Memoirs on Differential Equations and Mathematical Physics
Volume 41, 2007, 27–42

Robert Hakl and Sulkhan Mukhigulashvili

ON A PERIODIC BOUNDARY VALUE PROBLEM FOR THIRD ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. Unimprovable effective sufficient conditions are established for the unique solvability of the periodic problem

$$u'''(t) = \sum_{i=0}^{2} \ell_i(u^{(i)})(t) + q(t) \ \text{ for } \ 0 \le t \le \omega,$$
$$u^{(j)}(0) = u^{(j)}(\omega) \ \ (j = 0, 1, 2),$$

where $\omega > 0$, $\ell_i : C([0,\omega]) \to L([0,\omega])$ are linear bounded operators, and $q \in L([0,\omega])$.

2000 Mathematics Subject Classification. 34K06, 34K10.

Key words and phrases. Linear functional differential equation, ordinary differential equation, periodic boundary value problem.

[illegible]

$$u'''(t) = \sum_{i=0}^{2} \ell_i(u^{(i)})(t) + q(t) \ \text{ [illegible] } \ 0 \le t \le \omega,$$
$$u^{(j)}(0) = u^{(j)}(\omega) \ \ (j = 0, 1, 2),$$

[illegible] $\omega > 0$, $\ell_i : C([0,\omega]) \to L([0,\omega])$ [illegible] $q \in L([0,\omega])$.

1. Introduction

Consider the equation

$$u'''(t) = \sum_{i=0}^{2} \ell_i(u^{(i)})(t) + q(t) \ \text{ for } \ 0 \le t \le \omega \tag{1.1}$$

with the periodic boundary conditions

$$u^{(j)}(0) = u^{(j)}(\omega) \ \ (j = 0, 1, 2), \tag{1.2}$$

where $\omega > 0$, $\ell_i : C\big([0,\omega]\big) \to L\big([0,\omega]\big)$ are linear bounded operators, and $q \in L\big([0,\omega]\big)$.

By a solution of the problem (1.1), (1.2) we understand a function $u \in \widetilde{C}^2\big([0,\omega]\big)$ which satisfies the equation (1.1) almost everywhere on $[0,\omega]$ and fulfils the conditions (1.2).

There are many works and interesting results on the existence and uniqueness of solution for the periodic boundary value problem for higher order ordinary differential equations (see, e.g., [1], [2], [4]– [6], [11]– [15], [17], [18], [25], [26], [29], [30] and the references therein). But an analogous problem for functional differential equations, even in the case of linear equations, remains still little investigated.

In 1972 H. H. Schaefer (see [28, Theorem 4]) proved that there exists a linear bounded operator $\ell : C\big([0,\omega]\big) \to L\big([0,\omega]\big)$ which is not strongly bounded, that is, it does not have the following property: there exists a summable function $\eta : [0,\omega] \to [0,+\infty[$ such that

$$|\ell(x)(t)| \le \eta(t)\|x\|_C \ \text{ for } \ 0 \le t \le \omega, \ \ x \in C\big([0,\omega]\big).$$

It is well–known (see, e.g., [16]) that the general boundary value problem for linear functional differential equations with a strongly bounded linear operator has the Fredholm property, i.e., it is uniquely solvable iff the corresponding homogeneous problem has only the trivial solution. The same property (Fredholmity) for functional differential equations with a nonstrongly bounded linear operator was not investigated till 2000. The first step was made in [3], [8] for scalar first order functional differential equations. Those results were generalized for the n-th order functional differential systems in [10].

Thus, in the present paper, we study the problem (1.1), (1.2) under the assumptions that ℓ_0 is a strongly bounded operator and ℓ_i $(i = 1, 2)$ are bounded, not necessarily strongly bounded, operators. We establish new unimprovable integral conditions sufficient for the unique solvability of the problem (1.1), (1.2).

Note that in [12], unlike the earlier known results, there are investigated, among others, the existence and uniqueness of an ω-periodic solution of the nonautonomous ordinary differential equation

$$u^{(n)}(t) = \sum_{k=0}^{n-1} p_k(t)u^{(k)}(t) + q(t) \ \text{ for } \ 0 \le t \le \omega$$

without the requirement on the function p_0 to be of constant sign. In this paper we improve the result of [12] for $n = 3$ in a certain way (see Corollary 2.3). Consequently, the obtained results are also new even if (1.1) is an ordinary differential equation

$$u'''(t) = \sum_{i=0}^{2} p_i(t)u^{(i)}(t) + q(t) \quad \text{for} \quad 0 \le t \le \omega. \tag{1.3}$$

For functional differential equations, one can name only a few papers devoted to the study of the periodic boundary value problem (see, e.g., [7], [19]– [24], [27]).

All the results will be formulated for the differential equation with deviating arguments

$$u'''(t) = \sum_{i=0}^{2} p_i(t)u^{(i)}(\tau_i(t)) + q(t) \quad \text{for} \quad 0 \le t \le \omega, \tag{1.4}$$

which is a particular case of the equation (1.1). Here p_i, $q \in L\big([0,\omega]\big)$ and $\tau_i : [0,\omega] \to [0,\omega]$ $(i = 0, 1, 2)$ are measurable functions.

The method used for the investigation of the considered problem is based on the method developed in our previous papers (see [7], [19]– [21], [23]). In particular, the results presented in the paper generalize the results obtained in [7], [23].

The following notation is used throughout:

N is the set of all natural numbers;

R is the set of all real numbers, $R_+ = [0, +\infty[$;

$C\big([0,\omega]\big)$ is the Banach space of continuous functions $u : [0,\omega] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : \ 0 \le t \le \omega\}$;

$\widetilde{C}^2\big([0,\omega]\big)$ is the set of functions $u : [0,\omega] \to R$ which are absolutely continuous together with their first and second derivatives;

$L\big([0,\omega]\big)$ is the Banach space of Lebesgue integrable functions $p : [0,\omega] \to R$ with the norm $\|p\|_L = \int\limits_0^\omega |p(s)|\, ds$;

If $\ell : C\big([0,\omega]\big) \to L\big([0,\omega]\big)$ is a linear operator, then

$$\|\ell\| = \sup\big\{\|\ell(x)\|_L : \ \|x\|_C \le 1\big\}.$$

Definition 1.1. We will say that a linear operator $\ell : C\big([0,\omega]\big) \to L\big([0,\omega]\big)$ is *nonnegative* if for any nonnegative $x \in C\big([0,\omega]\big)$ the inequality

$$\ell(x)(t) \ge 0 \quad \text{for} \quad 0 \le t \le \omega$$

is fulfilled.

We will say that a linear operator $\ell : C\big([0,\omega]\big) \to L\big([0,\omega]\big)$ is *monotone* if either ℓ or $-\ell$ is nonnegative.

2. Main Results

Theorem 2.1. *Let nonnegative operators ℓ_{01}, $\ell_{02} : C\big([0,\omega]\big) \to L\big([0,\omega]\big)$ and bounded operators ℓ_1, ℓ_2 be such that*

$$\ell_{01}(1)(t) \not\equiv \ell_{02}(1)(t), \tag{2.1}$$

$$\frac{\omega^2}{32} \int_0^\omega \ell_{01}(1)(s)\,ds < \beta_1, \tag{2.2}$$

$$\frac{\int_0^\omega \ell_{01}(1)(s)\,ds}{\beta_1 - \frac{\omega^2}{32} \int_0^\omega \ell_{01}(1)(s)\,ds} \le \int_0^\omega \ell_{02}(1)(s)\,ds, \tag{2.3}$$

and

$$\int_0^\omega \ell_{02}(1)(s)\,ds \le \frac{64\beta_1}{\omega^2} \left(1 + \sqrt{1 - \frac{\omega^2}{32\beta_1} \int_0^\omega \ell_{01}(1)(s)\,ds}\right), \tag{2.4}$$

where

$$\beta_1 = 1 - \frac{\omega}{4}\,\|\ell_1\| - \|\ell_2\|.$$

Then the problem $(1.1), (1.2)$ *with both*

$$\ell_0 = \ell_{01} - \ell_{02} \ \ \textit{and} \ \ \ell_0 = \ell_{02} - \ell_{01}$$

has a unique solution.

Theorem 2.2. *Let ℓ_{i1}, $\ell_{i2} : C\big([0,\omega]\big) \to L\big([0,\omega]\big)$ $(i = 0, 1, 2)$ be nonnegative operators, ℓ_1, ℓ_2 admit the representations*

$$\ell_1 = \ell_{11} - \ell_{12}, \quad \ell_2 = \ell_{21} - \ell_{22},$$

and let

$$\beta_1 = 1 - \frac{\omega}{4}\, \max\big\{\|\ell_{11}\|, \|\ell_{12}\|\big\} - \max\big\{\|\ell_{21}\|, \|\ell_{22}\|\big\}. \tag{2.5}$$

Let, moreover, the conditions (2.1)–(2.4) *be fulfilled. Then the problem* $(1.1), (1.2)$ *with both*

$$\ell_0 = \ell_{01} - \ell_{02} \ \ \textit{and} \ \ \ell_0 = \ell_{02} - \ell_{01}$$

has a unique solution.

If the operator ℓ_0 is monotone, then from Theorem 2.2 we get the following assertion which is nonimprovable in a certain sense (see Example 2.1).

Corollary 2.1. *Let a monotone operator ℓ_0 and strongly bounded operators ℓ_1, ℓ_2 be such that*

$$\ell_0(1)(t) \not\equiv 0, \quad \ell_i = \ell_{i1} - \ell_{i2} \ \ (i = 1, 2),$$

where ℓ_{i1}, $\ell_{i2} : C\big([0,\omega]\big) \to L\big([0,\omega]\big)$ *are nonnegative operators, and let*

$$\frac{\omega}{4}\max\big\{\|\ell_{11}\|, \|\ell_{12}\|\big\} + \max\big\{\|\ell_{21}\|, \|\ell_{22}\|\big\} < 1.$$

Let, moreover,

$$\int_0^\omega |\ell_0(1)(s)|\, ds \leq \frac{128\beta_1}{\omega^2}\,, \tag{2.6}$$

where β_1 *is given by* (2.5). *Then the problem* (1.1), (1.2) *has a unique solution.*

Now consider the equation with deviating arguments (1.4), where q, $p_i \in L\big([0,\omega]\big)$ and $\tau_i : [0,\omega] \to [0,\omega]$ are measurable functions.

Corollary 2.2. *Put*

$$\begin{gathered}\beta_2 = 1 - \frac{\omega}{4}\max\bigg\{\int_0^\omega [p_1(s)]_+\, ds, \int_0^\omega [p_1(s)]_-\, ds\bigg\} - \\ - \max\bigg\{\int_0^\omega [p_2(s)]_+\, ds, \int_0^\omega [p_2(s)]_-\, ds\bigg\}.\end{gathered} \tag{2.7}$$

Let, moreover, p_{01}, $p_{02} \in L\big([0,\omega]\big)$ *be nonnegative functions such that*

$$p_{01}(t) \not\equiv p_{02}(t), \tag{2.8}$$

$$\frac{\omega^2}{32}\int_0^\omega p_{01}(s)\, ds < \beta_2, \tag{2.9}$$

$$\frac{\int_0^\omega p_{01}(s)\, ds}{\beta_2 - \frac{\omega^2}{32}\int_0^\omega p_{01}(s)\, ds} \leq \int_0^\omega p_{02}(s)\, ds, \tag{2.10}$$

$$\int_0^\omega p_{02}(s)\, ds \leq \frac{64\beta_2}{\omega^2}\bigg(1 + \sqrt{1 - \frac{\omega^2}{32\beta_2}\int_0^\omega p_{01}(s)\, ds}\,\bigg). \tag{2.11}$$

Then the problem (1.4), (1.2) *with both*

$$p_0(t) = p_{01}(t) - p_{02}(t) \ \text{ and } \ p_0(t) = p_{02}(t) - p_{01}(t) \ \text{ for } \ 0 \leq t \leq \omega$$

has a unique solution.

If $\tau_i(t) = t$ $(i = 0, 1, 2)$, then we get the following assertion.

Corollary 2.3. *Let* p_{01}, p_{02}, p_1, $p_2 \in L\big([0,\omega]\big)$ *be such that all the assumptions of Corollary* 2.2 *are fulfilled. Let, moreover, either*

$$|p_1(t)| + |p_2(t)| \not\equiv 0, \tag{2.12}$$

or

$$p_{01}(t) \not\equiv 0 \ \ and \ \ p_{02}(t) \not\equiv 0. \tag{2.13}$$

Then the problem (1.3), (1.2) *with both*

$$p_0(t) = p_{01}(t) - p_{02}(t) \ \ and \ \ p_0(t) = p_{02}(t) - p_{01}(t) \ \ for \ \ 0 \le t \le \omega$$

has a unique solution.

Remark 2.1. In the paper [13] (see Proposition 1.1 therein), there was proved that if p_0 does not change its sign, then the problem $u''' = p_0(t)u + q(t)$, $u^{(j)}(0) = u^{(j)}(\omega)$ $(j = 0, 1, 2)$ is uniquely solvable iff $p_0(t) \not\equiv 0$. Hence it follows that if the conditions (2.12) and (2.13) in Corollary 2.3 are violated, i.e., if the function $p_0 = p_{01} - p_{02}$ does not change its sign and $p_i(t) \equiv 0$ $(i = 1, 2)$, then other conditions dealing with the smallness of p_0 in the integral sense are not important.

If $\ell_1 \equiv 0$ and $\ell_2 \equiv 0$, then from Theorem 2.1 we get the following assertion which has been published in [7].

Corollary 2.4. *Let nonnegative operators* ℓ_{01}, $\ell_{02} : C([0,\omega]) \to L([0,\omega])$ *be such that*

$$\ell_{01}(1)(t) \not\equiv \ell_{02}(1)(t),$$

$$\int_0^\omega \ell_{01}(1)(s)\,ds < \frac{32}{\omega^2},$$

$$\frac{\int_0^\omega \ell_{01}(1)(s)\,ds}{1 - \frac{\omega^2}{32}\int_0^\omega \ell_{01}(1)(s)\,ds} \le \int_0^\omega \ell_{02}(1)(s)\,ds,$$

and

$$\int_0^\omega \ell_{02}(1)(s)\,ds \le \frac{64}{\omega^2}\left(1 + \sqrt{1 - \frac{\omega^2}{32}\int_0^\omega \ell_{01}(1)(s)\,ds}\right).$$

Then the problem (1.1), (1.2) *with both*

$$\ell_0 = \ell_{01} - \ell_{02} \ \ and \ \ \ell_0 = \ell_{02} - \ell_{01}$$

has a unique solution.

If the operator ℓ_0 is monotone, then from Theorem 2.1 we get the following assertion which has been published in [23].

Corollary 2.5. *Let a monotone operator* ℓ_0 *and bounded operators* ℓ_1, ℓ_2 *be such that*

$$\ell_0(1)(t) \not\equiv 0$$

and

$$\frac{\omega}{4}\,\|\ell_1\| + \|\ell_2\| < 1.$$

Let, moreover,

$$\int_0^\omega |\ell_0(1)(s)|\,ds \le \frac{128}{\omega^2}\Big(1-\frac{\omega}{4}\|\ell_1\|-\|\ell_2\|\Big). \tag{2.14}$$

Then the problem (1.1), (1.2) *has a unique solution.*

It is shown in [23] that Corollary 2.5 is unimprovable in a certain sense. We present an appropriate example here for the sake of completeness.

Example 2.1. The example below shows that the conditions (2.6) and (2.14) in Corollaries 2.1 and 2.5, respectively, are optimal and they cannot be weakened.

Let $\omega=1$, $\alpha_k=\frac{1}{32}+\frac{1}{16\pi^2k^2}-\frac{1}{128k^2}$, $\beta_k=\frac{1}{8k}-\frac{1}{4}$, $k\in N$, and let the function $u_0\in\widetilde{C}^2\big([0,1]\big)$ be defined by the equality

$$u_0(t)=\begin{cases}\widetilde{u}(t) & \text{for } \ 0\le t\le 1/2\\ -\widetilde{u}(t-1/2) & \text{for } \ 1/2<t\le 1\end{cases},$$

where

$$\widetilde{u}(t)=\begin{cases}1-\dfrac{t^2}{2\alpha_{k_0}} & \text{for } \ 0\le t\le \dfrac14-\dfrac{1}{8k_0}\\ 1+\dfrac{\beta_{k_0}}{\alpha_{k_0}}t+\dfrac{\beta_{k_0}^2}{2\alpha_{k_0}}-\dfrac{1-\sin\pi k_0(1-4t)}{16\pi^2k_0^2\alpha_{k_0}} & \text{for } \ \dfrac14-\dfrac{1}{8k_0}<t\le\dfrac14+\dfrac{1}{8k_0}\\ -1-\dfrac{t(1-t)}{2\alpha_{k_0}}+\dfrac{1}{8\alpha_{k_0}} & \text{for } \ \dfrac14+\dfrac{1}{8k_0}<t\le\dfrac12\end{cases},$$

and $k_0\in N$ is such that

$$\frac{4}{(128+\varepsilon)\alpha_{k_0}}<1. \tag{2.15}$$

Then it is clear that $u_0^{(j)}(0)=u_0^{(j)}(1)$ $(j=0,1,2)$, and there exist constants $\lambda_1>0$, $\lambda_2>0$ such that

$$\Big(\frac{\lambda_1}{4}+\lambda_2\Big)\int_0^1 |u_0'''(s)|\,ds = 1-\frac{4}{(128+\varepsilon)\alpha_{k_0}}\,. \tag{2.16}$$

Now, let the measurable function $\tau:[0,1]\to[0,1]$ and the linear operators $\ell_i: C([0,1])\to L([0,1])$, $(i=0,1,2)$ be given by the equalities

$$\tau(t)=\begin{cases}0 & \text{for } \ u_0'''(t)>0\\ 1/2 & \text{for } \ u_0'''(t)\le 0\end{cases},$$

$$\ell_0(x)(t)=|u_0'''(t)|x(\tau(t)),\quad \ell_i(x)(t)=\lambda_i|u_0'''(t)|x\Big(\frac{i-1}{4}\Big)\ \ (i=1,2).$$

From (2.15) and (2.16) it follows that

$$1-\frac{1}{4}\|\ell_1\|-\|\ell_2\|=1-\Big(\frac{\lambda_1}{4}+\lambda_2\Big)\int\limits_0^1|u_0'''(s)|\,ds=\frac{4}{(128+\varepsilon)\alpha_{k_0}}$$

and

$$\int\limits_0^1\ell_0(1)(s)\,ds=\int\limits_0^1|u_0'''(s)|\,ds=\frac{4}{\alpha_{k_0}}=$$
$$=128\frac{4}{(128+\varepsilon)\alpha_{k_0}}+\frac{4\varepsilon}{(128+\varepsilon)\alpha_{k_0}}<128\Big(1-\frac{1}{4}\|\ell_1\|-\|\ell_2\|\Big)+\varepsilon.$$

Thus, all the assumptions of Corollaries 2.1 and 2.5 are satisfied except (2.6), resp. (2.14), instead of which the condition

$$\int\limits_0^\omega|\ell_0(1)(s)|\,ds\le\frac{128}{\omega^2}\Big(1-\frac{\omega}{4}\|\ell_1\|-\|\ell_2\|\Big)+\varepsilon$$

is fulfilled.

On the other hand, from the definition of the functions u_0, τ and the operators ℓ_i it follows that

$$u_0'''(t)=|u_0'''(t)|\operatorname{sgn}u_0'''(t)=|u_0'''(t)|u_0(\tau(t))=\ell_0(u_0)(t),$$
$$\ell_1(u_0')(t)+\ell_2(u_0'')(t)=\big(\lambda_1u_0'(0)+\lambda_2u_0''(1/4)\big)|u_0'''(t)|=0,$$

that is, u_0 and $u_1(t)\equiv0$ are different solutions of the problem (1.1), (1.2) with $\omega=1$, $q(t)\equiv0$.

3. Proofs

Let $v\in\widetilde C^2\big([0,\omega]\big)$. Then for $j=0,1,2$ put

$$M_j=\max\big\{v^{(j)}(t):\ t\in[0,\omega]\big\},\quad m_j=\max\big\{-v^{(j)}(t):t\in[0,\omega]\big\}.\tag{3.1}$$

The following lemma is a consequence of a more general result obtained in [9] (see Theorem 1.1 and Remark 1.1 therein).

Lemma 3.1. *Let $k\in\{0,1\}$, $v\in\widetilde C^2\big([0,\omega]\big)$, and*

$$v(t)\not\equiv\text{const},\quad v^{(j)}(0)=v^{(j)}(\omega)\ \ (j=0,1,2).$$

Then the estimate

$$M_k+m_k<\frac{\omega^{2-k}}{d_{2-k}}(M_2+m_2)$$

holds, where $d_1=4$ and $d_2=32$.

Lemma 3.2. *Let $\ell:C\big([0,\omega]\big)\to L\big([0,\omega]\big)$ be a nonnegative linear operator. Then for an arbitrary $v\in C\big([0,\omega]\big)$ the inequalities*

$$-m_0\ell(1)(t)\le\ell(v)(t)\le M_0\ell(1)(t)\ \ \text{for}\ \ 0\le t\le\omega$$

are fulfilled, where M_0 and m_0 are defined by (3.1).

Proof. It is clear that

$$v(t) - M_0 \le 0, \quad v(t) + m_0 \ge 0 \ \text{ for } \ 0 \le t \le \omega.$$

Then from nonnegativity of ℓ we get

$$\ell(v - M_0)(t) \le 0, \quad \ell(v + m_0)(t) \ge 0 \ \text{ for } \ 0 \le t \le \omega,$$

whence follows the validity of our lemma. □

The next lemma on the Fredholm property, in the case where ℓ_1 and ℓ_2 are bounded operators (not necessarily strongly bounded), immediately follows from [10, Theorem 1.1].

Lemma 3.3. *The problem* (1.1), (1.2) *is uniquely solvable iff the corresponding homogeneous problem*

$$v'''(t) = \sum_{i=0}^{2} \ell_i(v^{(i)})(t), \tag{3.2}$$

$$v^{(j)}(0) = v^{(j)}(\omega) \ \ (j = 0, 1, 2) \tag{3.3}$$

has only the trivial solution.

Lemma 3.4. *Let there exist α_1, $\alpha_2 \in R_+$ such that for every $x \in C([0,\omega])$ assuming both positive and negative values, the inequality*

$$\Big| \int_E \ell_j(x)(s)\, ds \Big| \le \alpha_j \max\big\{x(s_1) - x(s_2) : \ 0 \le s_1, s_2 \le \omega\big\} \ \ (j = 1, 2) \tag{3.4}$$

holds, where $E \subseteq [0,\omega]$ is an arbitrary measurable set. Let, moreover, ℓ_{01}, $\ell_{02} : C([0,\omega]) \to L([0,\omega])$ be nonnegative operators such that

$$\ell_{01}(1)(t) \not\equiv \ell_{02}(1)(t), \tag{3.5}$$

$\beta = 1 - \omega\alpha_1/4 - \alpha_2$, and

$$\frac{\omega^2}{32} \int_0^\omega \ell_{01}(1)(s)\, ds < \beta, \tag{3.6}$$

$$\frac{\int_0^\omega \ell_{01}(1)(s)\, ds}{\beta - \frac{\omega^2}{32}\int_0^\omega \ell_{01}(1)(s)\, ds} \le \int_0^\omega \ell_{02}(1)(s)\, ds, \tag{3.7}$$

$$\int_0^\omega \ell_{02}(1)(s)\, ds \le \frac{64\beta}{\omega^2}\left(1 + \sqrt{1 - \frac{\omega^2}{32\beta}\int_0^\omega \ell_{01}(1)(s)\, ds}\right). \tag{3.8}$$

Then the problem (3.2), (3.3) *with both*

$$\ell_0 = \ell_{01} - \ell_{02} \ \ \text{and} \ \ \ell_0 = \ell_{02} - \ell_{01}$$

has only the trivial solution.

Proof. First suppose that $\ell_0 = \ell_{01} - \ell_{02}$ and assume on the contrary that the problem (3.2), (3.3) has a nontrivial solution v. Let M_j and m_j $(j = 0, 1, 2)$ be defined by (3.1) and t_1, $t_2 \in [0, \omega[$ be such that

$$v''(t_1) = M_2, \quad v''(t_2) = -m_2. \tag{3.9}$$

By virtue of (3.5) we have that $v(t) \not\equiv \text{const}$. Therefore, in view of (3.3), it is clear that v' and v'' assume both positive and negative values, and thus

$$M_j > 0, \quad m_j > 0 \ \ (j = 1, 2). \tag{3.10}$$

According to (3.10), Lemma 3.1 with $k = 1$, and (3.4) we have

$$\left| \int_E \ell_1(v')(s) + \ell_2(v'')(s)\, ds \right| \le (1 - \beta)(M_2 + m_2) \tag{3.11}$$

for an arbitrary measurable set $E \subseteq [0, \omega]$.

Assume that $t_1 < t_2$ and let $I = [0, t_1] \cup [t_2, \omega]$. Then, in view of (3.9) and (3.3), it is clear that

$$\int_I v'''(s)\, ds = M_2 + m_2, \quad \int_{t_1}^{t_2} v'''(s)\, ds = -(M_2 + m_2).$$

Hence, on account of (3.2) and (3.11), we get

$$\beta(M_2 + m_2) \le \int_I \big(\ell_{01}(v)(s) - \ell_{02}(v)(s)\big)\, ds, \tag{3.12}$$

$$\beta(M_2 + m_2) \le \int_{t_1}^{t_2} \big(\ell_{02}(v)(s) - \ell_{01}(v)(s)\big)\, ds. \tag{3.13}$$

From (3.6) it follows that $\beta > 0$, and, in view of the fact that $v(t) \not\equiv \text{const}$, we have $\beta(M_0 + m_0) > 0$. Now the inequalities (3.12) and (3.13), according to Lemma 3.1 with $k = 0$, yield

$$0 < \beta(M_0 + m_0) < \frac{\omega^2}{32} \int_I \big(\ell_{01}(v)(s) - \ell_{02}(v)(s)\big)\, ds, \tag{3.14}$$

$$0 < \beta(M_0 + m_0) < \frac{\omega^2}{32} \int_{t_1}^{t_2} \big(\ell_{02}(v)(s) - \ell_{01}(v)(s)\big)\, ds, \tag{3.15}$$

respectively.

On the other hand, integration of (3.2) from 0 to ω, on account of (3.3) and (3.11), results in

$$0 \le (-1)^{k-1} \int_0^{\omega} \big(\ell_{01}(v)(s) - \ell_{02}(v)(s)\big)\, ds + (1 - \beta)(M_2 + m_2), \tag{3.16$_k$}$$

where $k = 1, 2$.

Now we will show that v assumes both positive and negative values. Assume on the contrary that v does not change its sign. It is sufficient to discuss the following two cases:

$$t_1 < t_2, \quad (-1)^{k-1}v(t) \ge 0 \ \text{ for } \ 0 \le t \le \omega \ \ (k = 1, 2). \tag{3.17$_k$}$$

If (3.17_1) is satisfied, then from (3.12), (3.14), and (3.16_1), in view of Lemma 3.2 (since ℓ_{01} and ℓ_{02} are nonnegative operators), we obtain

$$\beta(M_2 + m_2) \le M_0 \int_0^\omega \ell_{01}(1)(s)\,ds, \tag{3.18}$$

$$M_0\left(\beta - \frac{\omega^2}{32}\int_0^\omega \ell_{01}(1)(s)\,ds\right) < -m_0\beta, \tag{3.19}$$

$$-m_0 \int_0^\omega \ell_{02}(1)(s)\,ds \le M_0 \int_0^\omega \ell_{01}(1)(s)\,ds + (1-\beta)(M_2 + m_2). \tag{3.20}$$

From (3.18) and (3.20) we get

$$-m_0\beta \int_0^\omega \ell_{02}(1)(s)\,ds \le M_0 \int_0^\omega \ell_{01}(1)(s)\,ds,$$

which, together with (3.19), contradicts (3.7).

If (3.17_2) is satisfied, from (3.13), (3.15), and (3.16_2) we can obtain a contradiction to (3.7) in an analogous way.

Consequently, the contradictions obtained guarantee that v changes its sign and thus

$$M_0 > 0, \quad m_0 > 0. \tag{3.21}$$

Without loss of generality we can assume that $t_1 < t_2$, and therefore the inequalities (3.12)–(3.16_k) hold. Now, from (3.14) and (3.15), according to (3.21) and Lemma 3.2, we obtain

$$\beta M_0 + \beta m_0 < M_0\frac{\omega^2}{32}\int_I \ell_{01}(1)(s)\,ds + m_0\frac{\omega^2}{32}\int_I \ell_{02}(1)(s)\,ds, \tag{3.22}$$

$$\beta M_0 + \beta m_0 < M_0\frac{\omega^2}{32}\int_{t_1}^{t_2} \ell_{02}(1)(s)\,ds + m_0\frac{\omega^2}{32}\int_{t_1}^{t_2} \ell_{01}(1)(s)\,ds. \tag{3.23}$$

The inequalities (3.22) and (3.23), by virtue of (3.6) and (3.21), yield

$$0 < M_0\left(\beta - \frac{\omega^2}{32}\int_I \ell_{01}(1)(s)\,ds\right) < m_0\left(\frac{\omega^2}{32}\int_I \ell_{02}(1)(s)\,ds - \beta\right),$$

$$0 < m_0\left(\beta - \frac{\omega^2}{32}\int_{t_1}^{t_2} \ell_{01}(1)(s)\,ds\right) < M_0\left(\frac{\omega^2}{32}\int_{t_1}^{t_2} \ell_{02}(1)(s)\,ds - \beta\right).$$

Multiplying the last two inequalities, on account of the inequality $4AB \leq (A+B)^2$, we get

$$\beta^2 - \beta\frac{\omega^2}{32}\int\limits_0^\omega \ell_{01}(1)(s)\,ds < \frac{1}{4}\left(\frac{\omega^2}{32}\int\limits_0^\omega \ell_{02}(1)(s)\,ds - 2\beta\right)^2,$$

which contradicts (3.8). Therefore our assumption is invalid and the problem (3.2), (3.3) with $\ell_0 = \ell_{01} - \ell_{02}$ has only the trivial solution.

Now let $\ell_0 = \ell_{02} - \ell_{01}$. Put for every $x : [0,\omega] \to R$

$$\vartheta(x)(t) \overset{def}{=} x(\omega - t) \ \text{ for } \ 0 \leq t \leq \omega,$$

and

$$\widetilde{\ell}_{0i}(x)(t) \overset{def}{=} \vartheta\big(\ell_{0i}(\vartheta(x))\big)(t), \ \ \widetilde{\ell}_i(x)(t) \overset{def}{=} (-1)^{i-1}\vartheta\big(\ell_i(\vartheta(x))\big)(t) \ \ (i=1,2).$$

Then, obviously, the operators $\widetilde{\ell}_{0i}$ $(i = 1, 2)$ are also nonnegative and

$$\|\widetilde{\ell}_{0i}\| = \|\ell_{0i}\|, \quad \|\widetilde{\ell}_i\| = \|\ell_i\| \ \ (i = 1, 2). \tag{3.24}$$

On the other hand, v_1 is a solution of (3.2), (3.3) iff

$$v_2(t) \overset{def}{=} \vartheta(v_1)(t)$$

is a solution of the problem

$$v'''(t) = \sum_{i=0}^{2} \widetilde{\ell}_i(v^{(i)})(t), \quad v^{(j)}(0) = v^{(j)}(\omega) \ \ (j = 0, 1, 2) \tag{3.25}$$

with

$$\widetilde{\ell}_0 = \widetilde{\ell}_{01} - \widetilde{\ell}_{02}.$$

From (3.24) it follows that all the assumptions of the lemma are fulfilled for the problem (3.25). However, (3.25) has only the trivial solution, as was shown in the first part of this proof. Consequently, the problem (3.2), (3.3) has also only the trivial solution. □

Proof of Theorem 2.1. Put $\alpha_1 = \|\ell_1\|$, $\alpha_2 = \|\ell_2\|$. Then all the assumptions of Lemma 3.4 are fulfilled, and the conclusion of theorem follows from Lemma 3.3. □

Proof of Theorem 2.2. Put

$$\alpha_1 = \max\big\{\|\ell_{11}\|, \|\ell_{12}\|\big\}, \quad \alpha_2 = \max\big\{\|\ell_{21}\|, \|\ell_{22}\|\big\}.$$

Then all the assumptions of Lemma 3.4 are fulfilled, and the conclusion of theorem follows from Lemma 3.3. □

Corollary 2.1 follows immediately from Theorem 2.2 with $\ell_{01} \equiv 0$.

Proof of Corollary 2.2. Let

$$\ell_i(x)(t) \overset{def}{=} p_i(t)x(\tau_i(t)) \ \ (i=0,1,2),$$

$$\ell_{01}(x)(t) \overset{def}{=} p_{01}(t)x(\tau_0(t)), \quad \ell_{02}(x)(t) \overset{def}{=} p_{02}(t)x(\tau_0(t)),$$

$$\ell_{i1}(x)(t) \overset{def}{=} [p_i(t)]_+x(\tau_i(t)), \quad \ell_{i2}(x)(t) \overset{def}{=} [p_i(t)]_-x(\tau_i(t)) \ \ (i=1,2).$$

Then

$$\|\ell_{i1}\| = \int_0^\omega [p_i(t)]_+dt, \quad \|\ell_{i2}\| = \int_0^\omega [p_i(t)]_-dt \ \ (i=1,2)$$

and the conditions (2.7)–(2.11) yield the conditions (2.1)–(2.4) with β_1 defined by (2.5). Consequently, the assumptions of Theorem 2.2 are fulfilled. □

Acknowledgement

The research was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503 and the Grant No. 201/06/0254 of the Grant Agency of the Czech Republic.

References

1. P. Amster, P. De Nápoli, and M. C. Mariani, Periodic solutions of a resonant third-order equation. *Nonlinear Anal.* **60**(2005), No. 3, 399–410.
2. S. R. Baslandze and I. T. Kiguradze, On the unique solvability of a periodic boundary value problem for third-order linear differential equations. (Russian) *Differ. Uravn.* **42**(2006), No. 2, 153–158; English transl.: *Differ. Equ.* **42**(2006), No. 2, 165–171.
3. E. Bravyi, A note on the Fredholm property of boundary value problems for linear functional differential equations. *Mem. Differential Equations Math. Phys.* **20**(2000), 133–135.
4. A. Cabada, The method of lower and upper solutions for second, third, fourth, and higher order boundary value problems. *J. Math. Anal. Appl.* **185**(1994), No. 2, 302–320.
5. A. Cabada, The method of lower and upper solutions for third-order periodic boundary value problems. *J. Math. Anal. Appl.* **195**(1995), No. 2, 568–589.
6. G. T. Gegelia, On boundary value problems of periodic type for ordinary odd order differential equations. *Arch. Math.* (*Brno*) **20**(1984), No. 4, 195–203.
7. R. Hakl, Periodic boundary value problem for third order linear functional differential equations. *Nonlinear Oscil.* (in press).
8. R. Hakl, A. Lomtatidze, and I. P. Stavroulakis, On a boundary value problem for scalar linear functional differential equations. *Abstr. Appl. Anal.*, **2004**, No. 1, 45–67.
9. R. Hakl and S. Mukhigulashvili, On one estimate for periodic functions. *Georgian Math. J.* **12**(2005), No. 1, 97–114.
10. R. Hakl and S. Mukhigulashvili, On a boundary value problem for n-th order linear functional differential systems. *Georgian Math. J.* **12**(2005), No. 2, 229–236.
11. I. Kiguradze, Bounded and periodic solutions of higher-order linear differential equations. (Russian) *Mat. Zametki* **37**(1985), No. 1, 48–62.

12. I. Kiguradze, On periodic solutions of nth order ordinary differential equations. *Nonlinear Anal.* **40**(2000), No. 1-8, *Ser. A: Theory Methods*, 309–321.
13. I. Kiguradze and T. Kusano, On periodic solutions of higher-order nonautonomous ordinary differential equations. (Russian) *Differ. Uravn.* **35**(1999), No. 1, 72–78; English transl.: *Differential Equations* **35**(1999), No. 1, 71–77.
14. I. Kiguradze and T. Kusano, On conditions for the existence and uniqueness of a periodic solution of nonautonomous differential equations. (Russian) *Differ. Uravn.* **36**(2000), No. 10, 1301–1306; English transl.: *Differ. Equ.* **36**(2000), No. 10, 1436–1442.
15. I. Kiguradze and T. Kusano, On periodic solutions of even-order ordinary differential equations. *Ann. Mat. Pura Appl. (4)* **180**(2001), No. 3, 285–301.
16. I. Kiguradze and B. Půža, On boundary value problems for systems of linear functional-differential equations. *Czechoslovak Math. J.* **47(122)**(1997), No. 2, 341–373.
17. A. Lasota and Z. Opial, Sur les solutions périodiques des équations différentielles ordinaires. *Ann. Polon. Math.* **16**(1964), 69–94.
18. A. Lasota and F. H. Szafraniec, Sur les solutions périodiques d'une équation différentielle ordinaire d'ordre n. *Ann. Polon. Math.* **18**(1966), 339–344.
19. A. Lomtatidze and S. Mukhigulashvili, On periodic solutions of second order functional differential equations. *Mem. Differential Equations Math. Phys.* **5**(1995), 125–126.
20. S. Mukhigulashvili, On a periodic boundary value problem for second-order linear functional differential equations. *Bound. Value Probl.*, **2005**, No. 3, 247–261.
21. S. Mukhigulashvili, On periodic solutions of second order functional differential equations. *Ital. J. Pure Appl. Math.*, **2006**, No. 20, 29–50.
22. S. Mukhigulashvili, On the solvability of the periodic problem for nonlinear second-order function-differential equations. *Differ. Uravn.* **42**(2006), No. 3, 356–365; English transl.: *Differ. Equ.* **42**(2006), No. 3, 380–390.
23. S. Mukhigulashvili, On a periodic boundary value problem for third order linear functional differential equations. *Nonlinear Anal.* **66**(2007), No. 2, *Ser. A: Theory Methods*, 527–535.
24. S. Mukhigulashvili and B. Půža, On a periodic boundary value problem for third order linear functional differential equations. *Funct. Differ. Equ.* (in press).
25. J. J. Nieto, Periodic solutions for third order ordinary differential equations. *Comment. Math. Univ. Carolin.* **32**(1991), No. 3, 495–499.
26. P. Omari and M. Trombetta, Remarks on the lower and upper solutions method for second- and third-order periodic boundary value problems. *Appl. Math. Comput.* **50**(1992), No. 1, 1–21.
27. A. Ronto and A. Samoĭlenko, On the unique solvability of some boundary value problems for linear functional-differential equations in a Banach space. (Russian) *Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki*, **2002**, No. 9, 42–46.
28. H. H. Schaefer, Normed tensor products of Banach lattices. *Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces* (*Jerusalem,* 1972)*, Israel J. Math.* **13**(1972), 400–415.
29. G. Villari, Soluzioni periodiche di una classe di equazioni differenziali del terz'ordine quasi lineari. *Ann. Mat. Pura Appl. (4)* **73**(1966), 103–110.
30. M. Wypich, On the existence of a solution of nonlinear periodic boundary value problems. *Univ. Iagel. Acta Math.* **1991**, No. 28, 289–295.

(Received 14.02.2007)

Authors' addresses:

R. Hakl
Institute of Mathematics
Academy of Sciences of the Czech Republic
Žižkova 22, 616 62 Brno
Czech Republic
e-mail: hakl@ipm.cz

S. Mukhigulashvili
Current address:
Mathematical Institute
Academy of Sciences of the Czech Republic
Žižkova 22, 616 62 Brno
Czech Republic
E-mail: mukhig@ipm.cz

Permanent address:
A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: smukhig@rmi.acnet.ge