)} \cap F_j^{(k_2)} = \varnothing \quad \text{for} \quad k_1 \ne k_2.$$

Here $\varkappa$ is the same constant as in the statement of Lemma 5.4.1. Since

$$\mathrm{cap}\; F_j^{(k)} \le \mathrm{cap}\,(F_j \setminus F_{j+1}),$$

the sum

$$\sum_{j=0}^{\infty}(j+1)\sum_{k=1}^{L}\mathrm{cap}\; F_j^{(k)}$$

is sufficiently small.

Let $\mu_j^{(k)}$ be the equilibrium measure of the set $F_j^{(k)}$ (cf. [29, Chapter II]). We introduce the potential

$$V_j^{(k)}(x) = \int\limits_{F^{(k)}} N(x,y)\, d\mu_j^{(k)}(y),$$

where N is the Neumann function from Lemma 5.4.1.

By the definition of the function N, the potential $V_j^{(k)}$ satisfies both the equation

$$\Delta V_j^{(k)}(x) = \int\limits_{F_j(k)} \lambda(x)\, d\mu_j^{(k)}(y) = \lambda(x)\,\mathrm{cap}\; F_j^{(k)} \quad \text{in} \ \ G \tag{5.4.3}$$

and the boundary condition

$$\frac{\partial V_j^{(k)}}{\partial \nu} = 0 \ \text{ on } \ \partial G \setminus F_j^{(k)}. \tag{5.4.4}$$

We restrict ourselves to the case $n > 2$. For $n = 2$, one should replace everywhere $|x-y|^{2-n}$ by the fundamental solution of the operator $-\Delta + 1$. By Lemma 5.4.1,

$$\big|V_j^{(k)}(x)\big| \le k \int\limits_{F_j^{(k)}} \frac{d\mu_j^{(k)}(y)}{|x-y|^{n-2}}\,, \quad x \in G_{j-1} \setminus G_j.$$

Further, by Lemma 5.4.1(i),

$$\big|V_j^{(k)}(x)\big| \le k \,\mathrm{cap}\ F_j^{(k)} \quad x \in G_{j+1},$$

and by Lemma 5.4.1(ii), the estimate

$$\big|V_j^{(k)}(x)\big| \le c \int\limits_{F^{(k)}} (y_n + 1)\, d\mu_j^{(k)}(y) \le k(j+1)\,\mathrm{cap}\ F_j^{(k)}, \quad x \in G_{j-1} \setminus G_j$$

holds. Since

$$\int\limits_{F_j^{(k)}} \frac{d\mu_j^{(k)}(y)}{|x-y|^{n-2}} \le 1, \quad x \in \mathbb{R}^n,$$

(cf. [29, p. 175]), the above estimates imply

$$U = \sum_{j=0}^{\infty} \sum_{k=1}^{L} V_j^{(k)} \le k\Big(1 + \sum_{j=0}^{\infty}(j+1) \sum_{k=1}^{L} \mathrm{cap}\ F_j^{(k)}\Big).$$

We have

$$\int\limits_G (\nabla U)^2\, dx = \int\limits_{\partial G} d\mu(\xi) \int\limits_{\partial G} d\xi(\eta) \int\limits_G \nabla_x N(x,\xi) \nabla_x N(x,y)\, dx,$$

where $\mu = \sum\limits_{j,k} \mu_j^{(k)}$. By the definition of the function N,

$$\int\limits_G \nabla_x N(x,\xi) \nabla_x N(x,\eta)\, dx = N(\xi,\eta) - \int\limits_G \lambda(x) N(x,y)\, dx.$$

Hence

$$\int\limits_G (\nabla U)^2\, dx = \int\limits_{\partial G} U(\xi)\, d\mu(\xi) - \mu(G) \int\limits_G \lambda(x) U(x)\, dx.$$

Since the measure μ is finite and the function U is bounded, U has the finite Dirichlet integral. Besides, by (5.4.3) and (5.4.4), U is a solution of the problem

$$\Delta U(x) = \lambda(x) \sum_{j=0}^{\infty} \sum_{k=1}^{L} \operatorname{cap} F_j^{(k)} \ \text{ in } \ G,$$
$$\frac{\partial U}{\partial \nu} = 0 \ \text{ on } \ \partial G \setminus F.$$

In view of Lemma 5.4.1(iii),

$$V_j^{(k)}(x) = k \int\limits_{F_j^{(k)}} \frac{d\mu_j^{(k)}(y)}{|x-y|^{n-2}} \ \text{ for } \ \rho(x, F_j^{(k)}) < \frac{\varkappa}{2}\,,$$

where ρ stands for the distance. Therefore,

$$V_j^{(k)}(x) \geq k_0 = const > 0 \ \text{ quasi-everywhere in } \ F_j^{(k)}. \tag{5.4.5}$$

If $\rho(x, F_j^{(k)}) \geq \frac{\varkappa}{2}$, then it follows by Lemma 5.4.1 that

$$\left|V_j^{(k)}(x)\right| \leq k(j+1)\mu_j^{(k)}(F_j^{(k)}) \leq k(j+1) \operatorname{cap} F_j^{(k)}. \tag{5.4.6}$$

Let $x \in F_{j_0}^{(k_0)}$. We express $U(x)$ as

$$U(x) = \sum_{j,k}{}' \int\limits_{F_j^{(k)}} N(x,y)\, d\mu_j^{(k)}(y) + \sum_{j,k}{}'' \int\limits_{F_j^{(k)}} N(x,y)\, d\mu_j^{(k)}(y),$$

where the first sum is taken over j and k such that the sets $F_j^{(k)}$ have a nonempty intersection with the $\frac{\varkappa}{2}$-neighborhood $O_{j_0}^{(k_0)}$ of the set $F_{j_0}^{(k_0)}$. We have

$$\sum_{j,k}{}' \int\limits_{F_j^{(k)}} N(x,y)\, d\mu_j^{(k)}(y) =$$
$$= \int\limits_{F_{j_0}^{(k_0)}} N(x,y)\, d\mu_{j_0}^{(k_0)}(y) + \sum_{(j,k)\neq(j_0,k_0)}{}' \int\limits_{F_j^{(k)} \cap O_{j_0}^{(k_0)}} N(x,y)\, d\mu_j^{(k)}(y) +$$
$$+ \sum_{(j,k)\neq(j_0,k_0)} \int\limits_{F_j^{(k)} \cap O_{j_0}^{(k_0)}} N(x,y)\, d\mu_j^{(k)}(y).$$

By (5.4.5), the first integral on the right-hand side dominates k_0 and each of the integrals over $F_j^{(k)} \cap O_{j_0}^{(k_0)}$ is nonnegative. By Lemma 5.4.1, the inequality

$$\Bigg| \int\limits_{F_j^{(k)} \cap O_{j_0}^{(k_0)}} N(x,y)\, d\mu_j^{(k)}(y) \Bigg| \le k \operatorname{cap} F_j^{(k)}$$

holds. In view of (5.4.6), the integral over $F_j^{(k)}$ on $\sum''$ does not exceed $k(j+1) \operatorname{cap} F_j^{(k)}$. Thus,

$$U(x) \ge k - k \sum_{j=0}^{\infty} (j+1) \sum_{k=1}^{L} \operatorname{cap} F^{(k)} \ge \frac{k_0}{2}\,, \quad x \in F_{j_0}^{(k_0)}.$$

Since j_0 and k_0 are arbitrary, it follows that $U(x) \ge \frac{k_0}{2}$ quasi-everywhere on F. Let the point $x' = 0$ be at the distance 1 from $\partial\omega$. Then any point $(0, x_n)$ with $x_n > 1$ has the distance 1 from F. By (5.4.6),

$$\big| U(0,x_n) \big| = \sum_{j,k} \big| V_j^{(k)}(0,x_n) \big| \le k_1 \sum_{j,k} (j+1) \operatorname{cap} F_j^{(k)}.$$

From the very beginning, one may assume that the last sum is less than $\frac{k_0}{4k_1}$. Hence $U(0,x_n) < \frac{k_0}{4}$. Let ξ be an infinitely differentiable function in $\overline{G}$, nonnegative, equal to unity for $x_n \ge 2$ and vanishing for $x_n \le 1$. Since

$$\int\limits_G \nabla U \nabla v \, dx = 0, \quad v \in \overset{\circ}{L}{}_2^1(G;F),$$

it follows that the function $V = (U - \frac{k_0}{2})$ satisfies the equality

$$\int\limits_G \nabla U \nabla v \, dx = f(v) := \int\limits_G \nabla \xi (U \nabla v - v \nabla U)\, dx, \tag{5.4.7}$$

where f is a linear functional on $\overset{\circ}{L}{}_2^1(G;F)$ supported by the set $\{x \in G : 1 \le x_n \le 2\}$. Let S denote a function from the space $\overset{\circ}{L}{}_2^1(G;F)$, satisfying (5.4.7) for all $v \in \overset{\circ}{L}{}_2^1(G;F)$. Since $V - S$ is harmonic in G, satisfies the zero Neumann condition on $\partial G \setminus F$ and nonnegative quasi-everywhere on F, by Lemma 5.2.1 we have $V - S \ge 0$ on G. By the assumption, the point at infinity is regular, hence $S(x) \to 0$ as $x \to \infty$ for $x \in G$. On the other hand, for $x_n > 2$,

$$S(0,x_n) \le V(0,x_n) = U(0,x_n) - \frac{k_0}{2} < -\frac{k_0}{4}\,.$$

This contradiction proves that the point at infinity is irregular.

Proof of Lemma 5.4.1. Let Λ be a solution of the Neumann problem

$$\Delta\Lambda = \lambda \ \text{ in } \ G, \quad \frac{\partial\Lambda}{\partial\nu} = 0 \ \text{ on } \ \partial G,$$

and

$$\Lambda(x) = O(x_n) \ \text{ as } \ x_n \to \infty.$$

Since

$$\int_G \lambda\, dx = 1,$$

it follows that

$$\Lambda(x) = |\omega| x_n + const + O(e^{-\gamma_1 x_n}),$$

where $|\omega|$ is the $(n-1)$-dimensional measure of ω. This known relation can be checked either by the Fourier method or with the help of the Laplace transform in x_n. Let $\Gamma(x,y)$ be the fundamental solution of the Neumann problem in the cylinder $\omega \times \mathbb{R}^1$, i.e., the solution of the problem

$$\begin{aligned} -\Delta_x \Gamma(x,y) &= \delta(x-y), \quad x, y \in \mathbb{R}^1, \\ \frac{\partial\Gamma(x,y)}{\partial\nu_x} &= 0, \quad x \in \partial\omega \times \mathbb{R}^1, \quad y \in \partial\omega \times \mathbb{R}^1, \end{aligned}$$

such that $\Gamma(x,y) = O(x_n)$ for $|x_n| \to \infty$. By the Fourier method we have

$$\Gamma(x', x_n; y', y_n) = \frac{|x_n - y_n|}{2|\omega|} + const + \sum_{k=1}^{\infty} \frac{\varphi_k(x')\varphi_k(y')}{2\gamma_k^2}\, e^{-y_k|x_n - y_n|},$$

where $\{\gamma_k^2\}$ and $\{\varphi_k\}$ are the sequences of positive eigenvalues and orthogonal and normalized eigenvectors of the Laplace operator in ω with zero Neumann condition on $\partial\omega$. The series on the right-hand side converges in some weak sense which we do not specify. Using the well-known estimate

$$|\varphi_k| \le k_0 \gamma_k^M$$

with positive constants k_0 and M, we obtain

$$\left|\Gamma(x,y) - \frac{|x_n - y_n|}{2\omega|} - const\right| \le k e^{-\gamma_1|x-y|} \ \text{ for } \ |x_n - y_n| > \varkappa.$$

The validity of property (iii) for Γ is practically known: the basic fact is that the fundamental solution of the Neumann problem in the half-space is the sum of the fundamental solution of the Laplace operator in $\mathbb{R}^n$ and its reflection in the boundary hyperplane. It remains to note that

$$N(x,y) = \Gamma(x', x_n; y', y_n) + \Gamma(x', -x_n; y', -y_n) - \Lambda(x) + const. \qquad \square$$

5.5 Estimates for the Green Function and for the Harmonic Measure of the Zaremba Problem

In this section we collect some quantitative information on solutions of the Zaremba problem.

Lemma 5.5.1. *Let $f = 0$ on $G\backslash G_\tau$ and let u be the solution of the Zaremba problem. Then*

$$\int\limits_\omega u(x)^2\,dx' \geq \frac{Z(x_n)}{Z(y_n)} \int\limits_\omega u(y)^2\,dy' \;\; \text{for} \;\; \tau > x_n > y_n.$$

Proof. Our argument is close to the one used in the proof of Lemma 5.3.2, therefore, we only outline it briefly. Setting a cut-off function into (5.1.4), similarly to (5.3.3), we obtain

$$\int\limits_{G\backslash G_\sigma} (\nabla u)^2\,dx' = \frac{\mathfrak{F}(\sigma}{2}, \;\; 0 < \sigma < \tau, \;\; \mathfrak{F}(\sigma) = \|u\|^2_{L_2(S_\sigma)}.$$

In the same way as (5.3.5) follows from (5.3.3), we get the inequality

$$\int\limits_\omega dx' \int\limits_0^\sigma \big[u_t^2 + k\mathfrak{E}(t)u^2\big]\,dt \leq \mathfrak{F}'(\sigma), \;\; u_t(x',t) := \frac{\partial u(\varkappa',t)}{\partial t}\,.$$

By what we said at the end of Section 5.2 it follows that the functional

$$\xi \longrightarrow \int\limits_0^\sigma \Big[\Big(\frac{d\xi}{dt}\Big)^2 + k\,\mathfrak{E}(t)\xi^2\Big]\,dt.$$

defined on functions obeying the condition $\xi(\sigma) = u(x',\sigma)$, attains its minimum at the solution

$$t \longrightarrow u(x',\sigma)\,\frac{Z(t)}{Z(\sigma)}$$

of equation (5.3.1), and the value of that minimum is $u(x',\sigma)\,\frac{Z'(\sigma)}{Z(\sigma)}$. This implies the estimate

$$\frac{Z'(\sigma)}{Z(\sigma)}\,\mathfrak{F}(\sigma) \leq \mathfrak{F}'(\sigma).$$

Integrating this inequality, we complete the proof. □

Corollary 5.5.1 (the Fragmen–Lindelöf principle)**.** *If u is a solution of problem* (5.5.1)*, where f is a function with a compact support in G, then either u has the finite Dirichlet integral and*

$$\limsup_{x_n\to\infty} \frac{|u(x)|}{z(x_n)^{\frac{1}{2}}} < \infty \tag{5.5.1}$$

or

$$\liminf_{x_n \to \infty} \frac{\|u(\,\cdot\,, x_n)\|_{L_2(\omega)}}{Z(x_n)^{\frac{1}{2}}} > 0.$$

Proof. Relation (5.5.1) follows directly from Corollary 5.3.1. Let u be a solution of the Zaremba problem with the infinite Dirichlet integral. Let v stand for the solution of the same problem with the finite Dirichlet integral. We apply Lemma 5.5.1 to the difference $u - v$. The result follows due to the fact that $z(x_n) = o(Z(x_n))$, and v satisfies (5.3.2).

Let $y = (y', y_n) \in G$. By the Green function of the Zaremba problem we mean the solution of the problem

$$-\Delta_x g(x,y) = \delta(x-y) \quad \text{for} \quad x \in G,$$
$$\frac{\partial g}{\partial \nu}(x,y) = 0 \quad \text{for} \quad x \in \partial G \setminus F, \quad g(x,y) = 0 \quad \text{for} \quad x \in F$$

with the finite Dirichlet integral outside any neighborhood of the point y. The equation and the Neumann condition on $\partial G \setminus F$ should be understood in the sense of the integral identity

$$\int\limits_G \nabla_x g(x,y) \nabla v(x)\, dx = v(y), \quad v \in C_0^\infty(\overline{G} \setminus F),$$

and the Dirichlet condition on F should be valid quasi-everywhere. Subtracting from g the fundamental solution of the laplace operator , multiplied by a cut-off function supported near y, and using the unique solvability of the Zaremba problem in the class $\overset{\circ}{L}{}_2^1(G; F)$, we conclude that g exists and is unique. Let g_0 be the Green function of the Dirichlet problem in $g - g_0$. Since $\frac{\partial g_0}{\partial \nu} \geq 0$ on ∂G, we may apply Lemma 5.2.1 to the difference $g - g_0$. Hence $g \geq g_0$ on G, and thus $g \geq 0$.

The following assertion contains pointwise estimates of g. □

Proposition 5.5.1. *The Green function of the Zaremba problem admits the following estimates:*

(i) *if* $|x_n - y_n| > 1$*, then*

$$g(x,y) \leq \begin{cases} k\Big(\dfrac{z(x_n)}{z(y_n)}\Big)^{\frac{1}{2}} & \textit{for} \;\; x_n > y_n + 1, \\ k\Big(\dfrac{z(y_n)}{z(x_n)}\Big)^{\frac{1}{2}} & \textit{for} \;\; y_n > x_n + 1. \end{cases}$$

(ii) *if* $|x_n - y_n| \leq 1$*, then*

$$g(x,y) \leq \begin{cases} k|x-y|^{2-n} & \textit{for} \;\; n > 2, \\ k \log\Big(\dfrac{2}{|x-y|}\Big) & \textit{for} \;\; n = 2. \end{cases}$$

Proof. Two last inequalities are well known and we won's give their proof based on Lemma 5.2.1. Two first inequalities follow directly from (ii) and Corollary 5.3.1. □

Remark 5.5.1. Various estimates for u follow from Proposition 5.5.1 and the representation of the problem

$$-\Delta u = f \ \text{ in } \ G, \quad \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \partial G \setminus F, \quad u = 0 \ \text{ on } \ F$$

with the help of the Green function. For example, it is easy to check that

$$|u(x)| \le k\Big(z(x_n)^{\frac{1}{2}} \int\limits_0^{\tau_n} z(t)^{-\frac{1}{2}} F(t)\, dt + z(x_n)^{-\frac{1}{2}} \int\limits_{x_n}^{\infty} z(t)^{\frac{1}{2}} F(t)\, dt \Big)$$

for $|f(x)| \le F(x_n)$.

Let $\overset{\circ}{C}(F)$ be the space of continuous functions vanishing as $|x| \to \infty$ and endowed with the norm

$$\|u\|_{\overset{\circ}{C}(F)} = \sup\big\{ |u(x)| : \ x \in F \big\}.$$

By $\overset{\circ}{C}{}^{\infty}(F)$ we denote the space of traces on F of functions from the space $\overset{\circ}{C}{}^{\infty}_0(\overline{G})$ of functions which are smooth on $\overline{G}$ and have compact support. Consider the boundary value problem

$$\Delta v = 0 \ \text{ in } \ G, \quad \frac{\partial v}{\partial \nu} = 0 \ \text{ on } \ \partial G \setminus F, \quad v = \varphi \ \text{ on } \ F. \tag{5.5.2}$$

Given $\varphi \in \overset{\circ}{C}{}^{\infty}(F)$, this problem is readily reduced to that considered in Section 5.1 and therefore it is uniquely solvable in the class of functions with the finite norm (5.1.2). By Lemma 5.2.1 combined with the inequality $0 \le \varphi \le 1$ on F, one has $0 \le u \le 1$ on $G \setminus F$. Hence the solution of problem (5.5.2) can be represented in the form

$$v(x) = \int\limits_F \varphi(y) H(x, dy), \tag{5.5.3}$$

where $H(x, e)$ is the measure of a set $e \subset F$, and $0 \le H \le 1$. Equality (5.5.3) enables one to extend the inverse operator of problem (5.5.2) onto the space $\overset{\circ}{C}(F)$. The functions from the domain of the resulting extension of the operator (5.5.3) will be called solutions of problem (5.5.2) with continuous Dirichlet data.

Proposition 5.5.2. *If $x_n > s$, then*

$$H(x, F \setminus F_s) \le k \Big(\frac{z(x_n)}{z(s)} \Big)^{\frac{1}{2}}. \tag{5.5.4}$$

Proof. Since $0 \le H \le 1$, it suffices to show that (5.5.4) holds for $x_n > s+2$. Let φ be a function from the space $\overset{\circ}{C}{}^{\infty}(F)$ supported on $F \setminus F_{s+1}$. By Corollary 5.3.1,

$$\Big| \int\limits_F \varphi(y) H(x, dy) \Big| \le k \Big(\frac{z(x_n)}{z(s)} \Big)^{\frac{1}{2}} \max_{F \setminus F_s} |\varphi|,$$

which implies (5.5.4) due to arbitrariness of φ. □

Corollary 5.5.2. *Let $\varphi \in \overset{\circ}{C}(F)$ and let*

$$\gamma(s) = \sup \big\{ |\varphi(x)| : \ x \in F_s \big\}.$$

Then for any solution of problem (5.5.2) *with continuous Dirichlet data the estimate*

$$|v(x)| \le \gamma(x_n) + k z(x_n)^{\frac{1}{2}} \int\limits_0^{x_n} \frac{|d\gamma(s)|}{z(s)^{\frac{1}{2}}}, \quad x \in \overline{G} \setminus F,$$

holds.

Proof. By (5.5.3),

$$|v(x)| \le \int\limits_F \gamma(y_n) H(x, dy) \le \gamma(x_n) + \int\limits_F \big[\gamma(y_n) - \gamma(x_n)\big]_+ H(x, dy),$$

where ω_+ stands for the positive part of ω. The last integral can be written as

$$\int\limits_0^{x_n} \big[\gamma(s) - \gamma(x_n)\big] \, dH(x, F \setminus F_s).$$

Therefore,

$$|v(x) \le \gamma(x_n) - \int\limits_0^{x_n} H(x, F \setminus F_s) \, d\gamma(s).$$

It remains to apply inequality (5.5.4). □

Chapter 6

Behavior, Near the Boundary, of Solutions of the Dirichlet Problem for a Second-Order Elliptic Equation

6.1 Operator with Measurable Bounded Coefficients

In the present section estimates near the boundary point and at infinity are obtained for the solution of the Dirichlet problem, Green's function, and the $\mathscr{L}$-harmonic measure for an elliptic operator

$$\mathscr{L}u = (a^{ij}u_{x^i})_{x^j} \quad (a^{ij} = a^{ji};\ \ i, j = 1, 2, \dots, n).$$

The coefficients a^{ij} given in $\mathbb{R}^n$ $(n > 2)$ are measurable and satisfy the condition

$$\lambda\xi^2 \le a^{ij}\xi_i\xi_j \le \lambda^{-1}\xi^2, \tag{6.1.1}$$

where ξ is an arbitrary real vector in $\mathbb{R}^n$, and $\lambda = \mathit{const} \le 1$.

6.1.1 Notation and lemmas

We shall utilize the following notation: Ω is an open subset of $\mathbb{R}^n$; ∂E and CE are the boundary and complement of an arbitrary set $E \subset \mathbb{R}^n$; f_+, f_- are positive and negative parts of the function or charge f; $S_r = \{x :\ |x| \le r\}$, $C_r = S_r \cap C\Omega$; (r, ω) are spherical coordinates with center at the point $O \in \partial\Omega$; c is an constant depending only on λ and n.

Let $\Gamma(x)$ be the fundamental solution of the operator $\mathscr{L}$ in $\mathbb{R}^n$ with a singularity at the point O; $\rho(x) = [\Gamma(x)]^{(2-n)^{-1}}$, $T_r = \{x :\ \rho(x) \le r\}$. As has been shown in [32, 63], there exists a constant α depending only on λ and n such that in $\mathbb{R}^n$,

$$2\alpha|x| \le \rho(x) \le (2\alpha)^{-1}|x|, \tag{6.1.2}$$

which is equivalent to the imbedding $S_{2\alpha r} \subset T_r \subset S_{(2\alpha)^{-1} r}$.

Let us introduce additional notation:

$$K_{r_1,r_2} = S_{r_1} \setminus S_{r_2}, \quad Q_{r_1,r_2} = T_{r_1} \setminus T_{r_2},$$
$$\mathscr{M}_r(u) = r^{-n} \int\limits_{K_{a^{-1}r,ar}} u^2\, dx, \tag{6.1.3}$$

$\operatorname{cap}(E)$ is the harmonic capacity of the set E, $\gamma(r) = r^{2-n} \operatorname{cap}(C_r)$ is the relative capacity of $C\Omega$ in the sphere S_r.

In order not to complicate the exposition, we consider the coefficients a^{ij} and the boundary Ω, infinitely differentiable during the proofs. However, since the constants in all the estimates are independent of this assumption, by utilizing more or less standard approximation techniques all the fundamental results (Theorems 6.1.3–6.1.6) may be carried over to the general case. The restriction $n > 2$ is introduced for simplicity of presentation. Let us just note that the method applies below is applicable also to general second order elliptic equations with divergent principal part.

In this section, u denotes a function from the space $L_2^{(1)}(S_\delta)$ ($\delta = const > 0$) which satisfies the equation $\mathscr{L}u = 0$ in $\Omega \cap S_\delta$ and is zero on C_δ.

Lemma 6.1.1. *Let*

$$\mathscr{J}(r) \equiv (2-n)^{-1} \int\limits_{\partial T_r} u^2 a^{ij} \Gamma_{x^i} n_j\, ds_x, \tag{6.1.4}$$

where $r < \delta$ and $\{n_j\}$ are projections of the unit exterior normal to ∂T_r onto the coordinate axes. Then

$$2r^{1-n} \int\limits_{T_r} a^{ij} u_{x^i} u_{x^j}\, dx = \mathscr{J}'(r). \tag{6.1.5}$$

Proof. Let us set

$$t = r^{2-n}.$$

Then

$$2\int\limits_{\Omega} (\Gamma - t)_+ a^{ij} u_{x^i} u_{x^j}\, dx = \int\limits_{\Omega} (\Gamma - t)_+ \mathscr{L}(u^2)\, dx =$$
$$= -\int\limits_{T_r} a^{ij} \Gamma_{x^i} (u^2)_{x^j}\, dx = -\int\limits_{\partial T_r} u^2 a^{ij} \Gamma_{x^i} n_j\, ds_x.$$

Differentiating with respect to r, we obtain (6.1.5). □

Lemma 6.1.2. *For $\alpha r < \delta$ the inequality $\mathscr{J}(r) \le c\mathscr{M}_r(u)$ is valid.*

Proof. Let us note that on ∂T_r

$$a^{ij}\Gamma_{x^i}n_j = -a^{ij}n_i n_j|\nabla\Gamma| \le 0$$

and that

$$\int\limits_{\partial T_r} a^{ij}\Gamma_{x^i}n_j\, ds_x = -1.$$

Now, the required estimate follows from (6.1.2) and the inequality

$$\max_{K_{(2\alpha)^{-1}r,2\alpha r}} u^2 \le c\mathscr{M}_r(u), \tag{6.1.6}$$

which is substantially due to Moser [57]. □

Lemma 6.1.3. *The inequality*

$$\mathscr{I}(r) \le c\mathscr{I}(R)\exp\Big(-c\int\limits_r^R \gamma(\tau)\,\frac{d\tau}{\tau}\Big) \tag{6.1.7}$$

is valid for $r < R < \delta$.

Proof. By virtue of Lemma 6.1.1 and the estimate (6.1.2),

$$\mathscr{I}'(r) \ge cr^{1-n}\int\limits_{T_r}(\nabla u)^2\,dx \ge cr^{1-n}\int\limits_{B_{\alpha r}}(\nabla u)^2\,dx. \tag{6.1.8}$$

Multiplying the inequality

$$c\,\mathrm{cap}\,(C_r)\int\limits_{\partial S_r} u^2\,d\omega \le \int\limits_{S_r}(\nabla u)^2\,dx, \tag{6.1.9}$$

proved in [37] (see also [38, p. 48]) by r^{n-1} and integrating between $\alpha^3 r$ and αr, we obtain

$$c\,\mathrm{cap}\,(C_{\alpha^3 r})\int\limits_{K_{\alpha r,\alpha^3 r}} u^2\,dx \le r^n\int\limits_{B_{\alpha r}}(\nabla u)^2\,dx,$$

which, together with Lemma 6.1.2 and the estimate (6.1.8), yields

$$\mathscr{I}'(r) \ge cr^{1-m}\,\mathrm{cap}\,(C_{\alpha^3 r})\mathscr{I}(\alpha^2 r).$$

Integrating between αr and r and using the monotonicity of $\mathscr{I}(r)$ (Lemma 6.1.1), we obtain

$$\mathscr{I}(r) \ge \mathscr{I}(\alpha r) + c\mathscr{I}(\alpha^3 r)\int\limits_{\alpha r}^{r}\mathrm{cap}\,(C_{\alpha^3\tau})\,\frac{d\tau}{\tau^{n-1}} \ge$$

$$\ge \mathscr{I}(\alpha^3 r)\Big[1 + c\int\limits_{\alpha r}^{r}\mathrm{cap}\,(C_{\alpha^3\tau})\frac{d\tau}{\tau^{n-1}}\Big].$$

Let us put $r = r_k = \alpha^{3k} R$ $(k = 0, 1, \dots)$ here. Then there is a constant c, such that

$$\mathscr{I}(r_k) \geq \mathscr{I}(r_{k+1}) \exp\left(c \int_{r_{k+2}}^{r_{k+1}} \gamma(\tau) \frac{d\tau}{\tau}\right).$$

Therefore, for any $k \geq 1$,

$$\mathscr{I}(R) \geq \mathscr{I}(r_k) \exp\left(c \int_{r_{k+1}}^{\alpha^3 R} \gamma(\tau) \frac{d\tau}{\tau}\right).$$

Hence, we obtain (6.1.7) by the estimate $\gamma(\tau) \leq 1$ and the monotonicity of $\mathscr{I}(r)$. □

Lemma 6.1.4. *Let $R < \delta$ and $r \leq \alpha^2 R$, where α is the constant from* (6.1.2). *Then the inequality*

$$\int_{S_r} (\nabla u)^2 \, dx \leq c \mathscr{I}(R) r^{n-2} \exp\left(-c \int_r^R \gamma(\tau) \frac{d\tau}{\tau}\right) \tag{6.1.10}$$

is valid.

Proof. By (6.1.5) and (6.1.2), we obtain

$$\mathscr{I}(r) \geq c \int_{\alpha r}^{r} \tau^{1-n} \, d\tau \int_{S_{\alpha\tau}} (\nabla u)^2 \, dx \geq c r^{2-n} \int_{S_{\alpha^3 r}} (\nabla u)^2 \, dx.$$

Now (6.1.10) follows from the inequality (6.1.7). □

6.1.2 Estimates of the "decreasing" solution

Theorem 6.1.1. *Let the function $u \in L_2^{(1)}(S_\delta)$ satisfy the equation $\mathscr{L} u = 0$ in $\Omega \cap S_\delta$ and be equal zero on C_δ. Then for $R < \alpha\delta$ and $r < \alpha^5 R$, the estimate*

$$\max_{S_r} |u| \leq c \mathscr{M}_R^{\frac{1}{2}}(u) \exp\left(-c \int_r^R \gamma(\tau) \frac{d\tau}{\tau}\right) \tag{6.1.11}$$

is valid.

Proof. Applying the formula of Kronrod (see [70, 14]):

$$\int_\Omega \Phi(x) |\nabla u| \, dx = \int_{-\infty}^{+\infty} dt \int_{u=t} \Phi(\tau) \, ds_x,$$

where $\Phi(x)$ is a Borel-measurable function, and the function $u(x)$ satisfies the Lipschitz condition, we obtain

$$A \equiv \int\limits_{Q_{\alpha^{-2}r,\alpha^2 r}} u^2 a^{ij}\Gamma_{x^i}\Gamma_{x^j}\,d\tau = \int\limits_{\alpha^2 r}^{\alpha^{-2}r} \mathscr{J}(\tau)\tau^{1-n}\,d\tau.$$

Applying Lemma 6.1.3, we hence deduce

$$A \le c\mathscr{J}(R)r^{2-n}\exp\Big(-c\int\limits_r^R \gamma(\tau)\,\frac{d\tau}{\tau}\Big).$$

According to Lemma 6.1.4, the same estimate is true for the integral

$$B \equiv \int\limits_{Q_{\alpha^{-2}r,\alpha^2 r}} \big[\Gamma-(\alpha^{-2}r)^{2-n}\big]^2 a^{ij}u_{x^i}u_{x^j}\,dx.$$

Hence, by setting $v = u[\Gamma-(\alpha^{-2}r)^{2-n}]_+$, we obtain

$$C \equiv \int\limits_{CT_{\alpha 1}} a^{ij}v_{x^i}v_{x^j}\,dx \le 2(A+B) \le$$

$$\le cr^{2-n}\mathscr{J}(R)\exp\Big(-c\int\limits_r^R \gamma(\tau)\,\frac{d\tau}{\tau}\Big). \tag{6.1.12}$$

On the other hand, since $v=0$ outside $S_{\alpha^{-3}r}$ it follows that

$$C \ge c\int\limits_{K_{\alpha^{-3}r,\alpha r}} (\nabla v)^2\,dx \ge cr^{-2}\int\limits_{K_{\alpha^{-3}r,\alpha r}} v^2\,dx \ge c\,r^{2-n}\mathscr{M}_r(u). \tag{6.1.13}$$

By the maximum principle and the inequality (6.1.6), it follows from (6.1.12) and (6.1.13) that

$$\max_{S_{\alpha r}} u^2 \le \max_{\partial T_r} u^2 \le c\mathscr{M}_r(u) \le c\mathscr{J}(R)\exp\Big(-c\int\limits_r^R \gamma(\tau)\,\frac{d\tau}{\tau}\Big). \tag{6.1.14}$$

Finally, let us note that according to Lemma 6.1.2, the inequality $\mathscr{J}(R) \le c\mathscr{M}_R(u)$ is valid and that together with (6.1.14) it proves the theorem. □

An estimate of the decrease of a solution with finite energy at infinity is given in the following theorem.

Theorem 6.1.2. *Let the function $u \in L_2^{(1)}(CS_\delta)$ satisfy the equation $\mathscr{L}u = 0$ in $\Omega \cap CS_\delta$ and be equal zero on $C\omega \cap CS_\delta$. Then for $r > \alpha^{-1}\delta$, $R > \alpha^{-5}r$, the estimate*

$$\max_{\Omega \setminus S_R} |u| \le c\mathscr{M}_r^{\frac{1}{2}}(u)\Big(\frac{r}{R}\Big)^{n-2} \exp\bigg(-c\int_r^R \gamma(\tau)\,\frac{d\tau}{\tau}\bigg) \tag{6.1.15}$$

is valid.

Proof. Let E^* denote the image of the set E under the inversion $x = y|y|^{-2}$. If u is a solution of the equation $\mathscr{L}u = 0$ in $\omega \cap CS_\delta$, then, as has been shown in [64], the function

$$v(x) = \frac{u(y)}{\Gamma(y)} \tag{6.1.16}$$

satisfies some uniformly elliptic equation $\mathscr{N}v = 0$ in $\Omega^* \cap S_\delta$ with an ellipticity constant depending only on λ. Moreover, from the proof presented in [64], it immediately follows that the Kelvin transform (6.1.16) retains the finiteness of the energy of the solution, i.e., that $v \in L_2^{(1)}(S_{\delta^{-1}})$. According to Theorem 6.1.1, the estimate

$$\max_{\Omega^* \cap S_{R^{-1}}} v^2 \le c\mathscr{M}_{r^{-1}}(v) \exp\bigg(-c\int_{R^{-1}}^{r^{-1}} \gamma^*(\tau)\,\frac{d\tau}{\tau}\bigg),$$

where $\gamma^*(\tau) = \tau^{2-n}\,\mathrm{cap}\,(C\Omega^* \cap S_\tau)$, is true for the function $v(y)$. Hence, from (6.1.2) we obtain

$$\max_{\Omega \cap S_R} |u| \le c\mathscr{M}_r^{\frac{1}{2}}(u)\Big(\frac{r}{R}\Big)^{n-2} \exp\bigg(-c\int_{R^{-1}}^{r^{-1}} \gamma^*(\tau)\,\frac{d\tau}{\tau}\bigg). \tag{6.1.17}$$

Let us set $\nu = [\log_2 R]$, $\mu = [\log_2 r]$. Then

$$\int_{R^{-1}}^{r^{-1}} \gamma^*(\tau)\,\frac{d\tau}{\tau} \ge \sum_{k=\mu+1}^{\nu} \int_{2^{-k-1}}^{2^{-k}} \gamma^*(\tau)\,\frac{d\tau}{\tau} \ge c\sum_{k=\mu+1}^{\nu} 2^{k(n-2)}\,\mathrm{cap}\,(E_k^*),$$

where $E_k = C_{2^{k+2}} \setminus C_{2^{k+1}}$. But, as is known (see [29, p. 353]),

$$2^{-2(k+2)(n-2)}\,\mathrm{cap}\,(E_k) \le \mathrm{cap}\,(E_k^*) \le 2^{-2(k+1)(n-2)}\,\mathrm{cap}\,(E_k).$$

Therefore,

$$\int_{R^{-1}}^{r^{-1}} \gamma^*(\tau)\,\frac{d\tau}{\tau} \ge c\sum_{k=\mu+1}^{\nu} 2^{-k(n-2)}\,\mathrm{cap}\,(E_k).$$

Furthermore, using the semi-additivity of the capacities, we obtain

$$\int\limits_{R^{-1}}^{r^{-1}} \gamma^*(\tau)\,\frac{d\tau}{\tau} \geq c\Big[\sum_{k=\mu+1}^{\nu} \gamma(2^{k+2}) - 2^{2-n}\sum_{k=\mu+1}^{\nu}\gamma(2^{k+1})\Big] \geq$$

$$\geq (1-2^{2-n})\sum_{k=\mu+2}^{\nu}\gamma(2^{k+1}) - \gamma(2^{k+2}) \geq c\int\limits_{r}^{R}\gamma(\tau)\,\frac{d\tau}{\tau} - c.$$

Hence, we obtain the inequality (6.1.15) from (6.1.17). □

6.1.3 Estimates of the "growing" solution and the Phragmen–Lindelöf principle

Theorem 6.1.3. *For all $\delta > 0$, let the function $u \in L_2^{(1)}(CS_\delta)$ satisfy the equation $\mathscr{L}u = 0$ in $\Omega \cap CS_\delta$ and equal zero on $C\Omega \cap CS_\delta$. Then for $r < \alpha^5 R$, the estimate*

$$\mathscr{M}_r^{\frac{1}{2}}(u) \geq c \max_{\Omega\setminus S_R} |u| \Big(\frac{R}{r}\Big)^{n-2} \exp\Bigg(c\int\limits_{r}^{R}\gamma(\tau)\,\frac{d\tau}{\tau}\Bigg) \tag{6.1.18}$$

is valid.

This inequality follows directly from Theorem 6.1.2. Analogously, from Theorem 6.1.1 we obtain the following assertion on the behavior of the growing solution at infinity.

Theorem 6.1.4. *For all $\delta > 0$, let the function $u \in L_2^{(1)}(S_\delta)$ satisfy the equation $\mathscr{L}u = 0$ in $\Omega \cap S_\delta$ and be equal zero on C_δ. Then for $R > \alpha^{-5}r$, the estimate*

$$\mathscr{M}_R^{\frac{1}{2}}(u) \geq c \max_{S_R} |u| \exp\Bigg(c\int\limits_{r}^{R}\gamma(\tau)\,\frac{d\tau}{\tau}\Bigg) \tag{6.1.19}$$

is valid.

From Theorems 6.1.1 and 6.1.4 we obtain the following modification of the Phragmen–Lindelöf principle (compare with [27]).

Corollary 6.1.1. *Let u be the solution of the equation $\mathscr{L}u = 0$ which equals zero on the portion of $\partial\Omega$ located outside some sphere and belonging* *to* $L_2^{(1)}(S_R \cap \Omega)$ *for any* $R < \infty$. *Then for any* $r > 0$ *one of the inequalities*

$$\liminf_{R\to\infty} \mathscr{M}_R^{\frac{1}{2}}(u) \exp\Big(- c \int\limits_r^R \gamma(\tau)\, \frac{d\tau}{\tau}\Big) > 0, \tag{6.1.20}$$

$$\limsup_{R\to\infty} \max_{S_R} |u| R^{n-2} \exp\Big(c \int\limits_r^R \gamma(\tau)\, \frac{d\tau}{\tau}\Big) < \infty \tag{6.1.21}$$

is satisfied.

It follows from Theorems 6.1.2 and 6.1.3 that an analogous alternative characterizes the behavior, near the point O, of the solution of the equation $\mathscr{L}u = 0$, which equals zero on $S_\delta \cap \partial\Omega$ for some δ.

6.1.4 Inhomogeneous boundary condition

Theorem 6.1.5. *Let* $\varphi \in C(\partial\Omega)$, *and let* u *be the solution of the equation* $\mathscr{L}u = 0$, *which satisfies the condition* $u = \varphi$ *on* $\partial\Omega$ (*see* [32]). *In addition, by definition, let* $\beta = \alpha^{-8}$, *and*

$$\omega^{\pm}(t) = \max_{|x|\le t} \big[\varphi(x) - \varphi(0)\big]_{\pm}. \tag{6.1.22}$$

Then the inequality

$$\big[u(x) - \varphi(0)\big]_{\pm} \le \omega^{\pm}\big(\beta|x|\big) + c \int\limits_{\beta|x|}^{\infty} \exp\Big(- c \int\limits_{|x|}^{t} \gamma(\tau)\, \frac{d\tau}{\tau}\Big)\, d\omega^{\pm}(t) \tag{6.1.23}$$

is valid.

Proof. Let $H(x, E)$ be the $\mathscr{L}$-harmonic measure of the set $E \subset \partial\Omega$ with respect to Ω. Then

$$u(x) - \varphi(0) = \int\limits_{\partial\Omega} \big[\varphi(y) - \varphi(0)\big] H(x, dy).$$

Obviously,

$$\big[u(x) - \varphi(0)\big]_{\pm} \le \int\limits_{\partial\Omega} \omega^{\pm}(|y|) H(x, dy) \le$$

$$\le \omega^{\pm}\big(\beta|x|\big) + \int\limits_{\partial\Omega} \big[\omega^{\pm}(|y|) - \omega^{\pm}\big(\beta|x|\big)\big]_{+} H(x, dy).$$

We hence obtain

$$\big[u(x)-\varphi(0)\big]_{\pm} \le \omega^{\pm}\big(\beta|x|\big) + c\int\limits_{\beta|x|}^{\infty} H(x,\partial\Omega\cap CS_t)\,d\omega^{\pm}(t). \tag{6.1.24}$$

Now, let us note that for fixed t the function $H(x,\partial\Omega\cap CS_t)$ satisfies the equation $\mathscr{L}u=0$ for $|x|<t$ and the zero boundary condition. Hence, the estimate

$$H(x,\partial\Omega\cap CS_t) \le c\exp\Big\{-c\int\limits_{|x|}^{t}\gamma(\tau)\,\frac{d\tau}{\tau}\Big\}, \tag{6.1.25}$$

where $\beta|x|<t$, follows from Theorem 6.1.1 and the inequality $H(x,E)\le 1$. There remains to substitute this inequality into (6.1.24). □

Remark 6.1.1. Meanwhile, the sufficient Wiener's condition

$$\int\limits_{0}^{1}\gamma(\tau)\,\frac{d\tau}{\tau}=\infty \tag{6.1.26}$$

for the regularity of the point O (see [32]) follows from (6.1.25) and (6.1.23). In fact, if the integral (6.1.26) diverges, then for any $\varepsilon>0$,

$$\limsup_{x\to 0}\big[u(x)-\varphi(0)\big]_{\pm} \le$$

$$\le c\limsup_{x\to 0}\int\limits_{\beta|x|}^{1}\exp\Big(-c\int\limits_{|x|}^{t}\gamma(\tau)\,\frac{d\tau}{\tau}\Big)\,d\omega^{\pm}(t)\le c\omega^{\pm}(\varepsilon).$$

Therefore, $u(x)\to\varphi(0)$ as $x\to 0$.

It also follows from Theorem 6.1.5 that the solution of the equation $\mathscr{L}u=0$ whose boundary values satisfy the Hölder condition at the point O, itself satisfies this condition if

$$\liminf_{x\to 0}\frac{1}{|\ln r|}\int\limits_{t}^{1}\gamma(\tau)\,\frac{d\tau}{\tau}>0. \tag{6.1.27}$$

The following theorem is proved analogously to Theorem 6.1.5.

Theorem 5′. *Let $\varphi\in C(\delta\Omega)$ and let u be a solution of the equation $\mathscr{L}u=0$ satisfying the condition $u=\varphi$ on $\partial\Omega$. Besides, let*

$$\sigma^{\pm}(t)\equiv\max_{|x|\ge t}\varphi_{\pm}(x)\underset{t\to\infty}{\longrightarrow}0.$$

Then the inequality

$$u_\pm(x) \le \sigma^\pm\big(\beta^{-1}|x|\big) + \\ + c \int\limits_0^{\beta^{-1}|x|} \Big(\frac{t}{|x|}\Big)^{n-2} \exp\Bigg(-c\int\limits_t^{|x|} \gamma(\tau)\,\frac{d\tau}{\tau}\Bigg)\, d\sigma^\pm(t) \quad (6.1.28)$$

is valid.

The estimate of the $\mathscr{L}$-harmonic measure

$$H(x, \partial\Omega \cap S_t) \le c\Big(\frac{t}{|x|}\Big)^{n-2} \exp\Bigg(-c\int\limits_t^{|x|} \gamma(\tau)\,\frac{d\tau}{\tau}\Bigg) \quad (6.1.29)$$

resulting from Theorem 6.1.2, where $\beta t \le |x|$ plays the part of the inequality (6.1.25) in the proof of this theorem.

6.1.5 Inhomogeneous equation

Theorem 6.1.6. *Let u, which equals zero on $\partial\Omega$, be a weak solution of the equation $\mathscr{L}u = f$, where f is a finite charge with a carrier in Ω (the existence of such a solution is proved in* [32]*). Then the inequality*

$$u_\mp(x) \le c \int\limits_{S_{\beta|x|}} r_{xy}^{2-n} f_\pm(dy) + \\ + c\int\limits_{\beta|x|}^{\infty} \exp\Bigg(-c\int\limits_{|x|}^{t} \gamma(\tau)\,\frac{d\tau}{\tau}\Bigg) t^{2-n}\, df_\pm(S_t) \quad (6.1.30)$$

is valid.

Proof. From the representation of the solution in terms of the Green's function $G(x, y)$ and from the inequality $G(x, y) \le c r_{xy}^{2-n}$ resulting from (6.1.2), we obtain

$$u_\mp(x) \le c \int\limits_{S_{\beta|x|}} r_{xy}^{2-n} f_\pm(dy) + \int\limits_{CS_{\beta|x|}} G(x, y) f_\pm(dy). \quad (6.1.31)$$

Since for fixed y, the function $G(x, y)$ satisfies the conditions of Theorem 6.1.1 for $|x| \le \beta|y|$, it follows that

$$G(x, y) \le c\mathscr{M}^{\frac{1}{2}}_{\alpha|y|}\big(G(\,\cdot\,, y)\big) \exp\Bigg(-c\int\limits_{|x|}^{|y|} \gamma(\tau)\,\frac{d\tau}{\tau}\Bigg).$$

Applying (6.1.2) to the Green's function, we find that the mean value of $G^2(\,\cdot\,,y)$ does not exceed $c|y|^{2(2-n)}$ on $K_{\alpha^2|y|}$. Therefore, for $\beta|x| \le |y|$,

$$G(x,y) \le c|y|^{2-n} \exp\left(-c \int\limits_{|x|}^{|y|} \gamma(\tau)\, \frac{d\tau}{\tau} \right), \tag{6.1.32}$$

which, together with (6.1.31), proves the theorem. □

The following estimate of the solution of the problem $\mathscr{L}u = f$, $u\big|_{\partial\Omega} = 0$, at infinity is obtained analogously:

$$\begin{aligned} u_{\mp}(x) \le c \int\limits_{CS_{\beta^{-1}|x|}} r_{xy}^{2-n} f_{\pm}(dy) + \\ + c|x|^{2-n} \int\limits_{0}^{\beta^{-1}|x|} \exp\left(-c \int\limits_{t}^{|x|} \gamma(\tau)\, \frac{d\tau}{\tau} \right) df_{\pm}(S_t). \end{aligned} \tag{6.1.33}$$

By comparing (6.1.23) and (6.1.30), as well as (6.1.28) and (6.1.33), we obtain estimates near the point O and at infinity for the solution of the problem $\mathscr{L}u = f$, $u\big|_{\partial\Omega} = \varphi$.

6.2 Modulus of Continuity of a Harmonic Function at a Boundary Point

In this section, the results obtained in Section 6.1 are improved for harmonic functions.

Let $n > 2$, $y \in \mathbb{R}^n$, $\mathscr{B}_R(y) = \{x \in \mathbb{R}^n : \ |x-y| < R\}$, $\mathscr{B}_R = \mathscr{B}_R(0)$ and Ω be a bounded domain in $\mathbb{R}^n$. By c, c_1, c_2 we denote possibly different positive constants which depend only on n. Further, let F be a closed subset of the ball $\mathscr{B}_R$ and u be a function from the Sobolev space $W_2^1(\mathscr{B}_R)$, harmonic on $\mathscr{B}_R \setminus F$ and equal to zero almost everywhere on F.

According to Section 6.1.1, for all ρ, $\rho \in (0,R)$, one has

$$\int\limits_{\partial\mathscr{B}_1} u^2(\rho,\omega)\, d\omega \le \exp\left(-c \int\limits_{\rho}^{R} \mathrm{cap}\,(F_r)\, \frac{dr}{r^{n-1}} \right) \int\limits_{\partial\mathscr{B}_1} u^2(R,\omega)\, d\omega, \tag{6.2.1}$$

where $F_r = F \cap \mathscr{B}_r$, cap is the Wiener capacity, $c = \frac{n-2}{n-1}$, and $d\omega$ is the area element of the boundary $\partial\mathscr{B}_1$. Estimates of this type have also been proved for solutions of linear elliptic second order equations with variable coefficients ([28, 38, 39, 61], etc), and also for certain linear equations of order higher than two [36, 51], and quasi-linear second order equations [40, 45]. From (6.2.1) one derives pointwise estimates for the modulus of the function

u, harmonic measure, and Green's function. A consequence of (6.2.1) is the sufficiency of the divergence of the Wiener series for the regularity of the boundary point. The following sufficient condition for Hölder continuity at the point $O \in \partial\Omega$ of the solution of the Dirichlet problem

$$\Delta v = 0 \text{ in } \Omega, \quad v = \phi \text{ on } \partial\Omega \tag{6.2.2}$$

with the function ϕ Hölder continuous at the point O also follows from (6.2.1):

$$\liminf_{\rho \in 0} \frac{1}{|\log \rho|} \int_{\rho}^{1} \operatorname{cap}(\mathscr{B}_r \setminus \Omega) \frac{dr}{r^{n-1}} > 0. \tag{6.2.3}$$

Under the additional requirement of decrease of the central projection of the set $\partial\mathscr{B}_r \setminus \Omega$ onto the sphere $\partial\mathscr{B}_r$ as $r \downarrow 0$ the condition (6.2.3) is equivalent to the inequality $\operatorname{cap}(\mathscr{B}_r \setminus \Omega) \geq c r^{n-2}$; according to [38], it is also necessary. For a rather long time it has been unclear about the question of necessity of (6.2.3) in general (cf. [45]). A negative answer to this question follows from the theorem proved below, which strengthens (6.2.1).

In the formulation of the theorem there occurs the function $r \longmapsto \delta(F_r, \mathscr{B}_r)$ defined as the infimum of those δ, $\delta > 0$, such that for all balls $\mathscr{B}_\delta(y)$ with centers on $\partial\mathscr{B}_r \setminus F$ one has

$$\operatorname{cap}\big(F_r \cap \mathscr{B}_\delta(y)\big) \geq \gamma \delta^{n-2}, \tag{6.2.4}$$

where γ is a small positive constant, depending only on n.

If $\operatorname{cap}(F_r) \geq \gamma(2r)^{n-2}$, then by definition, F_r is an essential subset of $\mathscr{B}_r$, and otherwise an inessential one. Since for essential F_r, (6.2.4) holds for all balls $\mathscr{B}_{2r}(y)$, $y \in \partial\mathscr{B}_r$, one has $\delta(F_r, \mathscr{B}_r) \leq 2r$.

Theorem 6.2.1. *For all ρ, $\rho \in (0, R)$, one has*

$$\begin{aligned} \int_{\partial\mathscr{B}_1} u^2(\rho, \omega)\, d\omega \leq \\ \leq \exp\bigg\{ -c\Big(\int_{I(\rho,R)} \operatorname{cap}(F_r) \frac{dr}{r^{n-1}} + \int_{E(\rho,R)} \frac{dr}{\delta(F_r, \mathscr{B}_r)} \Big) \bigg\} \times \\ \times \int_{\partial\mathscr{B}_1} u^2(R, \omega)\, d\omega, \end{aligned} \tag{6.2.5}$$

where $E(\rho, R) = \{r \in [\rho, R] : \ F_r$ is an essential subset of $\mathscr{B}_r\}$, and $I(\rho, R) = [\rho, R] \setminus E(\rho, R)$.

Remark 6.2.1. The second integral in the exponential in (6.2.5) makes sense, since the sets $E_a = \{r > 0 : \ \delta(r) > a\}$ are of type F_σ for all $a \geq 0$, i.e., the function $r \longmapsto \delta(r)$ is measurable. Indeed, fix $a \geq 0$ and let $r \in E_a$. Then there exist $\delta_2 > \delta_1 > a$ and $x \in \partial\mathscr{B}(0,r)$ such that

$$\text{cap}\,\big(\Omega^c \cap \mathscr{B}(0,r) \cap \mathscr{B}(x,\delta_2)\big) < k_0\delta_1^{n-2},$$

where Ω^c is the complement of Ω. For $|x-y| < \varepsilon = \delta_2 - \delta_1$ and $|y| \leq r$, this yields

$$\text{cap}\,\big(\Omega^c \cap \mathscr{B}\big(0,|y|\big) \cap \mathscr{B}(y,\delta_1)\big) < k_0\delta_1^{n-2},$$

i.e. $\delta(|y|) \geq \delta_1 > a$. Consequently, $\rho \in E_a$ for all $r - \varepsilon < \rho \leq r$.

For $r \in E_a$, let ε_r be the largest ε such that the set E_a contains the interval $(r-\varepsilon, r]$. The set

$$G = \bigcup_{r \in E_a} (r - \varepsilon_r, r) \subset E_a$$

is open and it is easily verified that the set $E_a \setminus G$ is at most countable. It follows that E_a is of type F_σ.

In the proof of theorem we have used the following assertion which contains bilateral estimates of the quantity

$$\lambda(r) = \inf \|\,\text{grad}\, u\|^2_{L_2(\mathscr{B}_r)} \|u\|^{-2}_{L_2(\partial\mathscr{B}_r)},$$

where the infimum is taken over all $u \in W_2^1(\mathscr{B}_r)$ which vanish almost everywhere on $F_r = F \cap \mathscr{B}_r$ (cf. [41]).

In what follows, the relation $a \sim b$ means that $a_1 a \leq b \leq c_2 a$.

Proposition 6.2.1. *If F_r is an inessential subset of $\mathscr{B}_r$, then $\lambda(r) \sim \text{cap}\,(F_r) r^{1-n}$, and if not, $\lambda(r) \sim \frac{1}{\delta(F_r, \mathscr{B}_r)}$.*

Proof. 1. The inequality

$$\lambda(r) \geq \frac{1}{2}\,(n-2)(n-1) r^{1-n}\,\text{cap}\,(F_r)$$

is proved in [37]. Let $\text{cap}\,(F_r) < \gamma(2r)^{n-2}$. We denote by w the capacity potential of the set F_r. We have

$$\lambda(r)\|1-w\|^2_{L_2(\partial\mathscr{B}_r)} \leq \int_{\mathscr{B}_r} (\text{grad}\, w)^2\, dx.$$

Consequently,

$$\lambda(r)\Big(\omega_n^{\frac{1}{2}} r^{\frac{n-1}{2}} - \|w\|_{L_2(\partial\mathscr{B}_r)}\Big) \leq \omega_n (n-2)\,\text{cap}\,(F_r),$$

where ω_n is the area of $\partial\mathscr{B}_1$. Since

$$\|w\|_{L_2(\partial\mathscr{B}_r)} \leq cr\|\,\text{grad}\,\omega\|_{L_2(\mathbb{R}^n)} \leq c_1 r (\text{cap}\ F_r)^{\frac{1}{2}},$$

one has $\lambda(r)r^{1-n}(1-c\gamma^{\frac{1}{2}}) \le c\,\mathrm{cap}\,F_r$.

2. Let cap $F_r \ge \gamma(2r)^{n-2}$. We construct a finite covering of the set $\partial\mathscr{B}_r \setminus F$ by balls $\mathscr{B}_\delta(y_i)$, $y_i \in \partial\mathscr{B}_r \setminus F$, where $\delta = \delta(F_r, \mathscr{B}_r) + \varepsilon$, $\varepsilon > 0$. It follows from the definition of $\delta(F_r, \mathscr{B}_r)$ that one can find a sufficiently small number ε such that $\mathrm{cap}\,(F_r \cap \mathscr{B}_\delta(y_i)) \ge \gamma\delta^{n-2}$. From this and the inequality

$$\int\limits_{\mathscr{B}_r\cap\mathscr{B}_\delta(y_i)} u^2\,dx \le \frac{c\delta^n}{\mathrm{cap}\,(F_r \cap \mathscr{B}_\delta(y_i))} \int\limits_{\mathscr{B}_r\cap\mathscr{B}_\delta(y_i)} (\mathrm{grad}\,u)^2\,dx$$

(cf., e.g., [41]) it follows that

$$\gamma\delta^{-2} \int\limits_{\mathscr{B}_r\cap\mathscr{B}_\delta(y_i)} u^2\,dx \le c \int\limits_{\mathscr{B}_r\cap\mathscr{B}_\delta(y_i)} (\mathrm{grad}\,u)^2\,dx.$$

But since

$$\delta^{-1} \int\limits_{\partial\mathscr{B}_r\cap\mathscr{B}_\delta(y_i)} u^2\,ds \le c\Bigg(\int\limits_{\mathscr{B}_r\cap\mathscr{B}_\delta(y_i)} (\mathrm{grad}\,u)^2\,dx + \delta^{-2} \int\limits_{\partial\mathscr{B}_r\cap\mathscr{B}_\delta(y_i)} u^2\,dx\Bigg),$$

one has

$$\gamma\delta^{-1} \int\limits_{\partial\mathscr{B}_r\cap\mathscr{B}_\delta(y_i)} u^2\,ds \le c \int\limits_{\mathscr{B}_r\cap\mathscr{B}_\delta(y_i)} (\mathrm{grad}\,u)^2\,dx.$$

Summing over i, we find

$$\gamma \int\limits_{\partial\mathscr{B}_r} u^2\,ds \le c\delta(F_r,\mathscr{B}_r) \int\limits_{\mathscr{B}_r\cap\mathscr{B}_\delta(y_i)} (\mathrm{grad}\,u)^2\,dx,$$

which is equivalent to the inequality

$$\lambda(r) \ge \frac{c\gamma}{\delta(F_r,\mathscr{B}_r)}\,.$$

3. As above, let cap $F_r \ge \gamma(2r)^{n-2}$. We set $\delta = \delta(F_r,\mathscr{B}_r) - \varepsilon$. Then one can find a ball $\mathscr{B}_\delta(y)$, $y \in \partial\mathscr{B}_r \setminus F$ such that $\mathrm{cap}\,(F_r \cap \mathscr{B}_\delta(y)) < \gamma\delta^{n-2}$. For any function u, $u \in W_2^1(\mathscr{B}r)$, $u = 0$ on F_r, we have

$$\lambda(r) \int\limits_{\partial\mathscr{B}_r} u^2\,ds \le \int\limits_{\mathscr{B}_r} (\mathrm{grad}\,u)^2\,dx.$$

Let $\eta \in C_0^\infty(\mathscr{B}_\delta(y))$, $\eta = 1$ on $\mathscr{B}_{\frac{\delta}{2}}(y)$, $|\,\mathrm{grad}\,\eta| \le \frac{c}{\delta}$. Then

$$\lambda(r) \int\limits_{\partial\mathscr{B}_r\cap\mathscr{B}_{\frac{\delta}{2}}(y)} u^2\,ds \le c \int\limits_{\mathscr{B}_r\cap\mathscr{B}_\delta(y)} \big[(\mathrm{grad}\,u)^2 + \delta^{-2}u^2\big]\,dx.$$

Since

$$\delta^{-2} \int\limits_{\mathscr{B}_r \cap \mathscr{B}_\delta(y)} u^2\, dx \le$$
$$\le c\Big(\int\limits_{\mathscr{B}_r \cap \mathscr{B}_\delta(y)} (\operatorname{grad} u)^2\, dx + \delta^{-1} \int\limits_{\partial\mathscr{B}_r \cap \mathscr{B}_{\frac{\delta}{2}}(y)} u^2\, ds \Big),$$

one has

$$\lambda(r) \int\limits_{\partial\mathscr{B}_r \cap \mathscr{B}_{\frac{\delta}{2}}(y)} u^2\, ds \le$$
$$\le c_1 \int\limits_{\mathscr{B}_r \cap \mathscr{B}_\delta(y)} (\operatorname{grad} u)^2\, dx + c_2 \delta^{-1} \int\limits_{\partial\mathscr{B}_r \cap \mathscr{B}_{\frac{\delta}{2}}(y)} u^2\, ds.$$

If $2c_2\delta^{-1} \ge \lambda(r)$, then we have obtained the upper bound needed for $\lambda(r)$. Let $2c_2\delta^{-1} < \lambda(r)$. Then

$$\lambda(r) \int\limits_{\partial\mathscr{B}_r \cap \mathscr{B}_{\frac{\delta}{2}}(y)} u^2\, ds \le 2c_1 \int\limits_{\mathscr{B}_r \cap \mathscr{B}_\delta(y)} (\operatorname{grad} u)^2\, dx.$$

We denote by w the capacity potential of the set $F_r \cap \mathscr{B}_\delta(y)$ and by ζ a function from $C_0^\infty(\mathscr{B}_{\frac{\delta}{2}}(y))$ such that $\zeta(y) = 1$, $|\operatorname{grad}\zeta| \le \frac{c}{\delta}$. Since $(1-w)\zeta = 0$ on F_r, one has

$$\lambda(r)\|1-w\|^2_{L_2(\partial\mathscr{B}_r \cap \mathscr{B}_{\frac{\delta}{4}}(y))} \le$$
$$\le c \int\limits_{\mathscr{B}_r \cap \mathscr{B}_{\frac{\delta}{2}}(y)} \big[(\operatorname{grad} w)^2 + \delta^{-2}(1-w)^2\big]\, dx \le$$
$$\le c\big(\operatorname{cap}(F_r \cap \mathscr{B}_\delta(y)) + \delta^{n-2}\big) \le c\delta^{n-2}. \tag{6.2.6}$$

Applying the inequalities

$$\int\limits_{\partial\mathscr{B}_r \cap \mathscr{B}_{\frac{\delta}{4}}(y)} w^2\, ds \le c\Big(\delta \int\limits_{\mathscr{B}_{\frac{\delta}{4}}(y)} (\operatorname{grad} w)^2\, dx + \delta^{-1} \int\limits_{\mathscr{B}_{\frac{\delta}{4}}(y)} w^2\, dx\Big),$$
$$\delta^{-2} \int\limits_{\mathscr{B}_{\frac{\delta}{4}}(y)} w^2\, dx \le \int \frac{w^2(x)}{|x-y|^2}\, dx \le c \int\limits_{\mathbb{R}^n} (\operatorname{grad} w)^2\, dx$$

successively, we conclude that

$$\int\limits_{\partial\mathscr{B}_r \cap \mathscr{B}_{\frac{\delta}{4}}(y)} w^2\, ds \le c\delta \operatorname{cap}\big(F_r \cap \mathscr{B}_\delta(y)\big) \le c\gamma\delta^{n-1}.$$

From this and (6.2.6) we get

$$c\lambda(r)(1 - c\gamma^{\frac{1}{2}})^2 \le c\delta^{n-2}.$$

Thus the proposition is proved. $\square$

Proof of the Theorem 6.2.1. For $r \in (0, R)$,

$$\int\limits_{\mathscr{B}_r} (\operatorname{grad} u)^2 \, dx = r^{n-1} \int\limits_{\partial \mathscr{B}_r} u \, \frac{\partial u}{\partial r} \, d\omega. \tag{6.2.7}$$

Hence one has

$$2\lambda(r) \int\limits_{\partial \mathscr{B}_r} u^2 \, d\omega \le \frac{d}{dr} \int\limits_{\partial \mathscr{B}_r} u^2 \, d\omega,$$

which, with Proposition 6.2.1, gives (6.2.5). $\square$

From (6.2.5) we derive a pointwise estimate for the function $|u|$. One verifies by integration by parts that one has

Lemma 6.2.1. *Let $\eta \in C_0^\infty(\mathscr{B}_\rho)$ and $r = |x - y|$. For $x \in \mathscr{B}_\rho \setminus F$, one has*

$$\frac{u^2(x)\eta(x)}{\omega_n(n-2)} + 2 \int\limits_{\mathscr{B}_\rho \setminus F} (\operatorname{grad} u)^2 \, \frac{\eta}{r^{n-2}} \, dy =$$
$$= \int\limits_{\mathscr{B}_\rho \setminus F} u^2 \big(r^{2-n} \Delta\eta - \operatorname{grad} r^{2-n} \cdot \operatorname{grad} \eta \big) \, dy. \tag{6.2.8}$$

From (6.2.8), assuming that $\eta = 1$ in a neighborhood of the ball $\mathscr{B}_{\frac{\rho}{2}}$, we deduce that one has

Corollary 6.2.1. *For all $x \in \mathscr{B}_{\frac{\rho}{2}} \setminus F$, we have*

$$|u(x)| \le c\rho^{-\frac{n}{2}} \|u\|_{L_2(\mathscr{B}_\rho \setminus \mathscr{B}_{\frac{\rho}{2}})}. \tag{6.2.9}$$

Since by (6.2.7) the function $\rho \to \|u(\rho, \cdot)\|_{L_2(\partial \mathscr{B}_1)}$ is nondecreasing, from (6.2.8) one gets $|u(x)| \le c\|u(\rho, \cdot)\|_{L_2(\partial \mathscr{B}_1)}$, where $x \in \mathscr{B}_{\frac{\rho}{2}} \setminus F$. From this and the theorem one gets

Corollary 6.2.2. *For $\rho \in (0, R)$ and $x \in \mathscr{B}_{\frac{\rho}{2}} \setminus F$, one has*

$$u^2(x) \le c \exp\bigg\{ -c\bigg(\int\limits_{I(\rho,R)} \operatorname{cap}(F_r) \, \frac{dr}{r^{n-1}} +$$
$$+ \int\limits_{E(\rho,R)} \operatorname{cap}(F_r) \, \frac{dr}{\delta(F_r, \mathscr{B}_r)} \bigg) \bigg\} \int\limits_{\partial \mathscr{B}_1} u^2(R, \omega) \, d\omega. \tag{6.2.10}$$

Noting that the integral over $I(\rho, R)$ in (6.2.10) does not exceed $c\log(\frac{R}{\rho})$, we get

Corollary 6.2.3. *If*

$$\lim_{\rho\to 0} \frac{1}{|\log\rho|} \int\limits_{E(\rho,R)} \frac{dr}{\delta(F_r, \mathscr{B}_r)} = \infty, \tag{6.2.11}$$

then $u(x) = o(|x|^N)$ for any positive N.

From (6.2.10), just as from (6.2.1) in [39], one derives a variety of information about the behavior of a harmonic function, Green's function, and harmonic measure at infinity and near a boundary point. Here we restrict ourselves to the question of the Hölder continuity of a solution u of (6.2.2).

Proposition 6.2.2. *Let $\overline{\Omega} \subset \mathscr{B}_1$, $\phi \in C(\partial\Omega)$ and $\phi(x) - \phi(0) \le const|x|^\alpha$, where $\alpha > 0$. If*

$$\liminf_{\rho\to 0} \frac{1}{|\log\rho|} \Bigg(\int\limits_{I(\rho,R)} \operatorname{cap}(F_r) \frac{dr}{r^{n-1}} + \int\limits_{E(\rho,R)} \frac{dr}{\delta(F_r, \mathscr{B}_r)} \Bigg) = \beta > 0, \tag{6.2.12}$$

where $F_r = \mathscr{B}_r \setminus \Omega$, there exists a positive constant γ depending on α, β, n such that $v(x) - \phi(0) \le const|x|^\gamma$.

Proof. One can assume that $\phi(0) = 0$. Let $H(x, E)$ be the harmonic measure of the set $E \subset \partial\Omega$ with respect to Ω. We have

$$v(x) = \int\limits_{\partial\Omega} \Phi(x) H(x, dy) \le$$

$$\le \int\limits_{\partial\Omega} |y|^\alpha H(x, dy) \le c|x|^\alpha + \int\limits_{\partial\Omega\setminus\mathscr{B}_{2|x|}} \big(|y|^\alpha - (2|x|)^\alpha\big) H(x, dy).$$

From this we have

$$|v(x)| \le c|x|^\alpha + \int\limits_{2|x|}^{1} H(x, \partial\Omega\setminus\mathscr{B}_t)\, d(t^\alpha). \tag{6.2.13}$$

Since the function $x \to H(x, \partial\Omega\setminus\mathscr{B}_t)$ is harmonic and satisfies the homogeneous Dirichlet condition on $\mathscr{B}_t \setminus \partial\Omega$, by (6.2.10) for $|x| < \frac{t}{2}$,

$$H(x, \partial\Omega\setminus\mathscr{B}_t) \le$$

$$\le c\exp\Bigg\{ -c\Bigg(\int\limits_{I(2|x|,t)} \operatorname{cap}(F_r) \frac{dr}{r^{n-1}} + \int\limits_{E(2|x|,t)} \frac{dr}{\delta(F_r, \mathscr{B}_r)} \Bigg)\Bigg\}. \tag{6.2.14}$$

From this estimate and (6.2.12) we find that

$$H(x, \partial\Omega \setminus \mathscr{B}_t) \le c\Big(\frac{|x|}{t}\Big)^{c\,\beta}.$$

Hence

$$|v(x)| \le c|x|^{\alpha} + c|x|^{c\,\beta} \int\limits_{2|x|}^{1} t^{\alpha - c\,\beta - 1}\, dt. \qquad \square$$

We shell show that the domain Ω can satisfy (6.2.12), while simultaneously (6.2.3) does not hold.

Example 6.2.1. Let

$$\mathscr{B}^{(\nu)} = \big\{x \in \mathbb{R}^n : \ |x| < \rho_\nu\big\}, \ \ \nu \ge 2, \ \ \log_2 \log_2 \rho_\nu^{-1} = \nu,$$

and let Ω be the union of spherical shells $\mathscr{B}^{(\nu)} \setminus \mathscr{B}^{(\nu+1)}$, joined by holes in the spheres $\partial\mathscr{B}^{(\nu)}$, and $F = \mathscr{B} \setminus \Omega$. The hole ρ is a geodesic ball with an arbitrary center and radius $\sigma_\nu = \rho_{\nu+1}^{1+\varepsilon}$, $\varepsilon > 0$. Let ρ be a small positive number and ν be an index such that $\rho_\nu \le \rho < \rho_{\nu-1}$. It is clear that $\operatorname{cap}(\mathscr{B}_r \setminus \Omega) \sim \rho_\nu^{n-2}$. Consequently,

$$\int\limits_{\rho}^{1} \operatorname{cap}(\mathscr{B}_\tau \setminus \Omega)\tau^{1-n}\, d\tau \sim \nu,$$

and since $\log_2 \rho^{-1} \sim 2^\nu$, the domain under consideration does not satisfy (6.2.3).

Now we note that there exists a constant $c > 1$ such that (6.2.4) does not hold for $\rho \in (c_0\rho_k, \rho_{k-1})$, $1 \le k \le \nu$. If now $\rho \in (\rho_k, c_0\rho_k)$, then for any ball $\mathscr{B}_{c(\rho-\rho_k+\delta_k)}(y)$ with center on $\partial\mathscr{B}_\rho$ (6.2.4) holds. Hence $\delta(F_\rho, \mathscr{B}_\rho) \le c(\rho - \rho_k + \delta_k)$. From this it follows that

$$\int\limits_{I(\rho,1)} \operatorname{cap}(F_k)\, \frac{dr}{r^{n-1}} + \int\limits_{E(\rho,1)} \frac{dr}{\delta(F_r, \mathscr{B}_r)} \ge$$

$$\ge c\sum_{k=1}^{\nu} \log\frac{\rho_k}{\delta_k} \sim c\,\varepsilon\, 2^\nu \sim c\,\varepsilon |\log\rho|^{-1}.$$

Thus, (6.2.12) holds.

Setting $\log_2 \sigma_\nu = -|\log\rho_\nu|^{1+\varepsilon}$, we get

$$\sum_{k=1}^{\nu} |\log_1 \rho_k|^{1+\varepsilon} \sim 2^{k(1+\varepsilon)}.$$

Hence for such a choice of diameters of the holes ω_ν (6.2.10) holds, guaranteeing the superpower convergence of the function u to zero (cf. Corollary 6.2.3).

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