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NECESSARY CONDITIONS OF OPTIMALITY FOR NEUTRAL VARIABLE STRUCTURE OPTIMAL PROBLEMS WITH DISCONTINUOUS INITIAL CONDITION

Abstract. An optimal control problem for variable structure dynamical systems governed by quasi-linear neutral differential equations with discontinuous initial condition is considered. The discontinuity of the initial condition means that at the initial moment the values of the initial function and the trajectory, generally speaking, do not coincide. Necessary conditions of optimality are obtained: for the optimal control and the initial function in the form of integral maximum principle; for the optimal initial, final and structure changing moments in the form of equalities and inequalities containing discontinuity effects. Besides, a variable structure neutral time-optimal linear problem of economical character is investigated.

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Key words and phrases. Neutral systems, optimal control, necessary conditions, variable structure systems, discontinuous initial condition.

რეზიუმე. განხილულია ოპტიმალური მართვის ამოცანა ცვლადსტრუქტურიანი დინამიკური სისტემებისათვის, რომლებიც აღიწერებიან წყვეტილი საწყისი პირობის შემცველი კვაზი-წრფივი ნეიტრალური ტიპის დიფერენციალური განტოლებებით. საწყისი პირობის წყვეტილობა ნიშნავს, რომ საწყის მომენტში საწყისი ფუნქციისა და ტრაექტორიის მნიშვნელობები, საზოგადოდ, არ ემთხვევა ერთმანეთს. მიღებულია ოპტიმალურობის აუცილებელი პირობები: ინტეგრალური მაქსიმუმის პრინციპის ფორმით მართვისა და საწყისი ფუნქციისათვის; უტოლობებისა და ტოლობების სახით საწყისი, საბოლოო და სისტემის გადართვის მომენტებისათვის, რომლებიც შეიცავენ წყვეტილობის ეფექტებს. გარდა ამისა გამოკვლეულია ეკონომიკური შინაარსის წრფივი ნეიტრალური ცვლადსტრუქტურიანი ოპტიმალური ამოცანა.

1. Introduction

Investigations of variable structure optimal control problems with delay is one of the important directions of the optimal control theory. The delay factor may arise in many practical problems in connection with expenditure of time for signal transmission. Variation of the structure of a system means that the system at some beforehand unknown moment may go over from one law of movement to another. Moreover, after variation of the structure the initial condition of the system depends on its previous state. This joins them into a single system with variable structure. Assume that the change of the system structure has to take place at a priori unknown moments of time. Such problems are important for various practical applications. For example, in economics it is needed to change invested capital at some unknown moments. In engineering a controlled apparatus is to start from another controlled apparatus, which may be cosmic, ground, submarine and etc. Optimal control problems for various classes of variable structure systems are investigated in [1]–[17]. Optimal problems for some classes of neutral differential equations and differential inclusions with discontinuous initial condition are considered in [18]–[22].

This work deals with necessary conditions of optimality for quasi-linear neutral variable structure control systems.

The rest of the paper is organized as follows. In Section 2 all necessary notation and auxiliary assertions are given. Therein necessary conditions of criticality are formulated in the form of Theorem 2.5, on the bases of which the main theorem is proved in Section 5.

In Section 3 the following control problem for neutral variable structure systems with discontinuous initial condition is considered:

$$\begin{aligned} \dot{x}_i(t) = \sum_{j=1}^{k_i} A_{ij}(t)\dot{x}_i(\eta_{ij}(t)) + \\ + f_i\big(t, x_i(\tau_{i1}(t)), \dots, x_i(\tau_{is_i}(t)), u_i(t)\big), \quad t \in [t_i, t_{i+1}], \qquad (1.1_i) \\ x_i(t) = \varphi_i(t), \ \ t \in [\tau_i, t_i), \quad x_i(t_i) = x_{i0} + g_i(t_i, x_{i-1}(t_i)), \qquad (1.2_i) \\ i = \overline{1, m} \quad (g_1 = 0) \end{aligned}$$

under the restrictions

$$q^p\big(t_1, \dots, t_{m+1}, x_{10}, \dots, x_{m0}, x_m(t_{m+1})\big) = 0, \quad p = \overline{1, m},$$

and the functional

$$q^0\big(t_1, \dots, t_{m+1}, x_{10}, \dots, x_{m0}, x_m(t_{m+1})\big) \to \min .$$

The set of differential equations (1.1_i), $i = \overline{1, m}$, is called a variable structure system. The initial conditions (1.2_i), $i = \overline{1, m}$, are called discontinuous since, generally speaking, $\varphi_i(t_i) \neq x_i(t_i)$.

Connection among solutions of equations of variable structure system is fulfilled by the conditions

$$x_i(t_i) = x_{i0} + g_i(t_i, x_{i-1}(t_i)), \quad i = \overline{1, m}.$$

The problem consists in finding an optimal element

$$(\widetilde{t}_1, \dots, \widetilde{t}_{m+1}, \widetilde{x}_{10}, \dots, \widetilde{x}_{m0}, \widetilde{\varphi}_1, \dots, \widetilde{\varphi}_m, \widetilde{u}_1, \dots, \widetilde{u}_m).$$

In that section main theorems are also formulated.

In Section 4 an economical problem of optimal distribution of the invested capital and determination of optimal investment periods for various branches of the economy are considered as an application of the results provided in the preceding section. The factor that should be taken into account is that the number of branches may be varying across investment periods. If we take into account the fact that the investment effects become actually appreciable in the course of a long time (delay), as well as the recurrent process factor, then such a process of economic development can be described as a linear neutral variable structure optimal control problem. For the formulated economical problem Theorem 4.1 is established (necessary conditions of optimality), which is a corollary of Theorem 3.3.

In Section 5 Theorem 3.1 is proved by the methods given in [23], [24].

2. Notation and Auxiliary Assertions

Let $J = [a, b] \subset R$ be a finite interval and $i \in \{1, \dots, m\}$; let R^{n_i} be the n_i-dimensional vector space of points $x_i = (x_i^1, \dots, x_i^{n_i})^*$, where $*$ is the sign of transposition; let $O_i \subset R^{n_i}$ be an open set and let E_{f_i} be the set of functions $f_i : J \times O_i^{s_i} \to R^{n_i}$ satisfying the following conditions: for almost all $t \in J$ the function $f_i(t, \cdot) : O_i^{s_i} \to R^{n_i}$ is continuously differentiable, for each $(x_{i1}, \dots, x_{is_i}) \in O_i^{s_i}$ the functions

$$f_i(t, x_{i1}, \dots, x_{is_i}), \quad \frac{\partial f_i(\cdot)}{\partial x_{ij}}, \quad j = \overline{1, s_i},$$

are measurable on J and for any function $f_i \in E_{f_i}$ and any compact $K_i \subset O_i$ there exists $m_{f_i,K_i}(\cdot) \in L(J, R_+)$, $R_+ = (0, \infty)$, such that for any $(x_{i1}, \dots, x_{is_i}) \in K_i^{s_i}$ and almost all $t \in J$

$$\big|f_i(t, x_{i1}, \dots, x_{is_i})\big| + \sum_{j=1}^{s_i} \left|\frac{\partial f_i(\cdot)}{\partial x_{ij}}\right| \le m_{f_i,K_i}(t).$$

Functions $f_{i1}, f_{i2} \in E_{f_i}$ will be called equivalent if for any fixed $(x_{i1}, \dots, x_{is_i}) \in O_i^{s_i}$ and almost all $t \in J$

$$f_{i1}(t, x_{i1}, \dots, x_{is_i}) - f_{i2}(t, x_{i1}, \dots, x_{is_i}) = 0.$$

The classes of equivalent functions of the set E_{f_i} form a vector space which will be denoted also by E_{f_i}; we will call also these classes functions and denote them also by f_i.

In the space E_{f_i} we introduce the family of subsets

$$B = \{V_{K_i,\delta} : K_i \subset O_i, \ \ \delta > 0\}.$$

Here K_i is an arbitrary compact set, $\delta > 0$ is an arbitrary number,

$$V_{K_i,\delta} = \{\delta f_i \in E_{f_i} : H(\delta f_i; K_i) \le \delta\},$$

$$H(\delta f_i; K_i) = \sup \left\{ \left| \int_{t'}^{t''} \delta f_i(t, x_{i1}, \dots, x_{is_i}) dt \right| : x_{ij} \in K_i, \ j = \overline{1, s_i}, \ t', t'' \in J \right\}.$$

The family B can be accepted as a basis of neighborhoods of zero of the space E_{f_i}. Hence, it defines uniquely a locally convex separate vector topology which transforms E_{f_i} into a topological vector space [25]. In what follows, we will suppose that the space E_{f_i} is supplied with this topology.

Let $\tau_{ij}(t)$, $t \in R$, $j = \overline{1, s_i}$, be absolutely continuous scalar functions satisfying the conditions $\tau_{ij}(t) \le t$, $\dot{\tau}_{ij}(t) > 0$; $\gamma_{ij}(t)$ be the inverse function to $\tau_{ij}(t)$. Next, let $\eta_{ij}(t)$, $t \in R$, $j = \overline{1, k_i}$, be continuously differentiable scalar functions satisfying the conditions $\eta_{ij}(t) < t$, $\dot{\eta}_{ij}(t) > 0$; $\rho_{ij}(t)$ be the inverse function to $\eta_{ij}(t)$; E_{φ_i} be the space of continuously differentiable functions $\varphi_i : J_i = [\tau_i, b] \to R^{n_i}$, $\tau_i = \min\{\tau_{i1}(a), \dots, \tau_{is_i}(a), \eta_{i1}(a), \dots, \eta_{ik_i}(a)\}$, with the norm $\|\varphi_i\| = |\varphi_i(a)| + \max\limits_{t \in J_i} |\dot{\varphi}_i(t)|$; $\Delta_i = \{\varphi_i \in E_{\varphi_i} : \varphi_i(t) \in O_i\}$ be the set of initial functions; $A_{ij}(t)$, $j = \overline{1, k_i}$, $t \in J$, be $n_i \times n_i$-dimensional continuous matrix functions; $g_i(t_i; x_{i-1}), (t_i, x_{i-1}) \in J \times O_{i-1}$, $i = \overline{2, m}$, be continuously differentiable functions.

Lemma 2.1 ([24], p. 10)**.** *Let $f_{i1}, f_{i2} \in E_{f_i}$ be equivalent functions. Then for any piecewise continuous function** $x_i(t) \in O_i$, $t \in J$, *the following equality is fulfilled*

$$\left| \int_{t'}^{t''} \Big[f_{i1}\big(t, x_i(\tau_{i1}(t)), \dots, x_i(\tau_{is_i}(t))\big) - \right.$$

$$\left. - f_{i2}\big(t, x_i(\tau_{i1}(t)), \dots, x_i(\tau_{is_i}(t))\big) \Big] dt \right| = 0, \quad \forall \ t', t'' \in J.$$

Now we introduce the set

$$B_i = \Big\{ \mu_i = (t_1, \dots, t_{i+1}, x_{10}, \dots, x_{i0}, \varphi_1, \dots, \varphi_i, f_1, \dots, f_i) \in J^{1+i} \times$$

$$\times \prod_{j=1}^{i} O_j \times \prod_{j=1}^{i} \Delta_j \times \prod_{j=1}^{i} E_{f_j} : t_1 < \dots < t_{i+1} \Big\}, \quad i = \overline{1, m},$$

where

$$\prod_{j=1}^{i} O_j = O_1 \times \dots \times O_i.$$

* Everywhere we assume that piecewise continuous functions have finite number of discontinuity points of the first kind.

To each element $\mu_m \in B_m$ we assign the set of differential equations

$$\dot{x}_i(t) = \sum_{j=1}^{k_i} A_{ij}(t)\dot{x}_i(\eta_{ij}(t)) + \\ + f_i\big(t, x_i(\tau_{ij}(t), \dots, x_i(\tau_{is_i}(t))\big), \quad t \in [t_i, t_{i+1}], \tag{2.1_i}$$

$$x_i(t) = \varphi_i(t), t \in [\tau_i, t_i), x_i(t_i) = x_{i0} + g_i(t_i, x_{i-1}(t_i)), \quad i = \overline{1, m}. \tag{2.2_i}$$

Here and everywhere we suppose, that $g_1 = 0$. On the right-hand side of the equation (2.1_i) we suppose any function from the equivalence class.

Definition 2.1. Let

$$\mu_m = (t_1, \dots, t_{m+1}, x_{10}, \dots, x_{1m}, \varphi_1, \dots, \varphi_m, f_1, \dots, f_m) \in B_m.$$

The set of functions $\big\{x_i(t) = x_i(t; \mu_i) \in O_i,\ t \in [\tau_i, t_{i+1}]\colon\ i = \overline{1, m}\big\}$, where $\mu_i \in B_i$, is called a solution corresponding to the element μ_m if the function $x_i(t)$ satisfies the condition (2.2_i) on the interval $[\tau_i, t_i]$ and the equation

$$x_i(t) = x_{i0} + g_i(t_i, x_{i-1}(t_i)) + \int\limits_{t_i}^{t} \Big[\sum_{j=1}^{k_i} A_{ij}(\xi)\dot{x}_i(\eta_{ij}(\xi)) + \\ + f_i\big(\xi, x_i(\tau_{i1}(\xi)), \dots, x_i(\tau_{is_i}(\xi))\big)\Big] d\xi$$

on the interval $[t_i, t_{i+1}]$.

It is obvious that the function $x_i(t)$, $t \in [t_i, t_{i+1}]$, is absolutely continuous and satisfies the equation (2.1_i) almost everywhere.

It follows from the local theorem of existence and uniqueness for the neutral equation [26] that to each element μ_m there corresponds a unique solution if the numbers $t_{i+1} - t_i$, $i = \overline{1, m}$, are small enough.

On the basis of Lemma 2.1 we can conclude that if f_{i1} and f_{i2} are equivalent functions, then $x_i(t; \mu_{i1}) = x_i(t; \mu_{i2})$, where

$$\mu_{ij} = (t_1, \dots, t_{i+1}, x_{10}, \dots, x_{i0}, \varphi_1, \dots, \varphi_i, f_{1j}, \dots, f_{ij}), \quad j = 1, 2.$$

The following theorem about continuous dependence of solution on initial data and right-hand side is proved by repeated application of an analogue of Theorem 1.2.1 ([24], p. 11) for quasi-linear equations of neutral type with several variable delays.

Theorem 2.1. *Let $\big\{\widetilde{x}_i(t) = x_i(t; \widetilde{\mu}_i),\ t \in [\tau_i, \widetilde{t}_{i+1}] : i = \overline{1, m}\big\}$ be the solution corresponding to the element*

$$\widetilde{\mu}_m = \big(\widetilde{t}_1, \dots, \widetilde{t}_{m+1}, \widetilde{x}_{10}, \dots, \widetilde{x}_{m0}, \widetilde{\varphi}_1, \dots, \widetilde{\varphi}_m, \widetilde{f}_1, \dots, \widetilde{f}_m\big) \in B_m, \ \ \widetilde{t}_{m+1} < b,$$

and let $K_{i1} \subset O_i$ be a compact set containing some neighborhood of the set $K_{i0} = \widetilde{\varphi}_i(J_i) \cup \widetilde{x}_i\big([\widetilde{t}_i, \widetilde{t}_{i+1}]\big)$. Then the following assertions are valid:

2.1) *there exist numbers $\delta_i > 0$, $i = 0, 1$, such that to each element*

$$\mu_m \in V(\widetilde{\mu}_m; K_{11}, \dots, K_{m1}, \delta_0, \alpha_0) = \prod_{i=1}^{m+1} \big(V(\widetilde{t}_i; \delta_0) \cap J\big) \times$$

$$\times \prod_{i=1}^{m} \big(V(\widetilde{x}_{i0};\delta_0)\cap O_i\big) \times \prod_{i=1}^{m} \big(V(\widetilde{\varphi}_i;\delta_0)\cap \Delta_i\big)\times$$
$$\times \prod_{i=1}^{m} \bigg[\widetilde{f}_i + \bigg(W(K_{i1};\alpha_0)\cap V_{K_{i1},\delta_0}\bigg)\bigg] \subset B_m$$

there corresponds the solution $\{x_i(t),\ t\in[\tau_i,t_{i+1}] : i=\overline{1,m}\}$. *Moreover, the function* $x_i(t)$ *is defined on the interval* $[\tau_i,\widetilde{t}_{i+1}+\delta_1]\subset J$ *and on the interval* $[\widetilde{t}_i,\widetilde{t}_{i+1}+\delta_1]$ *it satisfies the equation* (2.1_i) *and takes values from* $\operatorname{int} K_{i1}$*;*

2.2) *for an arbitrary* $\varepsilon>0$ *there exists a number* $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0]$ *such that the inequality*

$$\big|x_i(t)-\widetilde{x}_i(t)\big|\le\varepsilon,\quad \forall\, t\in[\theta_i,\widetilde{t}_{i+1}+\delta_1],\ \ \theta_i=\max\{t_i,\widetilde{t}_i\},\quad i=\overline{1,m},$$

is valid for any $\mu_m\in V(\widetilde{\mu}_m;K_{11},\dots,K_{m1},\delta_2,\alpha_0)$.

Here

$$V(\widetilde{t}_i;\delta_0)=\big\{t_i\in R : |\widetilde{t}_i-t_i|<\delta_0\big\},$$
$$V(\widetilde{x}_{i0};\delta_0)=\big\{x_{i0}\in R^{n_i} : |\widetilde{x}_{i0}-x_{i0}|<\delta_0\big\},$$
$$V(\widetilde{\varphi}_i;\delta_0)=\big\{\varphi_i\in E_{\varphi_i} : \|\widetilde{\varphi}_i-\varphi_i\|<\delta_0\big\},$$
$$W(K_{i1};\alpha_0)=\big\{\delta f_i\in E_{f_i} : H_1(\delta f_i;K_{i1})<\alpha_0\big\},$$

where

$$H_1(\delta f_i;K_{i1})=\sup\bigg\{\int_J \bigg[\big|\delta f_i(t,x_{i1},\dots,x_{is_i})\big|+$$
$$+\sum_{j=1}^{s_i}\bigg|\frac{\partial\delta f_i(\cdot)}{\partial x_{ij}}\bigg|\bigg]dt : x_{ij}\in O_i,\ \ j=\overline{1,s_i}\bigg\},$$

$\alpha_0>0$ *is a given number.*

Remark 2.1. Theorem 2.1 is valid if the set $V(\widetilde{\mu}_m;K_{11},\dots,K_{m1},\delta_0,\alpha_0)$ is replaced by the set

$$V(\widetilde{\mu}_m;K_{11},\dots,K_{m1},\delta_0)=\prod_{i=1}^{m+1}\big(V(\widetilde{t}_i;\delta_0)\cap J\big)\times\prod_{i=1}^{m}\big(V(\widetilde{x}_{i0};\delta_0)\cap O_i\big)\times$$
$$\times\prod_{i=1}^{m}\big(V(\widetilde{\varphi}_i;\delta_0)\cap\Delta_i\big)\times\prod_{i=1}^{m}\big(\widetilde{f}_i+W(K_{i1};\delta_0)\big)$$

since

$$\widetilde{f}_i+W(K_{i1};\delta_0)\subset\widetilde{f}_i+W(K_{i1};\alpha_0)\cap V_{K_{i1},\delta_0},\quad 0<\delta_0\le\alpha_0.$$

In the space

$$E_{\mu_i}=R^{1+i}\times\prod_{j=1}^{i}R^{n_j}\times\prod_{j=1}^{i}E_{\varphi_j}\times\prod_{j=1}^{i}E_{f_j}$$

we denote the set of variations

$$V_i = \Big\{ \delta\mu_i = (\delta t_1, \dots, \delta t_{i+1}, \delta x_{10}, \dots, \delta x_{i0}, \delta\varphi_1, \dots, \delta\varphi_i,$$

$$\delta f_1, \dots, \delta f_i) \in E_{\mu_i} : |\delta t_j| \le \alpha_1, j = \overline{1, i+1}; \ \ |\delta x_{j0}| \le \alpha_1,$$

$$\|\delta\varphi_j\| \le \alpha_1, \ \ \delta f_j = \sum_{k=1}^{m_j} \lambda_{jk}\delta f_{jk}, \ \ |\lambda_{jk}| \le \alpha_1, \ \ j = \overline{1, i} \Big\},$$

where $\alpha_1 > 0$ is a fixed number, $\delta f_{jk} \in E_{f_j}$, $k = \overline{1, m_j}$, are fixed points.

The following lemma is a corollary to Theorem 2.1.

Lemma 2.2. *Let* $\{\widetilde{x}_i(t), \ t \in [\tau_i, \widetilde{t}_{i+1}] : i = \overline{1, m}\}$ *be the solution corresponding to the element* $\widetilde{\mu}_m \in B_m$, $\widetilde{t}_{m+1} < b$; $K_{i1} \subset O_i$ *be a compact set containing some neighborhood of the set* K_{i0}. *Then there exist such numbers* $\varepsilon_1 > 0$, $\delta_1 > 0$ *that for an arbitrary* $(\varepsilon, \delta\mu_m) \in [0, \varepsilon_1] \times V_m$ *the element* $\widetilde{\mu}_m + \varepsilon\delta\mu_m \in B_m$ *and the solution* $\{x_i(t; \widetilde{\mu}_i + \varepsilon\delta\mu_i), \ t \in [\tau_i, \widetilde{t}_{i+1} + \varepsilon\delta t_{i+1}] : i = \overline{1, m}\}$ *corresponds to it, where* $\delta\mu_i \in V_i$. *Moreover, the solution* $x_i(t; \widetilde{\mu}_i + \varepsilon\delta\mu_i)$ *is defined on the interval* $[\tau_i, \widetilde{t}_{i+1} + \delta_1] \subset J$, *takes values from* $intK_{i1}$ *and on the interval* $[\widetilde{t}_i + \varepsilon\delta t_i, \widetilde{t}_{i+1} + \delta_1]$ *satisfies the corresponding differential equation almost everywhere.*

By virtue of uniqueness, the solution $x_i(t; \widetilde{\mu}_i)$ on the interval $[\tau_i, \widetilde{t}_{i+1} + \delta_1]$ is a continuation of the solution $\widetilde{x}_i(t)$. Therefore, in what follows we assume that the solution $\widetilde{x}_i(t)$ is already defined on the entire interval $[\tau_i, \widetilde{t}_{i+1} + \delta_1]$.

Lemma 2.2 makes it possible to define the increment of the solution $\widetilde{x}_i(t) = x_i(t; \widetilde{\mu}_i) : \Delta x_i(t; \varepsilon\delta\mu_i) = x_i(t; \widetilde{\mu}_i + \varepsilon\delta\mu_i) - \widetilde{x}_i(t), \ \ \forall(t, \varepsilon, \delta\mu_i) \in [\tau_i, \widetilde{t}_{i+1} + \delta_1] \times [0, \varepsilon_1] \times V_i$.

Theorems provided below play an important role in the proof of necessary conditions of optimality.

In order to formulate the theorems about variation of solutions we need the following notation:

$$\omega_{ij}^0 = \big(\widetilde{t}_i, \underbrace{\widetilde{x}_i(\widetilde{t}_i), \dots, \widetilde{x}_i(\widetilde{t}_i)}_{j}, \underbrace{\widetilde{\varphi}_i(\widetilde{t}_i), \dots, \widetilde{\varphi}_i(\widetilde{t}_i)}_{p_i - j}, \widetilde{\varphi}(\tau_{ip_i+1})(\widetilde{t}_i)), \dots, \widetilde{\varphi}_i(\tau_{is_i}(\widetilde{t}_i)))\big),$$

$$j = \overline{0, p_i}.$$

The role of number p_i will be found out below. If $j = 0$, then w_{i0}^0 does not contain $\widetilde{x}_i(\widetilde{t}_i)$, and if $j = p_i$, then $w_{ip_i}^0$ does not contain $\widetilde{\varphi}_i(\widetilde{t}_i)$.

Further, $\gamma_{ij} = \gamma_{ij}(\widetilde{t}_i)$,

$$\omega_{ij}^1 = \big(\gamma_{ij}, \widetilde{x}_i(\tau_{i1}(\gamma_{ij})), \dots, \widetilde{x}_i(\tau_{ij-1}(\gamma_{ij})), \widetilde{x}_i(\widetilde{t}_i),$$
$$\widetilde{\varphi}_i(\tau_{ij+1}(\gamma_{ij})), \dots, \widetilde{\varphi}_i(\tau_{is_i}(\gamma_{ij}))\big),$$
$$\omega_{ij}^2 = \big(\gamma_{ij}, \widetilde{x}_i(\tau_{i1}(\gamma_{ij})), \dots, \widetilde{x}_i(\tau_{ij-1}(\gamma_{ij})),$$
$$\widetilde{\varphi}_i(\widetilde{t}_i), \widetilde{\varphi}_i(\tau_{ij+1}(\gamma_{ij})), \dots, \widetilde{\varphi}_i(\tau_{is_i}(\gamma_{ij}))\big), \ \ j = \overline{p_i + 1, s_i},$$
$$\omega_{i2} = \big(\widetilde{t}_{i+1}, \widetilde{x}_i(\tau_{i1}(\widetilde{t}_{i+1})), \dots, \widetilde{x}_i(\tau_{is_i}(\widetilde{t}_{i+1}))\big).$$

Theorem 2.2. *Let the following conditions be fulfilled*

2.3) $\gamma_{ij} = \widetilde{t}_i$, $j = \overline{1,p_i}$, $\gamma_{ip_i+1} < \cdots < \gamma_{is_i}$, $\rho_{ij}(\widetilde{t}_i) < \widetilde{t}_{i+1}$, $j = \overline{1,k_i}$, $i = \overline{1,m}$;

2.4) *there exists a number* $\delta > 0$ *such that*

$$\gamma_{i1}(t) \le \cdots \le \gamma_{ip_i}(t), \ \ t \in (\widetilde{t}_i - \delta, \widetilde{t}_i], \ \ i = \overline{1,m};$$

2.5) *there exist finite limits*

$$\dot{\gamma}_{ij}^- = \dot{\gamma}_{ij}(\widetilde{t}_i-), \ \ j = \overline{1,s_i}, \ \ i = \overline{1,m};$$

$$\dot{\widetilde{x}}_{ij}^- = \dot{\widetilde{x}}_{ij}(\eta_{ij}(\widetilde{t}_i-)), \ \ j = \overline{1,s_i}, \ \ i = \overline{1,m-1};$$

$$\lim_{\omega_i \to \omega_{ij}^0} \widetilde{f}_i(\omega_i) = f_{ij}^-, \ \ \omega_i = (t, x_{i1}, \dots, x_{is_i}) \in (\widetilde{t}_i - \delta, \widetilde{t}_i] \times O_i^{s_i}, \ \ j = \overline{0,p_i}$$

$$\lim_{(\omega_{i1},\omega_{i2}) \to (\omega_{ij}^1,\omega_{ij}^2)} \big[\widetilde{f}_i(\omega_{i1}) - \widetilde{f}_i(\omega_{i2})\big] = f_{ij}^-, \ \ \omega_{i1}, \omega_{i2} \in (\gamma_{ij} - \delta, \gamma_{ij}] \times O_i^{s_i},$$

$$j = \overline{p_i+1, s_i}, \ \ i = \overline{1,m};$$

$$\lim_{\omega_i \to \omega_i^2} \widetilde{f}_i(\omega_i) = f_{is_i+1}^-, \ \ \omega_i \in (\widetilde{t}_{i+1} - \delta, \widetilde{t}_{i+1}] \times O_i^{s_i} \ \ i = \overline{1,m-1}.$$

Then there exist numbers $\varepsilon_2 > 0$, $\delta_2 > 0$ *such that for an arbitrary*

$$(t, \varepsilon, \delta\mu_i) \in \big[\widetilde{t}_{i+1} - \delta_2, \widetilde{t}_{i+1} + \delta_2\big] \times [0, \varepsilon_2] \times V_i^-, \ \ i = \overline{1,m},$$

where

$$V_i^- = \big\{\delta\mu_i \in V_i : \delta t_j \le 0, \ \ j = \overline{1, i+1}\big\},$$

the following formulas

$$\Delta x_i(t; \varepsilon\delta\mu_i) = \varepsilon \delta x_i(t; \delta\mu_i) + o(t; \varepsilon\delta\mu_i),^* \ \ i = \overline{1,m}, \tag{2.3}$$

are valid, where

$$\delta x_i(t; \delta\mu_i) = \bigg\{ Y_i(\widetilde{t}_i-; t)\Big[\dot{\widetilde{\varphi}}_i(\widetilde{t}_i) - \sum_{j=1}^{k_i} A_{ij}(\widetilde{t}_i)\dot{\widetilde{\varphi}}_i(\eta_{ij}(\widetilde{t}_i) +$$
$$+ \sum_{j=0}^{p_i} (\widehat{\gamma}_{ij+1}^- - \widehat{\gamma}_{ij}^-) f_{ij}^-\Big] - \sum_{i=p_i+1}^{s_i} Y_i(\gamma_{ij}-; t) f_{ij}^- \dot{\gamma}_{ij}^- +$$
$$+ \Phi_i(\widetilde{t}_i; t) \frac{\partial \widetilde{g}_i}{\partial x_{i-1}} \Big[\sum_{j=1}^{k_{i-1}} A_{i-1j}(\widetilde{t}_i)\dot{\widetilde{x}}_{i-1j}^- + f_{i-1s_{i-1}+1}^-\Big]\bigg\} \delta t_i + \beta_i(t; \delta\mu_i); \tag{2.4}$$

$$\widehat{\gamma}_{i0}^- = 1, \ \ \widehat{\gamma}_{ij}^- = \dot{\gamma}_{ij}^-, \ \ j = \overline{1,p_i}, \ \ \widehat{\gamma}_{ip_i+1}^- = 0;$$

$$\beta_i(t; \delta\mu_i) = \Phi_i(\widetilde{t}_i; t)\Big[\delta x_{i0} - \dot{\widetilde{\varphi}}_i(\widetilde{t}_i)\delta t_i + \frac{\partial \widetilde{g}_i}{\partial t_i}\delta t_i +$$

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* Here and in the sequel the symbol $o(t, \varepsilon\delta\mu)$ means that $\lim\limits_{\varepsilon\to 0} o(t; \varepsilon\delta\mu)/\varepsilon = 0$, uniformly for $(t, \delta\mu)$.

$$+\frac{\partial\widetilde{g}_i}{\partial x_{i-1}}\delta x_{i-1}(\widetilde{t}_i;\delta\mu_i)\Big]+\sum_{j=p_i+1}^{s_i}\int\limits_{\tau_{ij}(\widetilde{t}_i)}^{\widetilde{t}_i}Y_i(\gamma_{ij}(\xi);t)\frac{\partial\widetilde{f}_i[\gamma_{ij}(\xi)]}{\partial x_{ij}}\times$$

$$\times\dot{\gamma}_{ij}(\xi)\delta\varphi_i(\xi)d\xi+\sum_{j=1}^{k_i}\int\limits_{\eta_{ij}(\widetilde{t}_i)}^{\widetilde{t}_i}Y_i(\rho_{ij}(\xi);t)A_{ij}(\rho_{ij}(\xi))\dot{\rho}_{ij}(\xi)\dot{\delta}\varphi_i(\xi)d\xi+$$

$$+\int\limits_{\widetilde{t}_i}^{t}Y_i(\xi;t)\delta f_i[\xi]d\xi;$$

$$\widetilde{g}_i=g_i(\widetilde{t}_i,\widetilde{x}_{i-1}(\widetilde{t}_i)),\delta f_i[\xi]=\delta f_i\big(\xi,\widetilde{x}(\tau_{i1}(\xi)),\dots,\widetilde{x}(\tau_{is_i}(\xi))\big),$$

the matrix functions $\Phi_i(\xi;t)$ *and* $Y_i(\xi;t)$ *satisfy the system*

$$\begin{cases}\dfrac{\partial\Phi_i(\xi;t)}{\partial\xi}=-\displaystyle\sum_{j=1}^{s_i}Y_i(\gamma_{ij}(\xi);t)\frac{\partial\widetilde{f}_i[\gamma_{ij}(\xi)]}{\partial x_{ij}}\dot{\gamma}_{ij}(\xi), \quad \xi\in[\widetilde{t}_i-\delta,t],\\ Y_i(\xi;t)=\Phi_i(\xi;t)+\displaystyle\sum_{j=1}^{k_i}Y_i(\rho_{ij}(\xi);t)A_{ij}(\rho_{ij}(\xi))\dot{\rho}_{ij}(\xi)\end{cases}\tag{2.5}$$

and the initial condition

$$Y_i(\xi;t)=\Phi_i(\xi;t)=\begin{cases}I_i, & \xi=t,\\ \Theta_i, & \xi\neq t,\end{cases}\tag{2.6}$$

where I_i *is the identity matrix,* Θ_i *is the zero matrix.*

$\delta x_i(t;\delta\mu_i)$ is called the variation of the solution $\widetilde{x}_i(t)$ and the expression (2.4) is called the formula of variation.

Theorem 2.3. *Let the condition* 2.3) *of Theorem* 2.2 *and the following conditions be fulfilled:*

2.6) *there exists a number* $\delta>0$ *such that*

$$\gamma_{i1}(t)\le\cdots\le\gamma_{ip_i}(t), \quad t\in[\widetilde{t}_i,\widetilde{t}_i+\delta], \quad i=\overline{1,m};$$

2.7) *there exist finite limits:*

$$\dot{\gamma}_{ij}^{+}=\dot{\gamma}_{ij}(\widetilde{t}_i+), \quad j=\overline{1,s_i}, \quad i=\overline{1,m};$$

$$\dot{\widetilde{x}}_{ij}^{+}=\dot{\widetilde{x}}_{ij}(\eta_{ij}(\widetilde{t}_{i+1}+)), \quad j=\overline{1,s_i}, i=\overline{1,m-1},$$

$$\lim_{\omega_i\to\omega_{ij}^0}\widetilde{f}_i(\omega_i)=f_{ij}^{+}, \quad \omega_i\in[\widetilde{t}_i,\widetilde{t}_i+\delta)\times O_i^{s_i}, \quad j=\overline{0,p_i},$$

$$\lim_{(\omega_{i1},\omega_{i2})\to(\omega_{ij}^1,\omega_{ij}^2)}\big[\widetilde{f}_i(\omega_{i1})-\widetilde{f}_i(\omega_{i2})\big]=f_{ij}^{+},$$

$$\omega_{i1},\omega_{i2}\in(\gamma_{ij},\gamma_{ij}+\delta)\times O_i^{s_i}, \quad j=\overline{p_i+1,s_i}, \quad i=\overline{1,m};$$

$$\lim_{\omega_i\to\omega_i^2}\widetilde{f}_i(\omega_i)=f_{is_i+1}^{+}, \quad \omega_i\in[\widetilde{t}_{i+1},\widetilde{t}_{i+1}+\delta]\times O_i^{s_i} \quad i=\overline{1,m-1}.$$

Then there exist numbers $\varepsilon_2>0, \delta_2>0$ *such that for an arbitrary*

$$(t,\varepsilon,\delta\mu_i)\in\big[\widetilde{t}_{i+1}-\delta_2,\widetilde{t}_{i+1}+\delta_2\big]\times[0,\varepsilon_2]\times V_i^+,\ \ i=\overline{1,m},$$

where

$$V_i^+=\big\{\delta\mu_i\in V_i:\delta t_j\geq 0,\ \ j=\overline{1,i+1}\big\},$$

the formula (2.3) *is valid, where*

$$\delta x_i(t;\delta\mu_i)=\bigg\{Y_i(\widetilde{t}_i+;t)\Big[\dot{\widetilde{\varphi}}_i(\widetilde{t}_i)-\sum_{j=1}^{k_i}A_{ij}(\widetilde{t}_i)\dot{\widetilde{\varphi}}_i(\eta_{ij}(\widetilde{t}_i)+$$

$$+\sum_{j=0}^{p_i}(\widehat{\gamma}_{ij+1}^+-\widehat{\gamma}_{ij}^+)f_{ij}^+\Big]-\sum_{j=p_i+1}^{s_i}Y_i(\gamma_{ij}+;t)f_{ij}^+\dot{\gamma}_{ij}^+ +$$

$$+\Phi_i(\widetilde{t}_i;t)\frac{\partial\widetilde{g}_i}{\partial x_{i-1}}\Big[\sum_{j=1}^{k_{i-1}}A_{i-1j}(\widetilde{t}_i)\dot{\widetilde{x}}_{i-1j}^+ + f_{i-1s_{i-1}+1}^+\Big]\bigg\}\delta t_i+\beta_i(t;\delta\mu_i);$$

$$\widehat{\gamma}_{i0}^+=1,\quad \widehat{\gamma}_{ij}^+=\dot{\gamma}_{ij}^+,\ \ j=\overline{1,p_i},\ \ \widehat{\gamma}_{ip_i+1}^+=0.$$

Remark 2.2. Theorems 2.2 and 2.3 are proved by the method given in [24]. The matrix function $Y_i(\xi;t)$, $\xi\in[a,t]$, is piecewise continuous (see the second equation of the system (2.5) and the condition (2.6)).

Let $\eta_{ij}(t)=\eta_i^j(t)=\eta_i(\eta_i^{j-1}(t))$, $j=\overline{1,s_i}$, $\eta_i^0(t)=t$, where $\eta_i(t)$ is continuously differentiable and satisfies the conditions $\eta_i(t)<t$, $\dot{\eta}_i(t)>0$. Then the function $Y_i(\xi;t)$ is discontinuous with respect to $\xi\in[a,t]$ at the points of the set

$$J_i(t)=\big\{\eta_i^j(t)\in[a,t]:j=1,2,\dots\big\}.$$

The theorem formulated below is a corollary to Theorems 2.2 and 2.3.

Theorem 2.4. *Let* $\eta_{ij}(t)=\eta_i^j(t)$ *and the assumptions of Theorems* 2.2, 2.3 *be valid and, in addition, let*

$$\sum_{j=0}^{p_i}(\widehat{\gamma}_{ij+1}^- -\widehat{\gamma}_{ij}^-)f_{ij}^-=\sum_{j=0}^{p_i}(\widehat{\gamma}_{ij+1}^+-\widehat{\gamma}_{ij}^+)f_{ij}^+=f_{i0},$$

$$f_{ij}^-\dot{\gamma}_{ij}^-=f_{ij}^+\dot{\gamma}_{ij}^+=f_{ij},\ \ j=\overline{p_i+1,s_i},\ \ i=\overline{1,m},$$

$$\sum_{j=1}^{k_{i-1}}A_{i-1j}(\widetilde{t}_i)\dot{\widetilde{x}}_{i-1j}^- + f_{i-1s_{i-1}+1}^-=$$

$$=\sum_{j=1}^{k_{i-1}}A_{i-1j}(\widetilde{t}_i)\dot{\widetilde{x}}_{i-1j}^+ + f_{i-1s_{i-1}+1}^+=f_{i-1s_{i-1}+1},\ \ i=\overline{2,m},$$

$$\widetilde{t}_i,\gamma_{ij}\notin J_i(\widetilde{t}_{i+1}),\ \ i=\overline{1,m},\ \ j=\overline{1,s_i}.$$

Then there exist numbers $\varepsilon_2>0$, $\delta_2>0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu_i)\in[\widetilde{t}_{i+1}-\delta_2,\widetilde{t}_{i+1}+\delta_2]\times[0,\varepsilon_2]\times V_i$, $i=\overline{1,m}$, *the formula* (2.3) *is* *valid, where*

$$\delta x_i(t;\delta\mu_i)=\Big\{Y_i(\widetilde{t}_i;t)\Big[\dot{\widetilde{\varphi}}_i(\widetilde{t}_i)-\sum_{j=1}^{k_i}A_{ij}(\widetilde{t}_i)\dot{\widetilde{\varphi}}_i(\eta_{ij}(\widetilde{t}_i)+f_{i0}\Big]-$$
$$-\sum_{j=p_i+1}^{s_i}Y_i(\gamma_{ij};t)f_{ij}+\Phi_i(\widetilde{t}_i;t)\frac{\partial\widetilde{g}_i}{\partial x_{i-1}}f_{i-1s_{i-1}+1}\Big\}\delta t_i+\beta_i(t;\delta\mu_i).$$

Investigation of the optimal control problem considered in this work will be carried out by the scheme given in [23], [24], according to which an optimal control problem can be formulated as a problem of finding criticality conditions for a continuous and differentiable mapping defined on a quasi-convex filter. Necessary conditions of optimality are obtained from necessary conditions of criticality.

Below all necessary definitions are given and necessary conditions of criticality are formulated.

Let $E_z=E_x\times E_\zeta$ be a locally convex topological vector space of points $z=(x,\zeta)$ and E_x be a finite dimensional space.

Let a mapping

$$P:D\longrightarrow R^s \tag{2.7}$$

and filter Φ in E_z be given.

Definition 2.2. The mapping (2.7) is defined on the filter Φ if there exists an element $W\in\Phi$ such that $W\subset D$.

Definition 2.3. Let the mapping (2.7) be defined on the filter Φ. The mapping (2.7) is called critical on the filter Φ if for any point $\widetilde{z}$ belonging to every element of the filter Φ there exists an element $W\in\Phi$ such that $W\subset D$ and $P(\widetilde{z})\in\partial P(W)$.

Definition 2.4. The mapping (2.7) is continuous on the filter Φ if there exists an element $W\in\Phi$ such that $W\subset D$ and the restriction

$$P:W\longrightarrow R^s$$

is continuous.

Definition 2.5. Let $X\subset E_x$ be a locally convex subspace. The set $D\subset X\times E_\zeta$ is called finitely locally convex if for any arbitrary point $z_0=(x_0,\zeta_0)\in D$ and any arbitrary linear finite dimensional manifold $L_{\zeta_0}\subset E_\zeta$ passing through the point ζ_0 there exist convex neighborhoods $V_{x_0}\subset X$ and $V_{\zeta_0}\subset L_\zeta$ of the points x_0 and ζ_0, respectively, such that

$$V_{x_0}\times V_{\zeta_0}\subset D.$$

Definition 2.6. The mapping (2.7) has a differential at the point $\widetilde{z}=(\widetilde{x},\widetilde{\xi})\in D$ if there exists a linear mapping

$$dP_{\widetilde{z}}:E_{\delta z}=E_z-\widetilde{z}\longrightarrow R^s$$

such that for any manifold

$$L_{\widetilde{\zeta}} = \left\{ \widetilde{\zeta} + \sum_{i=1}^{k} \lambda_i \delta\zeta_i : \lambda_i \in R \right\} \subset E_\zeta$$

the following representation holds

$$P(\widetilde{z} + \varepsilon\delta z) - P(\widetilde{z}) = \varepsilon dP_{\widetilde{z}}(\delta z) + o(\varepsilon\delta z), \ \ \forall(\varepsilon, \delta z) \in (0, \varepsilon_0] \times V_{01} \times V_{11},$$

where $V_{01} \subset X - \widetilde{x}$ and $V_{11} \subset L_{\widetilde{\zeta}} - \widetilde{\zeta}$ are bounded and convex neighborhoods of zero; $\varepsilon_0 > 0$ is a number for which

$$\widetilde{z} + \varepsilon\delta z \in D, \ \ \forall\ (\varepsilon, \delta z) \in (0, \varepsilon_0] \times V_{01} \times V_{11},$$

and finally,

$$\lim_{\varepsilon \to 0} o(\varepsilon\delta z)/\varepsilon = 0, \ \ \text{uniformly for} \quad \delta z \in V_{01} \times V_{11}.$$

Definition 2.7. The filter Φ in E_z is called quasi-convex, if for any element $W \in \Phi$ and any natural number p there exists an element $W_1 = W_1(W; p) \in \Phi$ such that for arbitrary $p+1$ points $z_0, \ldots, z_p$ from W_1 and an arbitrary neighborhood of zero $V_{01} \subset E_z$ there exists a continuous mapping

$$\varphi : co\big(\{z_0, \ldots, z_p\}\big) \longrightarrow W$$

satisfying the condition

$$(z - \varphi(z)) \in V_{01}, \ \ \forall\ z \in co\big(\{z_0, \ldots, z_p\}\big).$$

It is obvious that every convex filter Φ in E_z is quasi-convex.

By $co[\Phi]$ we denote the convex filter, whose elements are sets $co(W)$, where W is an arbitrary element of the filter Φ and $\mathrm{cone}(M)$ denote the cone generated by the set M.

Theorem 2.5 (necessary condition of the criticality)**.** *Let the mapping* (2.7) *be continuous on* $co[\Phi]$ *and critical on* Φ. *Let the filter* Φ *be quasi-convex. Then for any point* $\widetilde{z}$ *belonging to all sets of the filter* Φ, *at which the mapping* (2.7) *has a differential, there exist an element* $\widetilde{W} \in \Phi$ *and a non-zero* s*-dimensional row-vector* $\pi = (\pi_1, \ldots, \pi_s)$ *such that*

$$\pi dP_{\widetilde{z}}(\delta z) \leq 0, \quad \forall\ \delta z \in \mathrm{cone}(\widetilde{W} - \widetilde{z}).$$

Let $G_i \subset R^{r_i}$ be an open set and the function $f_i(t, x_{i1}, \ldots, x_{is_i}, u_i) \in R^{n_i}$ satisfy the following conditions: it is continuous on $O_i^{s_i} \times G_i$ and continuously differentiable with respect to $x_{ij} \in O_i$, $j = \overline{1, s_i}$, for almost all $t \in J$; for each fixed $(x_{i1}, \ldots, x_{is_i}, u_i) \in O_i^{s_i} \times G_i$ the functions f_i, $\frac{\partial f_i}{\partial x_{ij}}$, $j = \overline{1, s_i}$, are measurable on J; for any compacts $K_i \subset O_i$ and $M_i \subset G_i$ there exists a function $m_{K_i, M_i}(\cdot) \in L(J, R_+)$ such that for any $(x_{i1}, \ldots, x_{is_i}, u_i) \in K_i^{s_i} \times M_i$ and for almost all $t \in J$

$$\big|f_i(t, x_{i1}, \ldots, x_{is_i}, u_i)\big| + \sum_{j=1}^{s_i} \left| \frac{\partial f_i(\cdot)}{\partial x_{ij}} \right| \leq m_{K_i, M_i}(t).$$

Now we consider the set

$$F_i = \big\{f_i(t, x_{i1}, \dots, x_{is_i}) = f_i(t, x_{i1}, \dots, x_{is_i}, u_i(t)) : u_i(\cdot) \in \Omega_i\big\}.$$

Here Ω_i is the set of measurable functions $u_i(t) \in U_i \subset G_i$, $t \in J$, satisfying the condition: the set $clu(J)$ is compact and lies in G_i, where U_i is an arbitrary set.

It is clear that the set F_i can be identified with a subset of the space E_{f_i}.

Let

$$\widetilde{f}_i(t, x_{i1}, \dots, x_{is_i}) = f_i(t, x_{i1}, \dots, x_{is_i}, \widetilde{u}_i(t)), \quad \text{where } \widetilde{u}_i(\cdot) \in \Omega_i.$$

In F_i we define the filter $\Phi_{\widetilde{f}_i}$ with the basis

$$\big\{W_{K_i,\delta} : K_i \subset O_i - \text{ compact set}, \delta > 0 - \text{arbitrary number}\big\},$$

where

$$W_{K_i,\delta} = \big\{f_i \in F_i : H_1(f_i - \widetilde{f}_i; K_i) \le \delta\big\}.$$

Lemma 2.3 ([24], p. 73)**.** *The filter $\Phi_{\widetilde{f}_i}$ is quasi-convex. Moreover, for an arbitrary element $W_{\widetilde{f}_i} \in \Phi_{f_i}$ the inclusion*

$$\operatorname{cone}\Big(\big[W^{(i)}(K_{i1}; \alpha_0)\big]_{W_{\widetilde{f}_i}} - \widetilde{f}_i\Big) \supset F_i - \widetilde{f}_i$$

holds, where

$$W^{(i)}(K_{i1}; \alpha_0) = \widetilde{f}_i + W(K_{i1}; \alpha_0),$$

$[W^{(i)}(K_{i1}; \alpha_0)]_{W_{\widetilde{f}_i}}$ is the closure of the set $W^{(i)}(K_{i1}; \alpha_0) \cap W_{\widetilde{f}_i}$ with respect to $W^{(i)}(K_{i1}; \alpha_0)$ in the topology induced in $W^{(i)}(K_{i1}; \alpha_0)$ by the topology from E_{f_i}; $K_{i1} \subset O_i$ is a compact set.

Lemma 2.4 ([24], p. 9)**.** *Let $\widetilde{x}_i(t) \in K_{i1}, t \in J$ be a piecewise-continuous function and let $\delta f_{ij} \in W(K_{i1}; \alpha_0)$, $j = 1, 2, \dots$; moreover, let*

$$\lim_{j\to\infty} H(\delta f_{ij}; K_{i1}) = 0.$$

Then

$$\lim_{j\to\infty} \sup \left\{\left|\int\limits_{t'}^{t''} \delta f_{ij}(t, \widetilde{x}_i(\tau_{i1}(t)), \dots, \widetilde{x}_i(\tau_{is_i}(t))dt\right| : \forall t', t'' \in J\right\} = 0.$$

Lemma 2.5. *Let $\psi_i(t) \in R^{n_i}$, $t \in [\widetilde{t}_i, \widetilde{t}_{i+1}] \subset J$ and $\widetilde{x}_i(t) \in K_{i1}$, $t \in [\tau_i, \widetilde{t}_{i+1}]$, be piecewise-continuous functions. Then the mapping*

$$\delta f_i \to \int\limits_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} \psi_i(t)\delta f_i(t, \widetilde{x}_i(\tau_{i1}(t)), \dots, \widetilde{x}_i(\tau_{is_i}(t)))dt, \tag{2.8}$$

is continuous on $W(K_{i1}; \alpha_0)$ in the topology induced from E_{f_i}.

Proof. Let $\delta f_{ij} \in W(K_{i1};\alpha_0)$, $j = 1,2,\dots$, and

$$\lim_{j\to\infty} H(\delta f_{ij}; K_{i1}) = 0.$$

The mapping (2.8) is continuous if

$$\lim_{j\to\infty} \int\limits_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} \psi_i(t)\delta f_{ij}(t, \widetilde{x}_i(\tau_{i1}(t)),\dots,\widetilde{x}_i(\tau_{is_i}(t)))dt = 0. \tag{2.9}$$

Let $[\theta_{ip},\theta_{ip+1}]$, $p = \overline{1,\nu_i}$, be the intervals of continuity of the function $\psi_i(t)$ and $\psi_i(t) = \psi_{ip}(t)$, $t \in [\theta_{ip},\theta_{ip+1}]$.

Then

$$\Big|\int\limits_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} \psi_i(t)\delta f_{ij}(t, \widetilde{x}_i(\tau_{i1}(t)),\dots,\widetilde{x}_i(\tau_{is_i}(t)))dt\Big| \le$$
$$\le \sum_{p=1}^{\nu_i}\Big|\int\limits_{\theta_{ip}}^{\theta_{ip+1}} \psi_{ip}(t)\delta f_{ij}(t, \widetilde{x}_i(\tau_{i1}(t)),\dots,\widetilde{x}_i(\tau_{is_i}(t)))dt\Big|. \tag{2.10}$$

There exists a sequence of continuously differentiable functions $Q_{ip}^l(t)$, $l = 1,2,\dots$, such that

$$\lim_{l\to\infty} \|Q_{ip}^l - \psi_{ip}\| = 0,$$

where

$$\|Q_{ip}^l - \psi_{ip}\| = \max_{t\in[\theta_{ip},\theta_{ip+1}]} \big|Q_{ip}^l(t) - \psi_{ip}(t)\big|.$$

We have

$$\Big|\int\limits_{\theta_{ip}}^{\theta_{ip+1}} \psi_{ip}(t)\delta f_{ij}(t, \widetilde{x}_i(\tau_{i1}(t)),\dots,\widetilde{x}_i(\tau_{is_i}(t)))dt\Big| \le \|\psi_{ip} - Q_{ip}^l\|\alpha_0 +$$
$$+\Big|\int\limits_{\theta ip}^{\theta_{ip+1}} Q_{ip}^l(t)\delta f_{ij}\big(t, \widetilde{x}_i(\tau_{i1}(t)),\dots,\widetilde{x}_i(\tau_{is_i}(t))\big)d\xi\Big|. \tag{2.11}$$

Integration by parts gives

$$\int\limits_{\theta_{ip}}^{\theta_{ip+1}} Q_{ip}^l(t)\delta f_{ij}\big(t, \widetilde{x}_i(\tau_{i1}(t)),\dots,\widetilde{x}_i(\tau_{is_i}(t))\big)dt =$$
$$= Q_{ip}^l(\theta_{ip+1})\int\limits_{\theta_{ip}}^{\theta_{ip+1}} \delta f_{ij}\big(t, \widetilde{x}_i(\tau_{i1}(t)),\dots,\widetilde{x}_i(\tau_{is_i}(t))\big)dt -$$

$$-\int\limits_{\theta_{ip}}^{\theta_{ip+1}} \dot{Q}_{ip}^{l}(t)\bigg(\int\limits_{\theta_{ip}}^{t} \delta f_{ij}\big(\xi,\widetilde{x}_i(\tau_{i1}(\xi)),\dots,\widetilde{x}_i(\tau_{is_i}(\xi))\big)d\xi\bigg)dt.$$

For every fixed $l = 1, 2, \dots$ on the basis of the preceding lemma we have

$$\lim_{j\to\infty} \int\limits_{\theta_{ip}}^{\theta_{ip+1}} Q_{ip}^{l}(t)\delta f_{ij}\big(t,\widetilde{x}_i(\tau_{i1}(t)),\dots,\widetilde{x}_i(\tau_{is_i}(t))\big)dt = 0. \tag{2.12}$$

From (2.10), taking into consideration (2.11) and (2.12), we obtain (2.9). □

3. Statement of the Problem. Necessary Conditions of Optimality

Let $\Delta_{i0} = \big\{\varphi_i \in E_{\varphi_i} : \varphi_i(t) \in N_i\big\}$ be sets of initial functions, $N_i \subset O_i$ be convex sets; scalar functions $q^p(t_1,\dots,t_{m+1},x_1,\dots,x_m,x_{m+1})$, $p=\overline{1,l}$, be continuously differentiable in all arguments: $t_i \in J$, $x_i \in O_i$, $i=\overline{1,m}$, $x_{m+1} \in O_m$.

We introduce the sets:

$$B_{i1} = \Big\{\sigma_i = (t_1,\dots,t_{i+1},x_{10},\dots,x_{i0},\varphi_1,\dots,\varphi_i,,u_1,\dots,u_i) \in$$
$$\in J^{1+i} \times \prod_{j=1}^{i} O_j \times \prod_{j=1}^{i} \Delta_{j0} \times \prod_{j=1}^{i} \Omega_i; t_1 < \dots < t_{i+1}\Big\}, \quad i=\overline{1,m}.$$

To each element $\sigma_m \in B_{m1}$ we assign the system of neutral differential equations with variable structure and discontinuous initial condition

$$\dot{x}_i(t) = \sum_{j=1}^{k_i} A_{ij}(t)\dot{x}_i(\eta_{ij}(t)) +$$
$$+ f_i\big(t,x_i(\tau_{i1}(t)),\dots,x_i(\tau_{is_i}(t)),u_i(t)\big), \quad t \in [t_i,t_{i+1}], \tag{3.1$_i$}$$
$$x_i(t) = \varphi_i(t), \ \ t \in [\tau_i,t_i), \ \ x_i(t_i) = x_{i0} + g_i(t_i,x_{i-1}(t_i)), \quad i=\overline{1,m}. \tag{3.2$_i$}$$

Here, as above, $g_1 = 0$.

The solution $\{x_i(t;\sigma_i),\ t \in [\tau_i,t_{i+1}] : i=\overline{1,m}\}$, where $\sigma_i \in B_{i1}$, corresponding to the element $\sigma_m \in B_{m1}$ is defined similarly (see Def.2.1).

Definition 3.1. The element $\sigma_m \in B_{m1}$ is called admissible, if the following conditions are fulfilled

$$q^p\big(t_1,\dots,t_{m+1},x_{10},\dots,x_{m0},x_m(t_{m+1})\big) = 0, \ \ p=\overline{1,m}, \tag{3.3}$$

where $x_m(t) = x_m(t;\sigma_m)$.

The set of admissible elements is denoted by B_0.

Definition 3.2. The element

$$\widetilde{\sigma}_m = (\widetilde{t}_1,\dots,\widetilde{t}_{m+1},\widetilde{x}_{10},\dots,\widetilde{x}_{m0},\widetilde{u}_1,\dots,\widetilde{u}_m) \in B_0$$

is called optimal, if there exist a number $\widetilde{\delta} > 0$ and compacts $\widetilde{K}_i \subset O_i$, $i = \overline{1,m}$, such that for an arbitrary element $\sigma_m \in B_0$ satisfying the condition

$$\sum_{i=1}^{m+1} \bigl|\widetilde{t}_i - t_i\bigr| + \sum_{i=1}^{m} \Bigl(\bigl|\widetilde{x}_{i0} - x_{i0}\bigr| + \bigl\|\widetilde{\varphi}_i - \varphi_i\bigr\| + H_1\bigl(\widetilde{f}_i - f_i; \widetilde{K}_i\bigr)\Bigr) \le \widetilde{\delta},$$

the following inequality is fulfilled

$$\begin{aligned} & q^0\bigl(\widetilde{t}_1, \dots, \widetilde{t}_{m+1}, \widetilde{x}_{10}, \dots, \widetilde{x}_{m0}, \widetilde{x}_m(\widetilde{t}_{m+1})\bigr) \le \\ & \le q^0\bigl(t_1, \dots, t_{m+1}, x_{10}, \dots, x_{m0}, x_m(t_{m+1})\bigr). \end{aligned} \tag{3.4}$$

Here

$$\begin{aligned} & \widetilde{x}_m(t) = x_m(t; \widetilde{\sigma}_m), \quad \widetilde{f}_i(\omega_i) = f_i(\omega_i, \widetilde{u}(t)), \\ & f(\omega_i) = f(\omega_i, u(t)), \quad \omega_i = (t, x_{i1}, \dots, x_{is_i}). \end{aligned}$$

The problem (3.1_i)–(3.2_i), (3.3), (3.4) is called the optimal problem of neutral type with variable structure and discontinuous initial condition, and it consists in finding an optimal element $\widetilde{\sigma}_m$.

Theorem 3.1. *Let $\widetilde{\sigma}$ be an optimal element, $\widetilde{t}_1 > a$ and conditions* 2.3)–2.5) *of Theorem* 2.2 *be fulfilled. Let, besides, there exist finite limits:*

$$\begin{gathered} \dot{\widetilde{x}}_{mj}^{-} = \dot{\widetilde{x}}_m(\eta_{mj}(\widetilde{t}_{m+1}-)), j = \overline{1,k_m}; \\ \lim_{\omega_m \to \omega_m^2} \widetilde{f}_m(\omega_m) = f_{ms_m+1}^{-}, \omega_m \in (a, \widetilde{t}_{m+1}] \times O_m^{s_m}, \end{gathered}$$

where $\omega_m^2 = \bigl(\widetilde{t}_{m+1}, \widetilde{x}_m(\tau_{m1}(\widetilde{t}_{m+1})), \dots, \widetilde{x}_m(\tau_{ms_m}(\widetilde{t}_{m+1}))\bigr)$. *Then there exist a vector* $\pi = (\pi_0, \dots, \pi_l) \ne 0$, $\pi_0 \le 0$, *and a solution* $\bigl\{(\psi_i(t), \chi_i(t)), t \in [\widetilde{t}_i - \delta, \gamma_{i0}] : i = \overline{1,m},\ \delta > 0\bigr\}^*$ *of the variable structure system*

$$\begin{cases} \dot{\chi}_i(t) = -\displaystyle\sum_{j=1}^{s_i} \psi_i(\gamma_{ij}(t)) \frac{\partial \widetilde{f}_i[\gamma_{ij}(t)]}{\partial x_{ij}}\, \dot{\gamma}_{ij}(t), \\ \psi_i(t) = \chi_i(t) + \displaystyle\sum_{j=1}^{k_i} \psi_i(\rho_{ij}(t)) A_{ij}(\rho_{ij}(t)) \dot{\rho}_{ij}(t), \quad t \in [\widetilde{t}_i - \delta, \widetilde{t}_{i+1}], \end{cases} \tag{3.5}$$

$$\psi_i(t) = \chi_i(t) = 0, \quad t \in (\widetilde{t}_{i+1}, \gamma_{i0}], \quad i = \overline{1,m},$$

such that the following conditions are fulfilled:

3.1) *the integral maximum principle for controls*

$$\int_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} \psi_i(t) \widetilde{f}_i[t]\, dt \ge \int_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} \psi_i(t) f_i\bigl(t, \widetilde{x}_i(\tau_{i1}(t)), \dots, \widetilde{x}_{is_i}(\tau_{is_i}(t), u_i(t)\bigr) dt$$

$$\forall\ u_i(\cdot) \in \Omega_i, \quad i = \overline{1,m};$$

[*] $\gamma_{i0} = \max\{\gamma_{i1}(b), \dots, \gamma_{is_i}(b), \rho_{i1}(b), \dots, \rho_{ik_i}(b)\}$, $\psi_i(t)$ is piecewise-continuous function.

3.2) *the integral maximum principle for initial functions*

$$\sum_{j=p_i+1}^{s_i}\int\limits_{\tau_{ij}(\widetilde{t}_i)}^{\widetilde{t}_i}\psi_i(\gamma_{ij}(t))\frac{\partial\widetilde{f}_i[\gamma_{ij}(t)]}{\partial x_{ij}}\,\dot{\gamma}_{ij}(t)\widetilde{\varphi}_i(t)dt+$$

$$+\sum_{j=1}^{k_i}\int\limits_{\eta_{ij}(\widetilde{t}_i)}^{\widetilde{t}_i}\psi_i(\rho_{ij}(t))A_{ij}(\rho_{ij}(t))\dot{\rho}_{ij}(t)\dot{\widetilde{\varphi}}_i(t)dt\geq$$

$$\geq\sum_{j=p_i+1}^{s_i}\int\limits_{\tau_{ij}(\widetilde{t}_i)}^{\widetilde{t}_i}\psi_i(\gamma_{ij}(t))\frac{\partial\widetilde{f}_i[\gamma_{ij}(t)]}{\partial x_{ij}}\,\dot{\gamma}_{ij}(t)\varphi_i(t)dt+$$

$$+\sum_{j=1}^{k_i}\int\limits_{\eta_{ij}(\widetilde{t}_i)}^{\widetilde{t}_i}\psi_i(\rho_{ij}(t))A_{ij}(\rho_{ij}(t))\dot{\rho}_{ij}(t)\dot{\varphi}_i(t)dt,\ \ \forall\,\varphi_i(\cdot)\in\Delta_{i0},\ i=\overline{1,m};$$

3.3) *the conditions for the functions* $\chi_i(t)$, $i=\overline{1,m}$,

$$\pi\frac{\partial\widetilde{Q}}{\partial x_{m+1}}=\chi_m(\widetilde{t}_{m+1})\,,\quad \pi\frac{\partial\widetilde{Q}}{\partial x_i}=-\chi_i(\widetilde{t}_i),\quad i=\overline{1,m},$$

$$\chi_{i-1}(\widetilde{t}_i)=\chi_i(\widetilde{t}_i)\frac{\partial\widetilde{g}_i}{\partial x_{i-1}}\,,\quad i=\overline{2,m};$$

3.4) *the conditions for the moments* $\widetilde{t}_i$, $i=\overline{1,m+1}$,

$$\pi\frac{\partial\widetilde{Q}}{\partial t_i}\geq-\psi_i(\widetilde{t}_i-)\Big[\dot{\widetilde{\varphi}}_i(\widetilde{t}_i)-\sum_{j=1}^{k_i}A_{ij}(\widetilde{t}_i)\dot{\widetilde{\varphi}}_i(\eta_{ij}(\widetilde{t}_i))+$$

$$+\sum_{j=0}^{p_i}\big(\widehat{\gamma}_{ij+1}^{-}-\widehat{\gamma}_{ij}^{-}\big)f_{ij}^{-}\Big]+\sum_{j=p_i+1}^{s_i}\psi_i(\gamma_{ij}-)f_{ij}^{-}\dot{\gamma}_{ij}^{-}-$$

$$-\chi_i(\widetilde{t}_i)\Big[\frac{\partial\widetilde{g}_i}{\partial t_i}-\dot{\widetilde{\varphi}}_i(\widetilde{t}_i)\Big]-\chi_{i-1}(\widetilde{t}_i)\times$$

$$\times\Big[\sum_{j=1}^{k_{i-1}}A_{i-1\,j}(\widetilde{t}_i)\dot{\widetilde{x}}_{i-1\,j}^{-}+f_{i-1\,s_{i-1}+1}^{-}\Big],\quad i=\overline{1,m};$$

$$\pi\frac{\partial\widetilde{Q}}{\partial t_{m+1}}\geq-\psi_m(\widetilde{t}_{m+1})\Big[\sum_{j=1}^{k_m}A_{mj}(\widetilde{t}_{m+1})\dot{\widetilde{x}}_{mj}^{-}+f_{ms_m+1}^{-}\Big].$$

Here

$$Q=(q^0,\dots,q^l)^*,\ \frac{\partial\widetilde{Q}}{\partial t_i}=\frac{\partial}{\partial t_i}\,Q\big(\widetilde{t}_1,\dots,\widetilde{t}_{m+1},\widetilde{x}_{10},\dots,\widetilde{x}_{m0},\widetilde{x}_{m1}(\widetilde{t}_{m+1})\big),$$

$$\widetilde{f}_i[t]=\widetilde{f}_i\big(t,\widetilde{x}(\tau_{i1}(t)),\dots,\widetilde{x}_i(\tau_{is_i}(t))\big).$$

Here and everywhere we suppose, that $\chi_0(\widetilde{t}_1)=0$.

Theorem 3.2. *Let $\widetilde{\sigma}_m$ be an optimal element, $\widetilde{t}_{m+1} < b$, and the conditions of Theorem* 2.3 *be fulfilled. Let, besides, there exist finite limits*

$$\dot{\widetilde{x}}^+_{mj} = \dot{\widetilde{x}}_{mj}\big(\eta_{mj}(\widetilde{t}_{m+1}+)\big), \quad j = \overline{1, k_m};$$
$$\lim_{\omega_m \to \omega^2_m} \widetilde{f}_m(\omega_m) = f^+_{ms_m+1}, \quad \omega_m \in [\widetilde{t}_{m+1}, b) \times O^{s_m}_i.$$

Then there exist a vector $\pi = (\pi_0, \dots, \pi_l) \neq 0$, $\pi_0 \leq 0$, and a solution $\big\{(\psi_i(t), \chi_i(t)),\ t \in [\widetilde{t}_i - \delta, \gamma_{i0}] : i = \overline{1,m},\ \delta > 0\big\}$ of the system (3.5) *such that the conditions* 3.1)–3.3) *are fulfilled. Moreover,*

$$\pi\frac{\partial \widetilde{Q}}{\partial t_i} \leq \psi_i(\widetilde{t}_i+)\Big[\dot{\widetilde{\varphi}}_i(\widehat{t}_i) - \sum_{j=1}^{k_i} A_{ij}(\widetilde{t}_i)\dot{\widetilde{\varphi}}_i(\eta_{ij}(\widetilde{t}_i)) +$$
$$+ \sum_{j=0}^{p_i}(\widehat{\gamma}^+_{ij+1} - \widehat{\gamma}^+_{ij})f^+_{ij}\Big] + \sum_{j=p_i+1}^{s_i} \psi_i(\gamma_{ij}+)f^+_{ij}\dot{\gamma}^+_{ij} - \chi_i(\widetilde{t}_i)\Big[\frac{\partial \widetilde{g}_i}{\partial t_i} - \dot{\widetilde{\varphi}}_i(\widetilde{t}_i)\Big] -$$
$$- \chi_{i-1}(\widetilde{t}_i)\left[\sum_{j=1}^{k_{i-1}} A_{i-1j}(\widetilde{t}_i)\dot{\widetilde{x}}^+_{i-1j} + f^+_{i-1s_{i-1}+1}\right], \quad i = \overline{1,m},$$
$$\pi\frac{\partial \widetilde{Q}}{\partial t_{m+1}} \leq -\psi_m(\widetilde{t}_{m+1})\left[\sum_{j=1}^{k_m} A_{mj}(\widetilde{t}_{m+1})\dot{\widetilde{x}}_{mj} + f^+_{ms_m+1}\right].$$

Theorem 3.3. *Let $\widetilde{\sigma}_m$ be an optimal element, $\widetilde{t}_1, \widetilde{t}_{m+1} \in (a,b)$ and the conditions of Theorem* 2.4 *be fulfilled. Let, besides,*

$$\sum_{j=1}^{k_m} A_{mj}(\widetilde{t}_{m+1})\dot{\widetilde{x}}^-_{mj} + f^-_{ms_m+1} = \sum_{j=1}^{k_m} A_{mj}(\widetilde{t}_{m+1})\dot{\widetilde{x}}^+_{mj} + f^+_{ms_m+1} = f_{ms_m+1}.$$

Then there exist a vector $\pi = (\pi_0, \dots, \pi_l) \neq 0$, $\pi_0 \leq 0$, and a solution $\big\{(\psi_i(t), \chi_i(t)),\ t \in [\widetilde{t}_i - \delta, \gamma_{i0}] : i = \overline{1,m},\ \delta > 0\big\}$ of the system (3.5) *such that the conditions* 3.1)–3.3) *are fulfilled. Besides,*

$$\pi\frac{\partial \widetilde{Q}}{\partial t_i} = \psi_i(\widetilde{t}_i)\Big[\dot{\widetilde{\varphi}}_i(\widehat{t}_i) - \sum_{j=1}^{k_i} A_{ij}(\widetilde{t}_i)\dot{\widetilde{\varphi}}_i(\eta_{ij}(\widetilde{t}_i)) + f_{i0}\Big] +$$
$$+ \sum_{j=p_i+1}^{s_i} \psi_i(\gamma_{ij})f_{ij} - \chi_i(\widetilde{t}_i)\left[\frac{\partial \widetilde{g}_i}{\partial t_i} - \dot{\widetilde{\varphi}}_i(\widetilde{t}_i)\right] - \chi_{i-1}(\widetilde{t}_i)f_{i-1s_i+1}, \quad i = \overline{1,m},$$
$$\pi\frac{\partial \widetilde{Q}}{\partial t_{m+1}} = -\psi_m(\widetilde{t}_{m+1})f_{ms_m+1}.$$

Some comments. From the integral maximum principle 3.1) by standard way follows the pointwise maximum principle

$$\psi_i(t)\widetilde{f}_i[t] = \max_{u_i \in U_i} \psi_i(t)f_i(t, \widetilde{x}(\tau_{i1}(t)), \dots, \widetilde{x}_i(\tau_{is_i}(t), u_i), \tag{3.6}$$

almost everywhere on $[\widetilde{t}_i, \widetilde{t}_{i+1}]$.

The addends

$$\sum_{j=0}^{p_i}(\widehat{\gamma}_{ij+1}^{-}-\widehat{\gamma}_{ij}^{-})f_{ij}^{-} \text{ and } \sum_{j=p_i}^{s_i}\psi_i(\gamma_{ij}-)f_{ij}^{-}\gamma_{ij}^{-}$$

in the condition 3.3) are the effects discontinuity of the initial condition.

Let $\widetilde{\varphi}_i(\widetilde{t}_i)=\widetilde{x}_i(\widetilde{t}_i)$. Then $f_{i0}^{-}=\cdots=f_{ip_i}^{-}$, $f_{ij}^{-}=0$, $j=\overline{p_i+1,s_i}$. In this case the conditions for moments $\widetilde{t}_i$, $i=\overline{1,m}$, are simple.

Let the functions $\dot{\gamma}_{ij}(t)$, $j=\overline{1,s_i}$, $\widetilde{u}_i(t)$, $f_i(t,x_{i1},\dots,x_{is_i},u_i)$ be piecewise-continuous in t. Then the conditions of Theorems 3.1 and 3.2 connected with existence of one-side limits are fulfilled.

Let

$$\dot{\gamma}_{ip_i}^{-}<\cdots<\dot{\gamma}_{i1}^{-},\quad \dot{\gamma}_{i1}^{+}<\cdots<\dot{\gamma}_{ip_i}^{+},\quad i=\overline{1,m}.$$

Then the conditions 2.4) and 2.6) are fulfilled, respectively.

Theorems 3.1 and 3.2 correspond to the cases, where the variations at the points $\widetilde{t}_i$, $i=\overline{1,m+1}$, take place on the left and on the right, respectively.

Theorem 3.3 corresponds to the case where at the points $\widetilde{t}_i$, $i=\overline{1,m+1}$, double-sided variations take place.

4. Economic Problem with Variable Structure

In this section we consider the problem of optimal distribution of the invested capital, as well as of optimal determination of investment periods, among various economy branches.

Let the economy be divided into n_i various branches, where $i=\overline{1,m}$ are different periods of investment. Suppose that x_i^j, $j=\overline{1,n_i}$, is the total volume of the output produced by the j-th branch in the corresponding investment period. $x_i=(x_i^1,\dots,x_i^{n_i})^*$ is the state vector of the system, u_i^j, $j=\overline{1,n_i}$, are proportionality coefficients of the invested capital for the j-th branch in the corresponding period, $u_i=(u_i^1,\dots,u_i^{n_i})^*$ is the control vector parameter. If we take into account that the investment gives the real result after a certain lapse of time $\tau_i>0$, $i=\overline{1,m}$ (delay), and also recurrent process factor, then the process of economic development, on the basis of the balance equation of V. Leontief's dynamical model, can be described as the following neutral optimal control problem:

$$\begin{aligned}\dot{x}_i(t)=&A_i(t)\dot{x}_i(t-\tau_i)+B_i(t)x_i(t-\tau_i)+\\&+D_i(t)u_i(t),\quad t\in[t_i,t_{i+1}],\end{aligned}\tag{4.1$_i$}$$

$$\begin{cases}x_i(t)=\varphi_{i0}(t), & t\in[t_i-\tau_i,t_i),\quad i=\overline{1,m},\\ x_1(t_1)=x_{10}, & x_i(t_i)=E_ix_{i-1}(t_i),\quad i=\overline{2,m},\end{cases}\tag{4.2$_i$}$$

with the final condition

$$x_m(t_{m+1})=x_{m1}\tag{4.3}$$

and the cost function

$$t_{m+1}-t_1\longrightarrow\min,\tag{4.4}$$

where t_1 is a fixed initial moment; $x_{11} \in R_+^{n_1}$ and $x_{m1} \in R_+^{n_m}$ are fixed points; $\varphi_{i0} : [a - \tau_i, b] \to R_+^{n_i}$ are fixed continuously differentiable initial functions; $A_i(t),\ B_i(t),\ D_i(t),\ t \in J,\ i = \overline{1,m}$, are $n_i \times n_i$-dimensional continuous matrix functions; E_i, $i = \overline{2,m}$, are $n_i \times n_{i-1}$ dimensional constant matrices;

$$u_i(t) \in U_i = \Big\{ u_i = (u_i^1, \dots, u_i^{n_i})^* : \sum_{j=1}^{n_i} u_i^j \le 1,\ \ 0 \le u_i^j \le 1,\ \ j = \overline{1,n} \Big\},\ \ t \in J,$$

are piecewise-continuous control functions.

Next,

$$A_i(t) = C_i^{-1} K_i^0(t), \quad B_i(t) = C_i^{-1} K_i^1(t), D_i(t) = C_i^{-1} I_i(t),$$

where C_i is a non-degenerate matrix of the material cost coefficients c_i^{pj}, $p, j = \overline{1, n_i}$ which show the output quantity which the p-th branch must invest in the j-th branch so that the latter could produce one unit of output during the i-th period of investment. $K_i^0(t)$ is a diagonal matrix with coefficients of intensity of the output quantity of the i-th branch on the main diagonal. $K_i^1(t)$ is a diagonal matrix with accumulation coefficients of the i-th branch on the man diagonal. $I_i(t)$ is the volume of credit or foreign capital.

Our aim is to choose such investment proportionality coefficients, i.e., such controls $u_i(t)$, $t \in [t_i, t_{i+1}]$, $i = \overline{1,m}$, and investment structure change moments t_i, $i = \overline{2,m}$, the final moment being t_{m+1}, that the corresponding solution $\{x_i(t) \in R_+^{n_i},\ t \in [t_i - \tau_i, t_{i+1}] : i = \overline{1,m}\}$ of the system (3.1_i) satisfies the condition (3.2_i), (3.3) and the effectiveness function (3.4) takes the least possible value.

Below, for the optimal control problem (4.1_i), (4.2_i), (4.3), (4.4) we formulate the necessary conditions of optimality, which follow from Theorem 3.3.

Theorem 4.1. *Let $\widetilde{u}_i(t)$, $t \in [\widetilde{t}_i, \widetilde{t}_{i+1}]$, $i = \overline{1,m}$, be optimal controls, $\widetilde{t}_i \in (a,b)$, $i = \overline{2,m}$, be optimal time moments at which the system structure changes, $\widetilde{t}_{m+1}$ be optimal final moment and let $\{\widetilde{x}_i(t) \in R_+^{n_i}, \widetilde{t}_i - \tau_i \le t \le \widetilde{t}_{i+1},\ i = \overline{1,m}\}$ be the corresponding solution and $\dot{\widetilde{x}}_i(t - \tau_i)$, $i = \overline{1,m}$, be continuous at the points $\widetilde{t}_{i+1}$, $i = \overline{1,m}$, respectively. Besides, let $\widetilde{t}_i + \tau_i < \widetilde{t}_{i+1}$, $\widetilde{t}_i, \widetilde{t}_i + \tau_i \notin \{\widetilde{t}_{i+1} - k\tau_i : k = 1, 2, \dots\}$, $i = \overline{1,m}$. Then there exists a solution $\{(\psi_i(t), \chi_i(t)),\ t \in [\widetilde{t}_i, \widetilde{t}_i + \tau_i]$: $i = \overline{1,m}\}$ of the variable structure system*

$$\begin{cases} \dot{\chi}_i(t) = -\psi_i(t + \tau_i) B_i(t + \tau_i), \\ \psi_i(t) = \chi_i(t) + \psi_i(t + \tau_i) A_i(t + \tau_i), \end{cases} \qquad t \in [\widetilde{t}_i, \widetilde{t}_{i+1}],$$

$$\psi_i(t) = \chi_i(t) = 0, t \in (\widetilde{t}_{i+1}, \widetilde{t}_{i+1} + \tau_i], \quad i = \overline{1,m},$$

where $\psi_i(t)$ is a piecewise-continuous function, such that $(\psi_m(t), \chi_m(t)) \neq 0$, $t \in [\widetilde{t}_m, \widetilde{t}_{m+1}]$ and the following conditions are fulfilled:

4.1) *the maximum principle*

$$\psi_i(t)B_i(t)\widetilde{u}_i(t)=\max_{u_i\in U_i}\psi_i(t)B_i(t)u_i,\quad t\in[\widetilde{t}_i,\widetilde{t}_{i+1}],\quad i=\overline{1,m};$$

4.2) *the condition for* $\chi_i(t)$, $i=\overline{2,m}$,

$$\chi_{i-1}(\widetilde{t}_i)=\chi_i(\widetilde{t}_i)E_i;$$

4.3) *the condition for the final moment* $\widetilde{t}_{m+1}$

$$\psi_m(\widetilde{t}_{m+1})\big[A_m(\widetilde{t}_{m+1})\dot{\widetilde{x}}_m(\widetilde{t}_{m+1}-\tau_m)+B_m(\widetilde{t}_{m+1})\widetilde{x}(\widetilde{t}_{m+1}-\tau_m)+$$
$$+D_m(\widetilde{t}_{m+1})\widetilde{u}_m(\widetilde{t}_{m+1})\big]\geq 0;$$

4.4) *the conditions for the moments* $\widetilde{t}_i$, $i=\overline{2,m}$,

$$\psi_i(\widetilde{t}_i)\big[\dot{\varphi}_{i0}(\widetilde{t}_1)-A_i(\widetilde{t}_i)\dot{\varphi}_{i0}(\widetilde{t}_1-\tau_i)-B_i(\widetilde{t}_i)\varphi_{i0}(\widetilde{t}_1-\tau_i)-D_i\widetilde{u}_i(\widetilde{t}_i)\big]-$$
$$-\psi_i(\widetilde{t}_i+\tau_i)\big[E_i\widetilde{x}_i(\widetilde{t}_i)-\varphi_{i0}(\widetilde{t}_i)\big]-\chi_i(\widetilde{t}_i)\dot{\varphi}_{i0}(\widetilde{t}_i)+$$
$$+\chi_{i-1}(\widetilde{t}_i)\big[A_{i-1}(\widetilde{t}_i)\dot{\widetilde{x}}_{i-1}(\widetilde{t}_i-\tau_i)+$$
$$+B_{i-1}(\widetilde{t}_i)\widetilde{x}_{i-1}(\widetilde{t}_i-\tau_i)+D_i(\widetilde{t}_i)\widetilde{u}_i(\widetilde{t}_i)\big]=0.$$

5. Proof of Theorem 3.1

Consider the topological vector space

$$E_{\mu_m}=R^{1+m}\times\prod_{i=1}^m R^{n_i}\times\prod_{i=1}^m E_{\varphi_i}\times\prod_{i=1}^m E_{f_i}=E_x\times E_\zeta$$

of the points $\mu_m=(x,\zeta)$, where

$$x=(t_1,\dots,t_{m+1},x_{10},\dots,x_{m0})\in E_x,\ \ \zeta=(\varphi_1,\dots,\varphi_m,f_1,\dots,f_m)\in E_\zeta.$$

The set

$$X_0=(a,\widetilde{t}_1]\times\prod_{i=1}^m(\widetilde{t}_i,\widetilde{t}_{i+1}]\times\prod_{i=1}^m O_i\subset E_x$$

is a locally convex subspace.

By $D_0\subset E_{\mu_m}$ we denote the set of the elements

$$\mu_m\in X_0\times\prod_{j=1}^m\Delta_{j0}\times\prod_{j=1}^m E_{f_j},$$

to each of which there corresponds the solution $\{x_i(t;\mu_i),\ t\in[\tau_i,t_{i+1}]:i=\overline{1,m}\}$. The set D_0 is not empty, because $\widetilde{\mu}_m\in D_0$.

Lemma 5.1. *The set D_0 is finitely locally convex.*

Proof. Let $\mu_{m0}=(x_0,\zeta_0)\in D_0$, where

$$x_0=(t_1^0,\dots,t_{m+1}^0,x_{10}^0,\dots,x_{m0}^0),\ \ \zeta_0=(\varphi_1^0,\dots,\varphi_m^0,f_1^0,\dots,f_m^0),$$

is an arbitrary fixed point and $L_{\zeta_0} \subset E_\zeta$ is a linear finite dimensional manifold passing through the point ζ_0, i.e.,

$$L_{\zeta_0} = \Big\{\zeta_0 + \delta\zeta : \delta\zeta = \sum_{j=1}^{k_0} \lambda_j \delta\zeta_j,\ \lambda_j \in R,\ j = \overline{1, k_0}\Big\},$$

where

$$\delta\zeta_j = (\delta\varphi_{1j}, \dots, \delta\varphi_{mj}, \delta f_{1j}, \dots, \delta f_{mj}) \in E_\zeta, \quad j = \overline{1, k_0},$$

are fixed points.

Let $\{x_i^0(t) = x_i(t; \mu_{i0}),\ t \in [t_i^0, t_{i+1}^0] : i = \overline{1, m}\}$ be the solution corresponding to the element μ_{m0} and $K_{i0} \subset O_i$ be a compact set containing some neighborhood of the set $\varphi_i^0(J_i) \cup x_i^0([t_i^0, t_{i+1}^0])$.

There exists a number $\delta_0 > 0$ such that to every element

$$\mu_m \in \big(V(t_1^0; \delta_0) \cap (a, \widetilde{t}_1]\big) \times \prod_{j=1}^{m} \big(V(t_{i+1}^0; \delta_0) \cap (\widetilde{t}_i, \widetilde{t}_{i+1}]\big) \times$$

$$\times \prod_{j=1}^{m} \big(V(x_{i0}^0; \delta_0) \cap O_i\big) \times \prod_{j=1}^{m} \big(V(\varphi_i^0; \delta_0) \cap \Delta_{i0}\big) \times \prod_{j=1}^{m} \big(f_i^0 + W(K_{i0}; \delta_0)\big)$$

there corresponds the solution $\{x_i(t; \mu_i) \in \operatorname{int} K_{i0}, t \in [t_i^0, t_{i+1}^0] : i = \overline{1, m}\}$ (see Remark 2.1).

Let a number $\delta_1 \in [0, \delta_0]$ be so small that the neighborhood of the point ζ_0

$$V_{\zeta_0} = \Big\{\zeta_0 + \sum_{j=1}^{k_0} \lambda_j \delta\zeta_j : |\lambda_j| \le \delta_1,\ j = \overline{1, k_0}\Big\}$$

is included in the set

$$\prod_{i=1}^{m} \big(V(\varphi_i^0; \delta_0) \cap \Delta_{i0}\big) \times \prod_{i=1}^{m} \big(f_i^0 + W(K_{i0}; \delta_0)\big).$$

Thus there exist convex neighborhoods

$$V_{x_0} = \big(V(t_1^0; \delta_0) \cap (a, \widetilde{t}_1]\big) \times \prod_{j=1}^{m} \big(V(t_{i+1}^0; \delta_0) \cap (\widetilde{t}_i, \widetilde{t}_{i+1}]\big) \times$$

$$\times \prod_{i=1}^{m} \big(V(x_{i0}^0; \delta_0) \cap O_i\big) \subset X_0$$

and $V_{\zeta_0} \subset L_{\zeta_0}$ such that $V_{x_0} \times V_{\zeta_0} \subset D_0$.

Hence the set D_0 is finitely locally convex. $\square$

On the set D_0 we define the mapping

$$T : D_0 \longrightarrow R^{n_m}$$

by the formula

$$T(\mu_m) = x_m(t_{m+1}; \mu_m).$$

Lemma 5.2. *The mapping T is differentiable at the point $\widetilde{\mu}_m=(\widetilde{x},\widetilde{\zeta})$:*

$$
\begin{aligned}
dT_{\widetilde{\mu}_m}(\delta\mu_m)=\sum_{i=1}^{m}\Big\{Y_i(\widetilde{t}_i-)\Big[\dot{\widetilde{\varphi}}_i(\widetilde{t}_i)-\sum_{j=1}^{k_i}A_{ij}(\widetilde{t}_i)\dot{\widetilde{\varphi}}_i(\eta_{ij}(\widetilde{t}_i)+\\
+\sum_{j=0}^{p_i}(\widehat{\gamma}_{ij+1}^{-}-\widehat{\gamma}_{ij}^{-})f_{ij}^{-}\Big]-\sum_{j=p_i+1}^{s_i}Y_i(\gamma_{ij}-)f_{ij}^{-}\dot{\gamma}_{ij}^{-}+\\
+\Phi_i(\widetilde{t}_i)\Big[\frac{\partial\widetilde{g}_i}{\partial t_i}-\dot{\widetilde{\varphi}}(\widetilde{t}_i)+\Phi_{i-1}(\widetilde{t}_i)\Big[\sum_{j=1}^{k_{i-1}}A_{i-1j}(\widetilde{t}_i)\dot{\widetilde{x}}_{ij}^{-}+\\
+f_{i-1s_{i-1}+1}^{-}\Big\}\delta t_i+f_{ms_m+1}^{-}\delta t_{m+1}+\sum_{i=1}^{m}\Phi_i(\widetilde{t}_i)\delta x_{i0}+\\
+\sum_{i=1}^{m}\Big\{\sum_{j=p_i+1}^{s_i}\int_{\tau_{ij}(\widetilde{t}_i)}^{\widetilde{t}_i}Y_i(\gamma_{ij}(t))\frac{\partial\widetilde{f}_i[\gamma_{ij}(t)]}{\partial x_{ij}}\dot{\gamma}_{ij}(t)\delta\varphi_i(t)dt+\\
+\sum_{j=1}^{k_i}\int_{\eta_{ij}(\widetilde{t}_i)}^{\widetilde{t}_i}Y_i(\rho_{ij}(t))A_{ij}(\rho_{ij}(t)\dot{\rho}_{ij}(t)\dot{\delta}\varphi_i(t)dt+\int_{\widetilde{t}_i}^{\widetilde{t}_{i+1}}Y_i(t)\delta f_i[t]dt\Big\},
\end{aligned}
\tag{5.1}
$$

where $\Phi_i(t)$, $Y_i(t)$, $i=\overline{1,m}$, are matrix functions satisfying the system

$$
\begin{cases}
\dot{\Phi}_i(t)=-\displaystyle\sum_{j=1}^{s_i}Y_i(\gamma_{ij}(t))\frac{\partial\widetilde{f}_i[\gamma_{ij}(t)]}{\partial x_{ij}}\dot{\gamma}_{ij}(t), \ \ t\in[\widetilde{t}_i-\delta,\widetilde{t}_{i+1}],\\
Y_i(t)=\Phi_i(t)+\displaystyle\sum_{j=1}^{k_i}Y_i(\rho_{ij}(t))A_{ij}(\rho_{ij}(t)\dot{\rho}_{ij}(t), \quad i=\overline{1,m},
\end{cases}
\tag{5.2}
$$

and the conditions

$$
\begin{cases}
\Phi_{i-1}(\widetilde{t}_i)=\Phi_i(\widetilde{t}_i)\dfrac{\partial\widetilde{g}_i}{\partial x_{i-1}}, \ \ Y_{i-1}(\widetilde{t}_i)=Y_i(\widetilde{t}_i)\dfrac{\partial\widetilde{g}_i}{\partial x_{i-1}}, \ \ i=\overline{2,m},\\
\Phi_i(t)=Y_i(t)=\Theta_i, \ \ t>\widetilde{t}_{i+1}, \ \ i=\overline{1,m}, \ \ \Phi_m(\widetilde{t}_{m+1})=I_m.
\end{cases}
\tag{5.3}
$$

Proof. Let $L_{\widetilde{\zeta}}\subset E_\zeta$ be a linear finite dimensional manifold and let

$$V_{01}\subset X_0-\widetilde{x}, \quad V_{11}\subset L_{\widetilde{\zeta}}-\widetilde{\zeta}$$

be bounded convex neighborhoods of zero, where

$$\widetilde{x}=(\widetilde{t}_1,\dots,\widetilde{t}_{m+1},\widetilde{x}_{10},\dots,\widetilde{x}_{m0}), \ \ \widetilde{\zeta}=(\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_m,\widetilde{f}_1,\dots,\widetilde{f}_m).$$

It follows from finitely locally convexity of the set D_0 and Theorem 2.2 the existence of a number $\varepsilon_0>0$ such that for an arbitrary $(\varepsilon,\delta\mu_m)\in[0,\varepsilon_0]\times V_{01}\times V_{11}$ we have $\widetilde{\mu}_m+\varepsilon\delta\mu_m\in D_0$ and

$$
\begin{aligned}
\Delta x_m(\widetilde{t}_{m+1}+\varepsilon\delta t_{m+1};\varepsilon\delta\mu_m)=\varepsilon\delta x_m(\widetilde{t}_{m+1}+\varepsilon\delta\mu_m;\delta\mu_m)+\\
+o(\widetilde{t}_{m+1}+\varepsilon\delta t_{m+1};\varepsilon\delta\mu_m),
\end{aligned}
$$

where the variation $\delta x_m(\widetilde{t}_{m+1}+\varepsilon\delta\mu_m;\delta\mu_m)$ is calculated by the formula (2.4).

We have

$$\begin{aligned}T(\widetilde{\mu}_m+\varepsilon\delta\mu_m)-T(\widetilde{\mu}_m)&=x_m(\widetilde{t}_{m+1}+\varepsilon\delta t_{m+1};\widetilde{\mu}_m+\varepsilon\delta\mu_m)-\widetilde{x}_m(\widetilde{t}_{m+1})=\\&=x_m(\widetilde{t}_{m+1}+\varepsilon\delta t_{m+1};\widetilde{\mu}_m+\varepsilon\delta\mu_m)-\widetilde{x}_m(\widetilde{t}_{m+1}+\varepsilon\delta t_{m+1})+\\&+\big[\widetilde{x}_m(\widetilde{t}_{m+1}+\varepsilon\delta t_{m+1})-\widetilde{x}_m(\widetilde{t}_{m+1})\big]=\\&=\varepsilon\delta x_m(\widetilde{t}_{m+1}+\varepsilon\delta\mu_{m+1};\delta\mu_m)+\int\limits_{\widetilde{t}_{m+1}}^{\widetilde{t}_{m+1}+\varepsilon\delta t_{m+1}}\widetilde{f}_m[t]dt+o(\varepsilon\delta\mu).\end{aligned}\tag{5.4}$$

It is easy to see that

$$\lim_{\varepsilon\to 0}\delta x_m(\widetilde{t}_{m+1}+\varepsilon\delta t_{m+1};\delta\mu_m)=\delta x_m(\widetilde{t}_{m+1};\delta\mu_m)\tag{5.5}$$

uniformly for $\delta\mu_m\in V_{01}\times V_{11}$ and

$$\int\limits_{\widetilde{t}_{m+1}}^{\widetilde{t}_{m+1}+\varepsilon\delta t_{m+1}}\widetilde{f}_m[t]dt=\varepsilon f^-_{ms_m+1}\delta t_{m+1}+o(\varepsilon\delta\mu_m).\tag{5.6}$$

Let us introduce the following notation:

$$\begin{cases}\Phi_m(t)=\Phi_m(t;\widetilde{t}_{m+1}),\ \ Y_m(t)=Y_m(t;\widetilde{t}_{m+1}),\\ \Phi_{i-1}(t)=\Phi_i(\widetilde{t}_i)\dfrac{\partial\widetilde{g}_i}{\partial x_{i-1}}\Phi_{i-1}(t;\widetilde{t}_i),\\ Y_{i-1}(t)=\Phi_i(\widetilde{t}_i)\dfrac{\partial\widetilde{g}_i}{\partial x_{i-1}}Y_{i-1}(t;\widetilde{t}_i),\ \ i=\overline{m,2}.\end{cases}\tag{5.7}$$

Obviously $\Phi_i(t),Y_i(t)$ satisfies the system (5.2) and the conditions (5.3) (see (2.5) and (2.6)).

From (5.4), taking into consideration (5.5), (5.6), (5.7) and the variation formulas (2.4), we get

$$\begin{aligned}T(\widetilde{\mu}_m+\varepsilon\delta\mu_m)-T(\widetilde{\mu}_m)=\varepsilon\Big[\delta x_m(\widetilde{t}_{m+1};\delta\mu_m)+\sum_{j=1}^{k_m}A_{m_j}(\widetilde{t}_{m+1})\dot{\widetilde{x}}^-_{m_j}+\\+f^-_{ms_m+1})\delta t_{m+1}\Big]+o(\varepsilon\delta\mu_m)=\varepsilon dT_{\widetilde{\mu}_m}(\delta\mu_m)+o(\varepsilon\delta\mu_m),\end{aligned}\tag{5.8}$$

where

$$\begin{aligned}\delta x_m(\widetilde{t}_{m+1};\delta\mu_m)=\Big\{Y_m(\widetilde{t}_m-)\Big[\dot{\widetilde{\varphi}}_m(\widetilde{t}_m)-\\-\sum_{j=1}^{k_m}A_{m_j}(\widetilde{t}_m)\dot{\widetilde{\varphi}}_m(\eta_{m_j}(\widetilde{t}_m)+\sum_{j=0}^{p_m}(\widehat{\gamma}^-_{mj+1}-\widehat{\gamma}^-_{mj})f^-_{ij}\Big]-\\-\sum_{j=p_m+1}^{s_m}Y_m(\gamma_{mj}-)f^-_{mj}\dot{\gamma}^-_{mj}+\Phi_m(\widetilde{t}_m)\Big(\frac{\partial\widetilde{g}_m}{\partial t_m}-\dot{\widetilde{\varphi}}_m(\widetilde{t}_m)\Big)+\end{aligned}$$

$$+\Phi_{m-1}(\widetilde{t}_m)\Big[\sum_{j=1}^{k_{m-1}} A_{m-1j}(\widetilde{t}_m)\dot{\widetilde{x}}_{m-1j}^{-} + f_{m-1s_{m-1}+1}^{-}\Big]\Big\}\delta t_m+$$
$$+\Phi_m(\widetilde{t}_m)\delta x_{m0} + \Phi_{m-1}(\widetilde{t}_m)\delta x_{m-1}(\widetilde{t}_m;\delta\mu_{m-1})+$$
$$+\sum_{j=p_m+1}^{s_m}\int\limits_{\tau_{mj}(\widetilde{t}_m)}^{\widetilde{t}_m} Y_m(\gamma_{mj}(t))\frac{\partial\widetilde{f}_m[\gamma_{mj}(t)]}{\partial x_{mj}}\dot{\gamma}_{mj}(t)\delta\varphi_m(t)dt+$$
$$+\sum_{j=1}^{k_{m-1}}\int\limits_{\eta_{mj}(\widetilde{t}_m)}^{\widetilde{t}_m} Y_m(\rho_{mj}(t))A_{mj}(\rho_{mj}(t))\dot{\rho}_{mj}(t)\dot{\delta\varphi}_m(t)dt + \int\limits_{\widetilde{t}_m}^{\widetilde{t}_{m+1}} Y_m(t)\delta f_m[t]dt.$$

Continuing this process with respect to the variation $\delta x_j(t;\delta\mu_i)$, $i=\overline{m-1,1}$, and grouping terms in suitable way, from (5.8) we obtain the formula (5.1). □

Now consider the space $E_z = R\times E_{\mu_m}$ of the points $z=(\xi,\mu_m)$.
Define the set

$$X=[0;\infty)\times X_0,\quad D=[0,\infty)\times D_0.$$

Obviously the set $D\subset X\times E_\zeta$ is finitely locally convex (see Lemma 5.1).
On the set D we define the mapping

$$P: D\longrightarrow R^{1+l}$$

by the formula

$$P(z)=Q\big(t_1,\dots,t_{m+1},x_{10},\dots,x_{0m},T(\mu_m)\big)+(\xi,0,\dots,0)^*,$$

where $Q=(q^0,\dots,q^l)^*$.

Lemma 5.3. *The mapping P is differentiable at the point $\widetilde{z}=(0,\widetilde{\mu}_m)$:*

$$dP_{\widetilde{z}}(\delta z)=\sum_{i=1}^{m}\bigg\{\Big[\frac{\partial\widetilde{Q}}{\partial t_i}+\frac{\partial\widetilde{Q}}{\partial x_{m+1}}\Big(Y_i(\widetilde{t}_i-)\dot{\widetilde{\varphi}}_i(\widetilde{t}_i)-$$
$$-\sum_{j=1}^{k_i}A_{ij}(\widetilde{t}_i)\dot{\widetilde{\varphi}}_i(\eta_{ij}(\widetilde{t}_i))+\sum_{j=0}^{p_i}(\widehat{\gamma}_{ij+1}^{-}-\widehat{\gamma}_{ij}^{-})f_{ij}^{-}\Big)-$$
$$-\sum_{j=p_i+1}^{s_i}\frac{\partial\widetilde{Q}}{\partial x_{m+1}}Y_i(\gamma_{ij}-)f_{ij}^{-}\dot{\gamma}_{ij}^{-}+\frac{\partial\widetilde{Q}}{\partial x_{m+1}}\Phi_i(\widetilde{t}_i)\Big(\frac{\partial\widetilde{g}_i}{\partial t_i}-\dot{\widetilde{\varphi}}_i(\widetilde{t}_i)\Big)+$$
$$+\frac{\partial\widetilde{Q}}{\partial x_{m+1}}\Phi_{i-1}(\widetilde{t}_i)\Big(\sum_{j=1}^{k_{i-1}}A_{i-1j}(\widetilde{t}_i)\dot{\widetilde{x}}_{i-1j}^{-}+f_{i-1s_{i-1}+1}^{-}\Big)\Big]\delta t_i+$$
$$+\Big(\frac{\partial\widetilde{Q}}{\partial x_i}+\frac{\partial\widetilde{Q}}{\partial x_{m+1}}\Phi_i(\widetilde{t}_i)\Big)\delta x_{i0}+$$

$$+\sum_{j=p_i+1}^{s_i}\int\limits_{\tau_{ij}(\widetilde{t}_i)}^{\widetilde{t}_i}\frac{\partial\widetilde{Q}}{\partial x_{m+1}}Y_i(\gamma_{ij}(t))\frac{\partial\widetilde{f}_i[\gamma_{ij}(t)]}{\partial x_{ij}}\dot{\gamma}_{ij}(t)\delta\varphi_i(t)dt+$$

$$+\sum_{j=1}^{k_i}\int\limits_{\tau_{ij}(\widetilde{t}_i)}^{\widetilde{t}_i}\frac{\partial\widetilde{Q}}{\partial x_{m+1}}Y_i(\rho_{ij}(t))A_{ij}(\rho_{ij}(t))\dot{\rho}_{ij}(t)\dot{\delta}\varphi_i(t)dt$$

$$+\int\limits_{\widetilde{t}_i}^{\widetilde{t}_{i+1}}\frac{\partial\widetilde{Q}}{\partial x_{m+1}}Y_i(t)\delta f_i[t]dt\Big\}+\Big[\frac{\partial\widetilde{Q}}{\partial t_{m+1}}+$$

$$+\frac{\partial\widetilde{Q}}{\partial x_{m+1}}\Big(\sum_{j=1}^{k_m}A_{mj}(\widetilde{t}_{m+1})\dot{\widetilde{x}}_{mj}^{-}+f_{ms_m+1}^{-}\Big)\Big]\delta t_{m+1}+(\delta\xi,0,\dots,0)^*, \quad (5.9)$$

$$\delta z=(\delta\xi,\delta\mu_m)\in E_z-\widetilde{z}.$$

Proof. Let $L_{\widetilde{\zeta}}\subset E_\zeta$ be an arbitrary finite-dimensional linear manifold and let

$$V_{01}\subset X-\widetilde{x},\quad V_{11}\subset L_{\widetilde{\zeta}}$$

be arbitrary convex bounded neighborhoods of zero, where

$$\widetilde{x}=(0,\widetilde{t}_1,\dots,\widetilde{t}_{m+1},\widetilde{x}_{10},\dots,\widetilde{x}_{m0}),\ \ \widetilde{\zeta}=(\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_m,\widetilde{f}_1,\dots,\widetilde{f}_m).$$

There exists a number $\varepsilon_0>0$ such that for $(\varepsilon,\delta z)\in[0,\varepsilon_0]\times V_{01}\times V_{11}$

$$\widetilde{z}+\varepsilon\delta z\in D$$

and the formula (5.8) is true.

We have

$$P(\widetilde{z}+\varepsilon\delta z)-P(\widetilde{z})=Q\big(\widetilde{t}_1+\varepsilon\delta t_1,\dots,\widetilde{t}_{m+1}+\varepsilon\delta t_{m+1},\widetilde{x}_{10}+\varepsilon\delta x_{10},\dots$$
$$\dots,\widetilde{x}_{m0}+\varepsilon\delta x_{m0},T(\widetilde{\mu}_m+\varepsilon\delta\mu_m)\big)-Q\big(\widetilde{t}_1,\dots,\widetilde{t}_{m+1},\widetilde{x}_{10},\dots$$
$$\dots,\widetilde{x}_{m0},T(\widetilde{\mu}_m)\big)+\varepsilon(\delta\xi,0,\dots,0)^*.$$

Let the number $\varepsilon_0>0$ be so small that

$$T(\widetilde{\mu}_m)+t\big(T(\widetilde{\mu}_m+\varepsilon\delta\mu_m)-T(\widetilde{\mu}_m)\big)\in K_{m1},$$
$$\forall(t,\varepsilon,\delta\mu_m)\in[0,1]\times[0,\varepsilon_0]\times V_m^-$$

(see Lemma 2.2).

Now we transform the difference

$$Q\big(\widetilde{t}_1+\varepsilon\delta t_1,\dots,\widetilde{t}_{m+1}+\varepsilon\delta t_{m+1},\widetilde{x}_{10}+\varepsilon\delta x_{10},\dots\dots,\widetilde{x}_{m0}+\varepsilon\delta x_{m0},$$
$$T(\widetilde{\mu}_m+\varepsilon\delta\mu_m)\big)-Q\big(\widetilde{t}_1,\dots,\widetilde{t}_{m+1},\widetilde{x}_{10},\dots\dots,\widetilde{x}_{m0},T(\widetilde{\mu}_m)\big)=$$
$$=\int\limits_0^1\frac{d}{dt}Q\big(\widetilde{t}_1+\varepsilon t\delta t_1,\dots,\widetilde{t}_{m+1}+\varepsilon t\delta t_{m+1},\widetilde{x}_{10}+\varepsilon t\delta x_{10},\dots$$
$$\dots,\widetilde{x}_{m0}+\varepsilon t\delta x_{m0},T(\widetilde{\mu}_m)+t(T(\widetilde{\mu}_m+\varepsilon\delta\mu_m)-T(\widetilde{\mu}_m))\big)dt=$$

$$= \varepsilon\Big[\sum_{i=1}^{m+1}\frac{\partial\widetilde{Q}}{\partial t_i}\delta t_i+\sum_{i=1}^{m}\frac{\partial\widetilde{Q}}{\partial x_i}\delta x_{i0}+\frac{\partial\widetilde{Q}}{\partial x_{m+1}}dT_{\widetilde{\mu}_m}(\delta\mu_m)\Big]+\alpha(\varepsilon\delta\mu_m),$$

where

$$\alpha(\varepsilon\delta\mu_m)=\varepsilon\int_0^1\Big\{\sum_{i=1}^{m+1}\Big[\Big(\frac{\partial\widetilde{Q}[\varepsilon;t]}{\partial t_i}-\frac{\partial\widetilde{Q}}{\partial t_i}\Big)\delta t_i+\sum_{i=1}^{m}\Big(\frac{\partial\widetilde{Q}[\varepsilon;t]}{\partial x_i}-\frac{\partial\widetilde{Q}}{\partial x_i}\Big)\delta x_{i0}+$$

$$+\Big[\frac{\partial\widetilde{Q}[\varepsilon;t]}{\partial x_{m+1}}-\frac{\partial\widetilde{Q}}{\partial x_{m+1}}\Big]dT_{\widetilde{\mu}_m}(\delta\mu_m)\Big\}dt+o(\varepsilon\delta\mu_m)\int_0^1\frac{\partial\widetilde{Q}[\varepsilon;t]}{\partial x_{m+1}}\,dt,$$

$$\frac{\partial\widetilde{Q}[\varepsilon;t]}{\partial t_i}=\frac{\partial\widetilde{Q}}{\partial t_i}\big(\widetilde{t}_1+\varepsilon t\delta t_1,\dots,\widetilde{t}_{m+1}+\varepsilon t\delta t_{m+1},\widetilde{x}_{10}+\varepsilon t\delta x_{10},\dots$$
$$\dots,\widetilde{x}_{m0}+\varepsilon t\delta x_{m0},T(\widetilde{\mu}_m)+t(T(\widetilde{\mu}_m+\varepsilon\delta\mu_m)-T(\widetilde{\mu}_m))\big).$$

It is not difficult to see that uniformly for $(t,\delta z)\in[0,1]\times V_{01}\times V_{11}$ we have

$$\lim_{\varepsilon\to0}\Big(\frac{\partial\widetilde{Q}[\varepsilon;t]}{\partial t_i}-\frac{\partial\widetilde{Q}}{\partial t_i}\Big)=0,\quad i=\overline{1,m+1};$$
$$\lim_{\varepsilon\to0}\Big(\frac{\partial\widetilde{Q}[\varepsilon;t]}{\partial x_i}-\frac{\partial\widetilde{Q}}{\partial x_i}\Big)=0,\quad i=\overline{1,m};$$
$$\lim_{\varepsilon\to0}\Big(\frac{\partial\widetilde{Q}[\varepsilon;t]}{\partial x_{m+1}}-\frac{\partial\widetilde{Q}}{\partial x_{m+1}}\Big)=0.$$

Therefore $\alpha(\varepsilon\delta z)=o(\varepsilon\delta z)$.

Thus

$$P(\widetilde{z}+\varepsilon\delta z)-P(\widetilde{z})=\varepsilon\Big[\sum_{i=1}^{m+1}\frac{\partial\widetilde{Q}}{\partial t_i}\delta t_i+\sum_{i=1}^{m}\frac{\partial\widetilde{Q}}{\partial x_i}\delta x_{i0}+$$
$$+\frac{\partial\widetilde{Q}}{\partial x_{m+1}}dT_{\widetilde{\mu_m}}(\delta\mu_m)+(\delta\xi,0,\dots,0)^*\Big]+o(\varepsilon\delta\mu_m).$$

Hence according to the equality (5.1) we obtain (5.9). □

In the space E_z we define the filter $\Phi_{\widetilde{z}}$ as the direct product of the filters

$$\Phi_{\widetilde{z}}=\Phi_{\widetilde{x}}\times\prod_{i=1}^{m}\Phi_{\widetilde{\varphi}_i}\times\prod_{i=1}^{m}\Phi_{\widetilde{f}_i},$$

where the filters $\Phi_{\widetilde{x}}$, $\Phi_{\widetilde{\varphi}_i}$, $i=\overline{1,m}$, are defined by the convex bases

$$\Big\{\big(V^0\cap[0,\infty)\big)\times\big(V_{\widetilde{t}_1}\cap(a,\widetilde{t}_1]\big)\times\prod_{i=1}^{m}\big(V_{\widetilde{t}_{i+1}}\cap(\widetilde{t}_i,\widetilde{t}_{i+1}]\big)\times$$
$$\times\prod_{i=1}^{m}V_{\widetilde{x}_{i1}}:V^0\subset R,\ \ V_{\widetilde{t}_i}\subset R,\ \ i=\overline{1,m+1},\ \ V_{\widetilde{x}_{i0}}\subset O_i,\ \ i=\overline{1,m}\ \ -$$

$$\text{convex neighborhood of points } 0 \in R,\ \widetilde{t}_i \in R,\ i = \overline{1, m+1},$$
$$\widetilde{x}_{i0} \subset 0,\ \ i = \overline{1, m}\Big\};$$
$$\big\{V_{\widetilde{\varphi}_i} \cap \Delta_{i0} : V_{\widetilde{\varphi}_i} \subset E_{\varphi_i} \ - \text{ convex neighborhood } \big\},\ \ i = \overline{1, m}.$$

The filters $\Phi_{\widetilde{f}_i}, i = \overline{1, m}$ are introduced in Section 2.

The filter $\Phi_{\widetilde{z}}$ is quasi-convex, because it is the direct product of convex filters $\Phi_{\widetilde{x}}$, $\Phi_{\widetilde{\varphi}_i}$, $i = \overline{1, m}$, on the quasi-convex filters $\Phi_{\widetilde{f}_i}$, $i = \overline{1, m}$ (see Lemma 2.3).

According to Theorem 2.1 and Remark 1.1, there exists a number $\delta_1 > 0$ such that

$$W = [0, \infty) \times \big(V(\widetilde{t}_1; \delta_1) \cap (a, \widetilde{t}_1]\big) \times \prod_{i=1}^{m} \big(V(\widetilde{t}_{i+1}; \delta_1) \cap (\widetilde{t}_i, \widetilde{t}_{i+1}]\big) \times$$
$$\times \prod_{i=1}^{m} V_{\widetilde{x}_{i0}} \times \prod_{i=1}^{m} \big(\widetilde{f}_i + W(K_{i1}; \delta_1)\big) \subset D.$$

Furthermore the mapping

$$P : W \longrightarrow R^{1+l}$$

is continuous in the topology induced from E_z.

Here $V(\widetilde{x}_{i0}, \delta_1) \subset O_i$ and $K_{i1} \subset O_i$ is a compact set containing some neighborhood of the set $\widetilde{\varphi}_i(J_i) \cup \widetilde{x}_i([\widetilde{t}_i, \widetilde{t}_{i+1}])$.

The element $W^{(i)}_{K_{i1}, \delta_1}$ of the filter $\Phi_{\widetilde{f}_i}$ is a subset of the convex set $\widetilde{f}_i + W(K_{i1}, \delta_1)$. Therefore

$$co\big(W_{\widetilde{z}}(K_{11}, \ldots, K_{m1}; \delta_1)\big) \subset W \subset D,$$

where

$$W_{\widetilde{z}}(K_{11}, \ldots, K_{m1}; \delta_1) = [0, \infty) \times \big(V(\widetilde{t}_1; \delta_1) \cap (a, \widetilde{t}_1]\big) \times$$
$$\times \prod_{i=1}^{m} \big(V(\widetilde{t}_{i+1}; \delta_1) \cap (\widetilde{t}_i, \widetilde{t}_{i+1}]\big) \times \prod_{i=1}^{m} V_{\widetilde{x}_{i0}} \times$$
$$\times \prod_{i=1}^{m} \big(\widetilde{\varphi}_i; \delta_1) \cap \Delta_{i0}\big) \times \prod_{i=1}^{m} W^{(i)}_{K_{i1}, \delta_1} \in \Phi_{\widetilde{z}}.$$

Consequently, there exists an element $W_{\widetilde{z}}(K_{11}, \ldots, K_{m1}; \delta_1)$ of the filter $\Phi_{\widetilde{z}}$ such that the mapping

$$P : \text{co}\left(W_{\widetilde{z}}(K_{11}, \ldots, K_{m1}; \delta_1)\right) \longrightarrow R^{1+l}$$

is continuous.

Thus the mapping P is defined and continuous on the filter $co([\Phi_{\widetilde{z}}])$.

The point $\widetilde{z} = (0, \widetilde{\mu}_m)$ belongs to all elements of the filter $\Phi_{\widetilde{z}}$. Furthermore,

$$P(\widetilde{z}) = (q^0(\widetilde{t}_1, \ldots, \widetilde{t}_{m+1}, \widetilde{x}_{10}, \ldots, \widetilde{x}_{m0}, \widetilde{x}_m(\widetilde{t}_{m+1})), 0, \ldots, 0)^*.$$

Now we introduce the set

$$B_1 = \Big\{\mu_m = (t_1, \dots, t_{m+1}, x_{10}, \dots, x_{m0}, \varphi_1, \dots, \varphi_m, f_1, \dots, f_m) : \\ f_i = f_i(t, x_{i1}, \dots, x_{im}, u_i(t)), i = \overline{1, m}; \\ (t_1, \dots, t_{m+1}, x_{10}, \dots, x_{m0}, \varphi_1, \dots, \varphi_m, u_1, \dots, u_m) \in B_0\Big\}.$$

For an arbitrary element

$$z = (\xi, \mu_m) \in W_{\widetilde{z}} \cap \big([0, \infty) \times B_1\big),$$

where $W_{\widetilde{z}} \in \Phi_{\widetilde{z}}$, we have

$$P(z) = \big(q^0(t_1, \dots, t_{m+1}, x_{10}, \dots, x_{m0}, x(t_{m+1}; \mu_m)), 0, \dots, 0\big)^* + \\ +(\xi, 0, \dots, 0)^*.$$

The element $\widetilde{\sigma}_m = (\widetilde{t}_1, \dots \widetilde{t}_{m+1}, \widetilde{x}_{10}, \dots, \widetilde{x}_{m0}, \widetilde{\varphi}_1, \dots, \widetilde{\varphi}_m, \widetilde{u}_1, \dots, \widetilde{u}_m) \in B_0$ is optimal, so there exists an element

$$W_{\widetilde{z}}(K_{12}, \dots, K_{m2}; \delta_2) \in \Phi_{\widetilde{z}},$$

where $\delta_2 \in (0, \widetilde{\delta})$, $K_{i2} \subset O_i$ is a compact set containing $\widetilde{K}_i$, such that for any element

$$z \in W_{\widetilde{z}}(K_{12}, \dots, K_{m2}; \delta_2) \cap ([0, \infty) \times B_1)$$

we have the inequality

$$q^0\big(\widetilde{t}_1, \dots \widetilde{t}_{m+1}, \widetilde{x}_{10}, \dots, \widetilde{x}_{m0}, \widetilde{x}_m(\widetilde{t}_{m+1})\big) \leq \\ \leq q^0\big(t_1, \dots, t_{m+1}, x_{10}, \dots, x_{m0}, x_m(t_{m+1}; \mu_m)\big) + \xi.$$

It is easy to see that

$$P\big(W_{\widetilde{z}}(K_{12}, \dots, K_{m2}; \delta_2) \cap ([0, \infty) \times B_1)\big) \subset L = \\ = \{(p^0, 0, \dots, 0)^* \in R^{1+l} : p^0 \in R\},$$

and the point $P(\widetilde{z})$ is a boundary point of the set

$$P\big(W_{\widetilde{z}}(K_{12}, \dots, K_{m2}; \delta_2) \cap ([0, \infty) \times B_1)\big)$$

with respect to the one-dimensional space L.

Thus

$$P(\widetilde{z}) \in \partial\big(P(W_{\widetilde{z}}(K_{12}, \dots, K_{m2}; \delta_2)) \cap L\big)$$

and, all the more,

$$P(\widetilde{z}) \in \partial P\big(W_{\widetilde{z}}(K_{12}, \dots, K_{m2}; \delta_2)\big).$$

Hence the mapping P is critical on the filter $\Phi_{\widetilde{z}}$.

The assumptions of Theorem 2.5 are fulfilled. Consequently, there exist a non-zero vector $\pi = (\pi_0, \dots, \pi_l) \neq 0$ and an element $\widehat{W}_{\widetilde{z}} \in \Phi_{\widetilde{z}}$ such that the following inequality is fulfilled

$$\pi dP_{\widetilde{z}}(\delta z) \leq 0 \quad \forall\, \delta z \in \operatorname{cone}(\widehat{W}_{\widetilde{z}} - \widetilde{z}), \tag{5.10}$$

where $dP_{\widetilde{z}}$ has the form (5.9).

We introduce the functions

$$\psi_i(t) = \pi \frac{\partial \widetilde{Q}}{\partial x_{m+1}}\, Y_i(t), \quad \chi_i(t) = \pi \frac{\partial \widetilde{Q}}{\partial x_{m+1}}\, \Phi_i(t), \quad i = \overline{1,m}, \tag{5.11}$$

which, as it is easy to see, satisfy the equation (3.5) and the conditions

$$\begin{cases} \psi_i(t) = \chi_i(t) = 0, \;\; t \in (\widetilde{t}_{i+1}, \gamma_{i0}], \;\; i = \overline{1,m}; \\ \psi_m(\widetilde{t}_{m+1}) = \chi_m(\widetilde{t}_{m+1}) = \pi \dfrac{\partial \widetilde{Q}}{\partial x_{m1}}\,, \\ \psi_{i-1}(\widetilde{t}_i) = \chi_{i-1}(\widetilde{t}_i) = \chi_i(\widetilde{t}_i)\frac{\partial \widetilde{g}_i}{\partial x_{i-1}}, \;\; i = \overline{2,m}. \end{cases} \tag{5.12}$$

From the inequality (5.10) according to (5.9), (5.11) and (5.12) we get

$$\sum_{i=1}^{m} \Big\{ \Big[\pi \frac{\partial \widetilde{Q}}{\partial t_i} + \psi_i(\widetilde{t}_i -)(\dot{\widetilde{\varphi}}_i(\widetilde{t}_i) - \sum_{j=1}^{k_i} A_{ij}(\widetilde{t}_i)\dot{\widetilde{\varphi}}_i(\eta_{ij}(\widetilde{t}_i)) +$$

$$+ \sum_{j=0}^{p_i} (\widehat{\gamma}_{ij+1}^{-} - \widehat{\gamma}_{ij}^{-}) f_{ij}^{-}) - \sum_{j=p_i+1}^{s_i} \psi_i(\gamma_{ij}^{-}) f_{ij}^{-} \dot{\gamma}_{ij}^{-} + \chi_i(\widetilde{t}_i) \times$$

$$\times \Big(\frac{\partial \widetilde{g}_i}{\partial t_i} - \dot{\widetilde{\varphi}}_i(\widetilde{t}_i) \Big) + \chi_{i-1}(\widetilde{t}_i) \Big(\sum_{j=1}^{k_{i-1}} A_{i-1j}(\widetilde{t}_i) \dot{\widetilde{x}}_{i-1j}^{-} + f_{i-1s_{i-1}+1}^{-} \Big) \Big] \delta t_i +$$

$$+ \Big(\pi \frac{\partial \widetilde{Q}}{\partial x_i} + \chi_i(\widetilde{t}_i) \Big) \delta x_{i0} + \sum_{j=p_i+1}^{s_i} \int\limits_{\tau_{ij}(\widetilde{t}_i)}^{\widetilde{t}_i} \psi_i(\gamma_{ij}(t)) \times$$

$$\times \frac{\partial \widetilde{f}_i[\gamma_{ij}(t)]}{\partial x_{ij}} \dot{\gamma}_{ij}(t) \delta\varphi_i(t) dt + \sum_{j=1}^{k_i} \int\limits_{\eta_{ij}(\widetilde{t}_i)}^{\widetilde{t}_i} \psi_i(\rho_{ij}(t)) A_{ij}(\rho_{ij}(t)) \times$$

$$\times \dot{\rho}_{ij}(t) \dot{\delta}\varphi_i(t) dt + \int\limits_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} \psi_i(t) \delta f_i[t] dt \Big\} +$$

$$+ \Big[\frac{\partial \widetilde{g}_i}{\partial t_{m+1}} + \psi_m(\widetilde{t}_{m+1}) \Big(\sum_{j=1}^{k_m} A_{mj}(\widetilde{t}_{m+1}) \dot{\widetilde{x}}_{mj}^{-} + f_{ms_m+1}^{-} \Big) \Big] \delta t_{m+1} +$$

$$+ \pi_0 \delta\xi \le 0 \;\; \forall\, \delta z \in \operatorname{cone}(\widehat{W}_{\widetilde{z}} - \widetilde{z}). \tag{5.13}$$

The condition $\delta z \in \operatorname{cone}(\widehat{W}_{\widetilde{z}} - \widetilde{z})$ is equivalent to the conditions $\delta\xi \in [0,\infty)$, $\delta t_i \in (-\infty, 0]$, $i = \overline{1, m+1}$; $\delta x_{i0} \in R^{n_i}$, $\delta\varphi_i \in \operatorname{cone}(\widehat{W}_{\widetilde{\varphi}_i} - \widetilde{\varphi}_i)$, $\delta f_i \in \operatorname{cone}(\widehat{W}_{\widetilde{f}_i} - \widetilde{f}_i)$, $i = \overline{1,m}$, where

$$\widehat{W}_{\widetilde{\varphi}_i} = \widehat{V}_{\widetilde{\varphi}_i} \cap \Delta_{i0}, \quad \widehat{W}_{\widetilde{f}_i} \in \Phi_{\widetilde{f}_i}, \quad i = \overline{1,m}.$$

Let $\delta t_i = 0$, $i = \overline{1, m+1}$, $\delta x_{i0} = \delta\varphi_i = \delta f_i = 0$, $i = \overline{1,m}$. Then

$$\pi_0 \delta\xi \le 0 \quad \forall\, \delta\xi \in [0,\infty).$$

Hence it follows

$$\pi_0 \leq 0.$$

Let $\delta\varphi_i = \delta f_i = \delta x_{i0} = 0$, $i = \overline{1,m}$, $\delta\xi = 0$. Then from (5.13) with $\delta t_i \in (-\infty, 0]$, $i = \overline{1, m+1}$, we obtain the conditions for the moments $\widetilde{t}_i$, $i = \overline{1, m+1}$ (see 3.4)).

Let $\delta t_i = 0$, $i = \overline{1, m+1}$, $\delta\varphi_i = \delta f_i = 0$, $i = \overline{1,m}$, $\delta\xi = 0$. Then with $\delta x_{i0} \in R^{n_i}$ we have

$$\pi \frac{\partial \widetilde{Q}}{\partial x_i} + \chi(\widetilde{t}_i) = 0, \quad i = \overline{1,m}. \tag{5.14}$$

Let $\delta t_i = 0$, $i = \overline{1, m+1}$, $\delta f_i = \delta x_{i0} = 0$, $i = \overline{1,m}$, $\delta\xi = 0$ $\delta\varphi_j(t) = 0$, $j = \overline{1,m}$, $j \neq i$. Then

$$\sum_{j=p_i+1}^{s_i} \int\limits_{\tau_{ij}(\widetilde{t}_i)}^{\widetilde{t}_i} \psi_i(\gamma_{ij}(t)) \frac{\partial \widetilde{f}_i[\gamma_{ij}(t)]}{\partial x_{ij}} \dot{\gamma}_{ij}(t) \delta\varphi_i(t) dt +$$

$$+ \sum_{j=1}^{k_i} \int\limits_{\eta_{ij}(\widetilde{t}_i)}^{\widetilde{t}_i} \psi_i(\rho_{ij}(t)) A_{ij}(\rho_{ij}(t)) \dot{\rho}_{ij}(t) \dot{\delta}\varphi_i(t) dt \leq 0, \ \forall\, \delta\varphi_i \in \operatorname{cone}(\widehat{W}_{\widetilde{\varphi}_i} - \widetilde{\varphi}_i).$$

From this inequality and the inclusion

$$\operatorname{cone}\big(\widehat{W}_{\widetilde{\varphi}_i} - \widetilde{\varphi}_i\big) \supset \Delta_{i0} - \widetilde{\varphi}_i$$

follows the integral maximum principle for the initial functions $\widetilde{\varphi}_i$, $i = \overline{1,m}$.

In (5.13) assume that $\delta t_i = 0$, $i = \overline{1, m+1}$, $\delta x_{i0} = \delta\varphi_i = 0$, $i = \overline{1,m}$, $\delta\xi = 0$, $\delta f_j[t] = 0$ $j = \overline{1,m}$, $j \neq i$. Then we get

$$\int\limits_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} \psi_i(t) \delta f_i[t] dt \leq 0, \quad \forall\, \delta f_i \in \operatorname{cone}\big(\widehat{W}_{\widetilde{f}_i} - \widetilde{f}_i\big). \tag{5.15}$$

It is clear that this inequality is the more so true with

$$\delta f_i \in \operatorname{cone}\Big(\big[\widehat{W}_{\widetilde{f}_i} \cap W^{(i)}(K_{i1}; \alpha_0)\big] - \widetilde{f}_i\Big),$$

where

$$W^{(i)}(K_{i1}; \alpha_0) = \widetilde{f}_i + W(K_{i1}; \alpha_0).$$

According to Lemma 2.5, the inequality (5.15) also is true with

$$\delta f_i \in \operatorname{cone}\Big(\big[W^{(i)}(K_{i1}; \alpha_0)\big]_{\widehat{W}_{\widetilde{f}_i}} - \widetilde{f}_i\Big).$$

But by virtue of Lemma 2.3 we can conclude that the inequality (5.15) takes place with

$$\delta f_i = f_i\big(t, x_{i1}, \dots, x_{is_i}, u_i(t)\big) - f_i(t, x_{i1}, \dots, x_{is_i}, \widetilde{u}_i(t)) \in \widetilde{F}_i - \widetilde{f}_i.$$

Taking this into consideration, from (5.15) we obtain the integral maximum principle 3.1) $\square$

Remark 5.1. Theorems 3.2 and 3.3 are proved analogously to Theorem 3.1. In this case for calculation of the differential we use Theorems 2.3 and 2.4, respectively. Moreover, we replace the set X_0 respectively by the sets

$$\prod_{i=1}^{m}[\widetilde{t}_i,\widetilde{t}_{i+1})\times[\widetilde{t}_{m+1},b)\times\prod_{i=1}^{m}O_i \ \text{ and } \ J^{m+1}\times\prod_{i=1}^{m}O_i.$$

References

1. H. H. Hall and E. D. Zambelii, On the optimization of multistage rockets. *JET Propulsion* **28**(1958), No. 7, 463–465.
2. V. A. Troickij, Optimization of motion for a two-stage rocket. (Russian) *Prikl. Mat. Mekh.* **29**(1965), 745–750. English transl.: *PMM, J. Appl. Math. Mech.* **29**(1965), 884–889.
3. V. V. Veličenko, Optimal control of composite systems. (Russian) *Dokl. Akad. Nauk SSSR* **176**(1967), 754–756.
4. V. A. Medvedev and V. N. Rozova, Optimal control of stepped systems. *Avtom. Telemekh.* **1972**, No. 3, 15–23; English transl.: *Autom. Remote Control* **33**(1972), 359–366.
5. J. Warga, Optimal control of differential and functional equations. *Academic Press, New York–London,* 1972.
6. I. K. Gogodze, About optimal control of systems, which changes his structure on contiguous intervals of time. (Russian) *Trudy Inst. Sistem Upravleniya Akad. Nauk Gruzin. SSR* **16**(1980), 5–9.
7. G. L. Kharatishvili, The maximum principle in optimal problems with switching. (Russian) *Trudy Inst. Sistem Upravleniya Akad. Nauk Gruzin. SSR* **19**(1980), No. 1, 5–17.
8. V. G. Boltyanskiĭ, The problem of optimization with change of phase space. (Russian) *Differentsial'nye Uravneniya* **19**(1983), No. 3, 518–521.
9. M. S. Nikol'skiĭ and A. Akhmadaliev, A linear problem of terminal control of composite systems. (Russian) *Vestnik Moskov. Univ. Ser. XV Vychisl. Mat. Kibernet.* **1984**, No. 1, 71–77.
10. L. T. Ashchepkov, Optimal control of discontinuous systems. (Russian) *Nauka, Sibirsk. Otdel., Novosibirsk,* 1987.
11. A. I. Arsenashvili, An optimal control problem for step-by-step systems with delay. (Russian) *Trudy Inst. Sistem Upravleniya Akad. Nauk Gruzin. SSR* **24**(1985), No. 1, 17–28.
12. N. M. Avalishvili, An existence theorem for a class of optimal problems with switching and variable delays. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **116**(1984), No. 1, 33–35.
13. G. Kharatishvili and T. Tadumadze, The problem of optimal control for nonlinear systems with variable structure, delays and piecewise continuous prehistory. *Mem. Differential Equations Math. Phys.* **11**(1997), 67–88.
14. Sh. E. Magerramov and K. B. Mansimov, Optimization of a class of discrete step control systems. (Russian) *Zh. Vychisl. Mat. Mat. Fiz.* **41**(2001), No. 3, 360–366; English transl.: *Comput. Math. Math. Phys.* **41**(2001), No. 3, 334–339.
15. T. Tadumadze and A. Arsenashvili, Optimal control of delayed variable structure systems with discontinuous initial condition. *Appl. Comput. Math.* **4**(2005), No. 1, 48–53.
16. A. I. Arsenashvili, Necessary conditions for optimality for systems with variable structure with deviating argument of neutral type. (Russian) *Optimal problems in systems with variable structure* (Russian), 72–93, *Tbilis. Gos. Univ., Inst. Prikl. Mat., Tbilisi,* 1985.

17. A. I. Arsenashvili and T. A. Tadumadze, Necessary conditions for optimality for controllable systems of neutral type with variable structure and a continuous condition of succession. (Russian) *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **27**(1988), 9–45.
18. B. Sh. Mordukhovich and A. M. Sasonkin, Duality and conditions for optimality in problems of control of functional-differential systems of neutral type. (Russian) *Differentsial'nye Uravneniya* **21**(1985), No. 5, 788–798; English transl.: *Differ. Equations* **21**(1985), 532–540.
19. B. Sh. Mordukhovich, Approximation methods in problems of optimization and control. (Russian) *Nauka, Moscow,* 1988.
20. B. S. Mordukhovich and L. Wang, Optimal control of neutral functional-differential inclusions. *SIAM J. Control Optim.* **43**(2004), No. 1, 111–136 (electronic).
21. N. Gorgodze, Necessary conditions of optimality in neutral type optimal problems with non-fixed initial moment. *Mem. Differential Equations Math. Phys.* **19**(2000), 150–153.
22. I. Ramishvili, The linearized maximum principle for quasi-linear neutral optimal problems with discontinuous initial condition and variable delays in controls. *Mem. Differential Equations Math. Phys.* **31**(2004), 145–148
23. R. V. Gamkrelidze and G. L. Kharatishvili, Extremal problems in linear topological spaces. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **33**(1969), 781–839.
24. G. L. Kharatishvili and T. A. Tadumadze, Formulas for the variation of a solution and optimal control problems for differential equations with retarded arguments. (Russian) *Sovrem. Mat. Prilozh.* No. 25, *Optimal. Upr.,* (2005), 3–166.
25. A. P. Robertson and W. J. Robertson, Topological vector spaces. *Cambridge Tracts in Mathematics and Mathematical Physics,* No. 53. *Cambridge University Press, New York,* 1964.
26. J. Hale, Theory of functional differential equations. Second edition. *Applied Mathematical Sciences,* Vol. 3. *Springer-Verlag, New York–Heidelberg,* 1977.

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