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Short Communications

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ON THE EXISTENCE OF BOUNDED SOLUTIONS FOR SYSTEMS OF NONLINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. Sufficient conditions are given for the existence of bounded solutions for the systems of nonlinear generalized ordinary differential equations.

[illegible]

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Let $a_{mik} : \mathbb{R} \to \mathbb{R}$ $(m = 1, 2;\ i, k = 1, \dots, n)$ be nondecreasing functions, $a_{ik}(t) \equiv a_{1ik}(t) - a_{2ik}(t)$, $A = (a_{ik})_{i,k=1}^n$, $A_m = (a_{mik})_{i,k=1}^n$ $(m = 1, 2)$; $f = (f_k)_{k=1}^n : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a vector-function belonging to the Carathéodory class corresponding to the matrix-function A.

In this paper we investigate the question of existence of solutions for the system of generalized ordinary differential equations

$$dx(t) = dA(t) \cdot f(t, x(t)), \tag{1}$$

where $x = (x_i)_{i=1}^n$, satisfying one of the following two conditions

$$\sup\Big\{ \sum_{i=1}^n |x_i(t)| :\ t \in \mathbb{R} \Big\} < \infty \tag{2}$$

and

$$\sup\Big\{ \sum_{i=1}^n |x_i(t)| :\ t \in \mathbb{R}_+ \Big\} < \infty. \tag{3}$$

We give sufficient conditions for the existence of solutions of the boundary value problems (1), (2) and (1), (3). Analogous results are contained in [9]–[14] for systems of ordinary differential and functional differential equations.

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The theory of generalized ordinary differential equations enables one to investigate ordinary differential, impulsive and difference equations from a common point of view (see [1]–[8], [15]).

Throughout the paper the following notation and definitions will be used.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $[a,b]$ $(a, b \in \mathbb{R})$ is a closed segment.

$\mathbb{R}^{n\times m}$ is the set all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the set of all real column n-vectors $x = (x_i)_{i=1}^{n}$.

$\bigvee\limits_a^b(X)$ is the total variation of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$, i.e., the sum of total variations of the latter's components.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$ at the point t (we will assume $X(t) = X(a)$ for $t \le a$ and $X(t) = X(b)$ for $t \ge b$, if necessary);

$$d_1X(t) = X(t) - X(t-), \quad d_2X(t) = X(t+) - X(t).$$

$\mathrm{BV}([a,b], \mathbb{R}^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a,b] \to \mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X) < +\infty$).

$\mathrm{BV}_{loc}(\mathbb{R}, \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : \mathbb{R} \to \mathbb{R}^{n\times m}$ for which $\bigvee\limits_a^b(X) < +\infty$) for every $a, b \in \mathbb{R}$ $(a < b)$.

$s_j : \mathrm{BV}([a,b], \mathbb{R}) \to \mathrm{BV}([a,b], \mathbb{R})$ $(j = 0, 1, 2)$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0,$$
$$s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau) \quad \text{and} \quad s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \ \text{ for } \ a < t \le b$$

and

$$s_0(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \ \text{ for } \ t \in [a,b].$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int\limits_s^t x(\tau)\,dg(\tau) = \int\limits_{]s,t[} x(\tau)\,ds_0(g)(\tau) + \sum_{s<\tau\le t} x(\tau)d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)d_2g(\tau),$$

where $\int\limits_{]s,t[} x(\tau)\,ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu(s_0(g))$ corresponding to the function $s_0(g)$.

If $a = b$, then we assume

$$\int\limits_a^b x(t)\,dg(t) = 0.$$

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\, dg(\tau) = \int_s^t x(\tau)\, dg_1(\tau) - \int_s^t x(\tau)\, dg_2(\tau) \ \text{ for } \ s \le t.$$

$L([a,b],\mathbb{R};g)$ is the set of all functions $x : [a,b] \to \mathbb{R}$ measurable and integrable with respect to the measures $\mu(g_i)$ $(i = 1,2)$, i.e. such that

$$\int_a^b |x(t)|\, dg_i(t) < +\infty \ \ (i = 1,2).$$

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to \mathbb{R}^{l\times n}$ is a nondecreasing matrix-function and $D \subset \mathbb{R}^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to D$ such that $x_{kj} \in L([a,b],\mathbb{R};g_{ik})$ $(i = 1,\dots,l;\ k = 1,\dots,n;\ j = 1,\dots,m)$;

$$\int_s^t dG(\tau)\cdot X(\tau) = \left(\sum_{k=1}^n \int_s^t x_{kj}(\tau) dg_{ik}(\tau)\right)_{i,j=1}^{l,m} \ \text{ for } \ a \le s \le t \le b,$$

$$S_j(G)(t) \equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n} \ \ (j = 0,1,2).$$

If $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}^{n\times m}$, then $K([a,b]\times D_1, D_2; G)$ is the Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that for each $i \in \{1,\dots,l\}$, $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$: a) the function $f_{kj}(\cdot,x) : [a,b] \to D_2$ is $\mu(g_{ik})$-measurable for every $x \in D_1$; b) the function $f_{kj}(t,\cdot) : D_1 \to D_2$ is continuous for $\mu(g_{ik})$-almost every $t \in [a,b]$, and $\sup\big\{|f_{kj}(\cdot,x)| : \ x \in D_0\big\} \in L([a,b],\mathbb{R};g_{ik})$ for every compact $D_0 \subset D_1$.

If $G_j : [a,b] \to \mathbb{R}^{l\times n}$ $(j = 1,2)$ are nondecreasing matrix-functions, $G = G_1 - G_2$ and $X : [a,b] \to \mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau) = \int_s^t dG_1(\tau)\cdot X(\tau) - \int_s^t dG_2(\tau)\cdot X(\tau) \ \text{ for } \ s \le t,$$

$$S_k(G) = S_k(G_1) - S_k(G_2) \ \ (k = 0,1,2),$$

$$L([a,b],D;G) = \bigcap_{j=1}^2 L([a,b],D;G_j),$$

$$K([a,b]\times D_1, D_2; G) = \bigcap_{j=1}^2 K([a,b]\times D_1, D_2; G_j).$$

$L_{loc}(\mathbb{R}, D; G)$ is the set of all matrix-functions $X = \mathbb{R} \to D$ such that its restriction on $[a, b]$ belongs to $L([a, b], D; G)$ for every a and b from $\mathbb{R}$ $(a < b)$.

$K_{loc}(\mathbb{R} \times D_1, D_2; G)$ is the set of all matrix-functions $F = (f_{kj})_{k,j=1}^{n,m} : \mathbb{R} \times D_1 \to D_2$ such that its restriction on $[a, b]$ belongs to $K([a, b], D; G)$ for every a and b from $\mathbb{R}$ $(a < b)$.

The inequalities between the matrices are understood componentwise.

A vector-function $x \in \mathrm{BV}_{loc}(\mathbb{R}, \mathbb{R}^n)$ is said to be a solution of the system (1) if

$$x(t) = x(s) + \int\limits_s^t dA(\tau) \cdot f(\tau, x(\tau)) \ \text{ for } \ s \le t \ \ (s, t \in \mathbb{R}).$$

Theorem 1. *Let there exist numbers* $\sigma_i \in \{-1, 1\}$ $(i = 1, \dots, n)$, *vector-functions* $\alpha_m = (\alpha_{mi})_{i=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}, \mathbb{R}^n)$ $(m = 1, 2)$ *and matrix-functions* $(\beta_{mik})_{i,k=1}^n$, $\beta_{mik} \in L_{loc}(\mathbb{R}, \mathbb{R}; a_{jik})$ $(m, j = 1, 2;\ i, k = 1, \dots, n)$ *such that*

$$\alpha_{mi}(t) \equiv \alpha_{mi}(0) + \sum_{k=1}^n \Bigg(\int\limits_0^t \beta_{mik}(\tau)\, da_{1ik}(\tau) - \int\limits_0^t \beta_{3-mik}(\tau)\, da_{2ik}(\tau) \Bigg) \ (m=1,2;\ i=1,\dots,n), \tag{4}$$

$$\alpha_1(t) \le \alpha_2(t) \quad \textit{for } \ t \in \mathbb{R}, \tag{5}$$

$$\begin{gathered}(-1)^m \sigma_i \big(f_k(t, x_1, \dots, x_{i-1}, \alpha_{ji}(t), x_{i+1}, \dots, x_n) - \beta_{mik}(t) \big) \le 0 \\ \textit{for } \ \mu(a_{1+|m-j|ik})\textit{-almost all } \ t \in \mathbb{R} \ \textit{ and} \\ \alpha_1(t) \le (x_l)_{l=1}^n \le \alpha_2(t) \ \ (m, j = 1, 2;\ \ i, k = 1, \dots, n), \\ (-1)^m \Bigg(x_i - (-1)^j \sum_{k=1}^n f_k(t, x_1, \dots, x_n) d_j a_{ik}(t) - \alpha_{mi}(t) - (-1)^j d_j \alpha_{mi}(t) \Bigg) \le 0 \\ \textit{for } \ t \in \mathbb{R}, \ \ \alpha_1(t) \le (x_l)_{l=1}^n \le \alpha_2(t) \ \textit{ and} \\ (-1)^j \sigma_i > 0 \ \ (m, j = 1, 2;\ \ i = 1, \dots, n)\end{gathered} \tag{6}$$

and

$$\sup \big\{ |\alpha_{mi}(t)| : \ t \in \mathbb{R} \big\} < \infty \ \ (m = 1, 2;\ \ i = 1, \dots, n). \tag{7}$$

Then the problem (1), (2) *is solvable.*

Corollary 1. *Let the matrix-function* $A(t) = (a_{ik})_{i,k=1}^n$ *be nondecreasing on* $\mathbb{R}$ *and let there exist numbers* $\sigma_i \in \{-1, 1\}$ $(i = 1, \dots, n)$, *vector-functions* $\alpha_m = (\alpha_{mi})_{i=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}, \mathbb{R}^n)$ $(m = 1, 2)$ *and matrix-functions* $(\beta_{mik})_{i,k=1}^n$, $\beta_{mik} \in L_{loc}(\mathbb{R}, \mathbb{R}; a_{jik})$ $(m, j = 1, 2;\ i, k = 1, \dots, n)$ *such that*

$$\alpha_{mi}(t) \equiv \alpha_{mi}(0) + \sum_{k=1}^n \Bigg(\int\limits_0^t \beta_{mik}(\tau) da_{1ik}(\tau) \Bigg) \ \ (m=1,2;\ \ i,k=1,\dots,n), \tag{8}$$

the conditions $(5)-(7)$ *hold, and the inequalities*

$$(-1)^m \sigma_i\big(f_k(t,x_1,\dots,x_{i-1},\alpha_{ji}(t),x_{i+1},\dots,x_n)-\beta_{jik}(t)\big)\le 0 \quad (j=1,2;\ \ i,k=1,\dots,n)$$

are fulfilled for $\mu(a_{ik})$*-almost all* $t\in\mathbb{R}$ *and* $\alpha_1(t)\le (x_l)_{l=1}^n\le\alpha_2(t)$*. Then the problem* $(1),(2)$ *is solvable.*

Theorem 2. *Let there exist numbers* $\sigma_i\in\{-1,1\}$ $(i=1,\dots,n)$*, vector-functions* $\alpha_m=(\alpha_{mi})_{i=1}^n\in \mathrm{BV}_{loc}(\mathbb{R}_+,\mathbb{R}^n)$ $(m=1,2)$ *and matrix-functions* $(\beta_{mik})_{i,k=1}^n$*,* $\beta_{mik}\in L_{loc}(\mathbb{R}_+,\mathbb{R};a_{jik})$ $(m,j=1,2$*;* $i,k=1,\dots,n)$ *such that*

$$\alpha_1(t)\le\alpha_2(t) \ \ \text{for}\ \ t\in\mathbb{R}_+, \tag{9}$$

$$(-1)^m \sigma_i\big(f_k(t,x_1,\dots,x_{i-1},\alpha_{ji}(t),x_{i+1},\dots,x_n)-\beta_{mik}(t)\big)\le 0$$
$$\text{for}\ \ \mu(a_{1+|m-j|ik})\text{-almost all}\ \ t\in\mathbb{R}_+\ \ \text{and}$$
$$\alpha_1(t)\le (x_l)_{l=1}^n\le\alpha_2(t)\ \ (m,j=1,2;\ \ i,k=1,\dots,n),$$

$$(-1)^m\bigg(x_i-(-1)^j\sum_{k=1}^n f_k(t,x_1,\dots,x_n)d_j a_{ik}(t)-\alpha_{mi}(t)-(-1)^j d_j\alpha_{mi}(t)\bigg)\le 0$$
$$\text{for}\ \ t\in\mathbb{R}_+,\ \ \alpha_1(t)\le (x_l)_{l=1}^n\le\alpha_2(t)\ \ \text{and}$$
$$(-1)^j\sigma_i>0 \quad (m,j=1,2;\ \ i=1,\dots,n) \tag{10}$$

and

$$\sup\big\{|\alpha_{mi}(t)|:\ t\in\mathbb{R}_+\big\}<\infty\ \ (m=1,2;\ \ i=1,\dots,n). \tag{11}$$

Then the problem $(1),(3)$ *is solvable.*

Corollary 2. *Let the matrix-function* $A(t)=(a_{ik})_{i,k=1}^n$ *be nondecreasing on* $\mathbb{R}_+$ *and let there exist numbers* $\sigma_i\in\{-1,1\}$ $(i=1,\dots,n)$*, vector-functions* $\alpha_m=(\alpha_{mi})_{i=1}^n\in \mathrm{BV}_{loc}(\mathbb{R}_+,\mathbb{R}^n)$ $(m=1,2)$ *and matrix-functions* $(\beta_{mik})_{i,k=1}^n$*,* $\beta_{mik}\in L_{loc}(\mathbb{R}_+,\mathbb{R};a_{jik})$ $(m,j=1,2$*;* $i,k=1,\dots,n)$ *such that the conditions* (8)–(11) *hold, and the inequalities*

$$(-1)^m \sigma_i\big(f_k(t,x_1,\dots,x_{i-1},\alpha_{ji}(t),x_{i+1},\dots,x_n)-\beta_{jik}(t)\big)\le 0 \quad (j=1,2;\ \ i,k=1,\dots,n)$$

are fulfilled for $\mu(a_{ik})$*-almost all* $t\in\mathbb{R}_+$ *and* $\alpha_1(t)\le (x_l)_{l=1}^n\le\alpha_2(t)$*. Then the problem* $(1),(3)$ *is solvable.*

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