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**CORRECT BOUNDARY VALUE PROBLEMS  
FOR SOME CLASSES OF SINGULAR ELLIPTIC  
DIFFERENTIAL EQUATIONS ON A PLANE**

**Abstract.** The investigation of differential equations of the type

$$\frac{\partial^n \omega}{\partial \bar{z}^n} + a_{n-1} \frac{\partial^{n-1} \omega}{\partial \bar{z}^{n-1}} + a_{n-2} \frac{\partial^{n-2} \omega}{\partial \bar{z}^{n-2}} + \cdots + a_0 \omega = 0$$

with sufficiently smooth coefficients  $a_0, a_1, \dots, a_{n-1}$  (the theory of meta-analytic functions) traces back to the work of G. Kolosov [6]. Subsequently, a vast number of papers in this direction were published by many authors. The present work deals with some singular cases of the above-given equation. Correct boundary value problems are pointed out, and their in a sense complete analysis is given.

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**Key words and phrases.** Meta-analytic functions, singular, elliptic, differential, equation, correct, boundary, problems.

**რეზიუმე.** საკმარისად გლუვკოეფიციენტებთან

$$\frac{\partial^n \omega}{\partial \bar{z}^n} + a_{n-1} \frac{\partial^{n-1} \omega}{\partial \bar{z}^{n-1}} + a_{n-2} \frac{\partial^{n-2} \omega}{\partial \bar{z}^{n-2}} + \cdots + a_0 \omega = 0$$

დიფერენციალურ განტოლებათა კვლევა (მეტაანალიზური ფუნქციათა თეორია) სათავეს იღებს გ. კოლოსოვის [6] ნაშრომში. შემდგომ ამ მიმართულებით გამოქვეყნდა სხვადასხვა ავტორთა მიერ შესრულებულ გამოკვლევათა დიდი რაოდენობა.

წინამდებარე ნაშრომში შეისწავლება აღნიშნული განტოლების ზოგიერთი სინგულარული შემთხვევა; ამ განტოლებათათვის მითითებულია კორექტული სასაზღვრო ამოცანები და მოცემულია მათი გარკვეული აზრით სრული ანალიზი.

*In Memory of Professor G. Manjavidze*

1<sup>0</sup>. In the domain  $G$  containing the origin of the plane of a complex variable  $z = x + iy$  we consider a differential equation of the type

$$E_\nu \omega \equiv z^{2\nu} \frac{\partial^2 \omega}{\partial \bar{z}^2} + Az^\nu \frac{\partial \omega}{\partial \bar{z}} + B\omega = 0, \tag{1}$$

where  $A$  and  $B$  are given complex numbers,  $\nu \geq 2$  is a given natural number and, as usual,  $\frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$ . To avoid a more simple case we assume that

$$B \neq 0. \tag{2}$$

The function  $\omega(z)$  is said to be a solution of the equation (1), if it belongs to the class  $C^2(G \setminus \{0\})$  and satisfies (1) at every point of the domain  $G \setminus \{0\}$ . We denote by  $\mathbb{K}$  the set of such functions; it should be noted that it is wide enough.

Every non-trivial (not identically equal to zero) function from the set  $\mathbb{K}$ , being a classical solution of an elliptic differential equation in the neighborhood of any non-zero point of the domain  $G$ , has an isolated singularity at the point  $z = 0$ . The analysis of the structure of the functions  $\omega \in \mathbb{K}$  shows highly complicated nature of their behaviour (in the vicinity of the singular point  $z = 0$ ) and, undoubtedly, is of independent interest because it allows one to obtain a priori estimates of solutions and of their derivatives which in turn are necessary for the correct statement and for the investigation of boundary value problems. A highly complicated nature of behaviour of solutions in the vicinity of the origin can be explained first by the fact that the equation (1) at the point  $z = 0$  degenerates up to the zero order.

For every function  $\omega(z) \in \mathbb{K}$  we introduce into consideration the following natural characteristic, i.e., the function of the real argument  $\rho > 0$ ,

$$T_\omega(\rho) \equiv \max_{0 \leq \varphi \leq 2\pi} \left\{ \left| \omega(\rho e^{i\varphi}) \right| + \left| \frac{\partial \omega}{\partial \bar{z}}(\rho e^{i\varphi}) \right| \right\}. \tag{3}$$

According to Theorem 1 proven below, we in particular conclude that for every non-trivial solution  $\omega(z)$  the function (3) increases more rapidly not only than an arbitrary power of  $\frac{1}{\rho}$  as  $\rho \rightarrow 0$ , but more rapidly than the function  $\exp\{\frac{\delta}{\rho^{\nu-1}}\}$  for certain positive numbers  $\delta$ .

With the equation (1) is tightly connected the characteristic equation

$$\lambda^2 + A\lambda + B = 0,$$

where  $\lambda$  is an unknown complex number, which, by (2), has two non-zero, possibly coinciding, roots; we denote them by  $\lambda_1$  and  $\lambda_2$ , and in what follows it will be assumed that

$$|\lambda_1| \leq |\lambda_2|. \tag{4}$$

Having in hand the roots  $\lambda_1$  and  $\lambda_2$ , we can factorize the operator  $E_\nu$  in the form

$$E_\nu = \left( z^\nu \frac{\partial}{\partial \bar{z}} - \lambda_1 I \right) \circ \left( z^\nu \frac{\partial}{\partial \bar{z}} - \lambda_2 I \right),$$

and immediately obtain that every function  $\omega(z) \in \mathbb{K}$  under the condition

$$\lambda_1 \neq \lambda_2$$

is representable as

$$\omega(z) = \phi(z) \exp \left\{ \frac{\lambda_1 \bar{z}}{z^\nu} \right\} + \psi(z) \exp \left\{ \frac{\lambda_2 \bar{z}}{z^\nu} \right\}, \quad (5)$$

and under the condition

$$\lambda_0 \equiv \lambda_1 = \lambda_2$$

as

$$\omega(z) = [\phi(z)\bar{z} + \psi(z)] \exp \left\{ \frac{\lambda_0 \bar{z}}{z^\nu} \right\}, \quad (6)$$

where  $\phi(z)$  and  $\psi(z)$  are arbitrary holomorphic functions in the domain  $G \setminus \{0\}$ ;  $z = 0$  is an isolated singular point for  $\phi(z)$  and  $\psi(z)$ .

2<sup>0</sup>. Below we will need the following two statements whose proof is based on the well-known Fragman–Lindelöf principle (see, e.g., [1], [2], and also [3]).

**Lemma 1.** *Let  $\phi(z)$  be a function holomorphic in the deleted neighborhood of the point  $z = 0$  and such that*

$$\phi(z) = 0 \quad (\exp\{g(z)\}), \quad z \rightarrow 0, \quad (7)$$

where

$$g(z) = \frac{1}{|z|^{k-2}} \{\delta + a \cos(k \arg z) + b \sin(k \arg z)\},$$

$k \geq 3$  is natural,  $\delta, a, b$  are real numbers, and

$$\delta = \sqrt{a^2 + b^2} \cos \pi \beta, \quad \beta = \max \left\{ 0, \frac{k-4}{2k-4} \right\}.$$

Then the function  $\phi(z)$  is identically equal to zero.

**Lemma 2.** *Let  $\phi$  a function holomorphic in the deleted neighborhood of the point  $z = 0$  and such that the condition (7) is fulfilled with*

$$g(z) = \frac{1}{|z|} \{\sqrt{a^2 + b^2} + a \cos(3 \arg z) + b \sin(3 \arg z)\}$$

and  $a, b$  real numbers. Then the function  $\phi(z)$  has the removable singularity at the point  $z = 0$ .

3<sup>0</sup>. The following theorem holds (cf. [3]).

**Theorem 1.** *Let  $\delta$  be a real number such that  $\delta < |\lambda_1| \cos \pi \beta$ , where*

$$\beta = \max \left\{ 0, \frac{\nu-3}{2\nu-2} \right\}. \quad (8)$$

Then for every non-trivial solution  $\omega(z) \in \mathbb{K}$

$$\overline{\lim}_{\rho \rightarrow 0^+} \frac{T_\omega(\rho)}{\exp \left\{ \frac{\delta}{\rho^{\nu-1}} \right\}} = +\infty. \quad (9)$$

*Proof.* First, let  $\lambda_1 \neq \lambda_2$ . Then differentiating the general solution (5) with respect to  $\bar{z}$ , we have

$$\frac{\partial \omega}{\partial \bar{z}} = \frac{\lambda_1}{z^\nu} \phi(z) \exp \left\{ \frac{\lambda_1 \bar{z}}{z^\nu} \right\} + \frac{\lambda_2}{z^\nu} \psi(z) \exp \left\{ \frac{\lambda_2 \bar{z}}{z^\nu} \right\},$$

which together with (5) provides us with

$$\begin{aligned} \phi(z) \exp \left\{ \frac{\lambda_1 \bar{z}}{z^\nu} \right\} &= \frac{1}{\lambda_1 - \lambda_2} \left( \lambda_1 \omega - z^\nu \frac{\partial \omega}{\partial \bar{z}} \right), \\ \psi(z) \exp \left\{ \frac{\lambda_2 \bar{z}}{z^\nu} \right\} &= \frac{1}{\lambda_1 - \lambda_2} \left( \lambda_2 \omega - z^\nu \frac{\partial \omega}{\partial \bar{z}} \right). \end{aligned} \tag{10}$$

Let for some solution  $\omega(z) \in \mathbb{K}$  the condition (9) be violated; this means that there exist positive numbers  $M$  and  $\rho_0$  such that

$$T_\omega(\rho) \leq M \exp \left\{ \frac{\delta}{\rho^{\nu-1}} \right\}, \quad 0 < \rho < \rho_0,$$

whence, with regard for (3), we obtain

$$\begin{aligned} |w(\rho e^{i\varphi})| &\leq M \cdot \exp \left\{ \frac{\delta}{\rho^{\nu-1}} \right\}, \\ \left\| \frac{\partial \omega}{\partial \bar{z}}(\rho e^{i\varphi}) \right\| &\leq M \cdot \exp \left\{ \frac{\delta}{\rho^{\nu-1}} \right\}, \quad 0 < \rho < \rho_0, \quad 0 \leq \varphi \leq 2\pi. \end{aligned} \tag{11}$$

In its turn, from (11) and (10) it follows the existence of a positive number  $M_0$  such that

$$\begin{aligned} |\phi(z)| &\leq M_0 \exp \left\{ \frac{1}{|z|^{\nu-1}} [\delta - |\lambda_1| \cos(\psi_1 - (\nu + 1)\varphi)] \right\}, \\ |\psi(z)| &\leq M_0 \exp \left\{ \frac{1}{|z|^{\nu-1}} [\delta - |\lambda_2| \cos(\psi_2 - (\nu + 1)\varphi)] \right\}, \\ 0 < |z| &< \rho_0, \quad 0 \leq \varphi \leq 2\pi, \end{aligned} \tag{12}$$

where  $\varphi = \arg z$ ,  $\psi_k = \arg \lambda_k$ ,  $k = 1, 2$ .

From the inequalities (12) by virtue of Lemma 1 we find that  $\phi(z) \equiv \psi(z) \equiv 0$ , i.e., the solution  $\omega(z)$  is trivial.

Let now  $\lambda_0 \equiv \lambda_1 = \lambda_2$ . Then differentiating the general solution (6) with respect to  $\bar{z}$ , we have

$$\frac{\partial \omega}{\partial \bar{z}} = \left[ \phi(z) \left( 1 + \frac{\lambda_0 \bar{z}}{z^\nu} \right) + \frac{\lambda_0}{z^\nu} \psi(z) \right] \exp \left\{ \frac{\lambda_0 \bar{z}}{z^\nu} \right\},$$

which together with (6) provides us with

$$\begin{aligned} z^\nu \phi(z) \exp \left\{ \frac{\lambda_0 \bar{z}}{z^\nu} \right\} &= z^\nu \frac{\partial \omega}{\partial \bar{z}} - \lambda_0 \omega, \\ z^\nu \psi(z) \exp \left\{ \frac{\lambda_0 \bar{z}}{z^\nu} \right\} &= (z^\nu + \lambda_0 \bar{z}) \omega - \bar{z} z^\nu \frac{\partial \omega}{\partial \bar{z}}. \end{aligned} \tag{13}$$

The formulas (13) obtained above are analogous to the formulas (10) which makes it possible to repeat our reasoning and conclude that the non-trivial solutions  $\omega(z) \in \mathbb{K}$  are unable to violate the condition (9).  $\square$

From the above-proven theorem it immediately follows that for every non-trivial solution  $\omega(z) \in \mathbb{K}$

$$\overline{\lim}_{\rho \rightarrow 0^+} \frac{T_\omega(\rho)}{\exp \left\{ \frac{\delta}{\rho^\sigma} \right\}} = +\infty,$$

where  $\delta$  is any real number, and the real number  $\sigma < \nu - 1$ .

4<sup>0</sup>. Theorem 1 admits generalizations to more general systems of differential equations of the type

$$\sum_{k=0}^m z^{\nu k} A_k \frac{\partial^k \omega}{\partial \bar{z}^k} = 0, \quad (14)$$

where  $\nu \geq 2$ ,  $m \geq 1$  are given natural numbers,  $A_k$ ,  $k = 0, 1, \dots, m$ , are given complex square matrices of dimension  $n \times n$ , and

$$\det A_0 \neq 0, \quad \det A_m \neq 0, \quad (15)$$

$$A_k A_j = A_j A_k, \quad j, k = 0, 1, \dots, m. \quad (16)$$

Under a solution of the system (14) we mean the vector function  $\omega(z) = (\omega_1(z), \omega_2(z), \dots, \omega_n(z))$  belonging to the class  $C^m(G \setminus \{0\})$  and satisfying (14) at every non-zero point of the domain  $G$ .

By  $\Lambda$  we denote the set of all possible complex roots of the polynomial

$$\sum_{k=0}^m \tau_k \lambda^k = 0,$$

where the coefficient  $\tau_k$  is some eigenvalue of the matrix  $A_k$ ,  $k = 0, 1, \dots, m$ . Introduce the number

$$\delta_0 \equiv \min_{\lambda \in \Lambda} |\lambda|,$$

which by (15) satisfies the inequality  $\delta_0 > 0$ .

The following theorem holds.

**Theorem 1\***. *Let  $\psi(z)$  be a function analytic in some deleted neighborhood of the point  $z = 0$  and having possibly arbitrary isolated singularities (concentration of singularities of the function  $\psi(z)$  at the point  $z = 0$  is not excluded). Further, let  $\delta$ ,  $\sigma$  be real numbers such that either  $\sigma < \nu - 1$  ( $\sigma$  is arbitrary) or  $\sigma = \nu - 1$ ,  $\delta < \delta_0 \cos \pi \beta$  where the number  $\beta$  is given by the formula (8). Then there are no non-trivial solutions of the system (14) satisfying the asymptotic condition*

$$\tilde{T}_\omega(|z|) = 0 \left( |\psi(z)| \exp \left\{ \frac{\delta}{|z|^\sigma} \right\} \right), \quad z \rightarrow 0,$$

where

$$\tilde{T}_\omega(\rho) \equiv \max_{0 \leq \varphi \leq 2\pi} \sum_{k=1}^n \sum_{p=0}^{m-1} \left| \frac{\partial^p \omega_k}{\partial \bar{z}^p} (\rho e^{i\varphi}) \right|, \quad \rho > 0.$$

5<sup>0</sup>. Everywhere below  $G$  will denote a finite domain (containing the origin of coordinates of the complex plane) with the boundary  $\Gamma$  consisting

of a finite number of simple, closed, non-intersecting Lyapunov contours. In the sequel, we will consider a special case of the equation (1), when  $\nu = 2$ , i.e., we consider the equation

$$z^4 \frac{\partial^2 \omega}{\partial \bar{z}^2} + Az^2 \frac{\partial \omega}{\partial \bar{z}} + B\omega = 0, \tag{17}$$

and study the following two boundary value problems.

**Problem  $R(\delta, \sigma)$ .** On the contour  $\Gamma$  there are prescribed Hölder continuous functions  $a(t), \gamma(t)$  where the function  $\gamma(t)$  is real and  $a(t) \neq 0, t \in \Gamma$ . Real positive numbers  $\delta, \sigma$  are also given. It is required to find a continuously extendable to  $\overline{G} \setminus \{0\}$  solution of the equation (17) satisfying both the asymptotic condition

$$\overline{\lim}_{\rho \rightarrow 0} \frac{T_\omega(\rho)}{\exp \left\{ \frac{\delta}{\rho^\sigma} \right\}} < +\infty \tag{18}$$

and the boundary condition

$$\operatorname{Re}\{a(t)\omega(t)\} = \gamma(t), \quad t \in \Gamma. \tag{19}$$

**Problem  $Q(\delta, \sigma)$ .** On the contour  $\Gamma$  there are prescribed Hölder continuous functions  $\gamma_k(t), a_{k,m}(t), k, m = 1, 2$ , where  $\gamma_1(t), \gamma_2(t)$  are real and

$$\det \|a_{k,m}(t)\| \neq 0, \quad t \in \Gamma.$$

Real positive numbers  $\delta, \sigma$  are also given. It is required to find a continuously extendable (together with its derivative  $\frac{\partial \omega}{\partial \bar{z}}$ ) to  $\overline{G} \setminus \{0\}$  solution of the equation (17) satisfying both the condition (18) and the boundary condition

$$\operatorname{Re}\{a_{k,1}(t)\omega(t) + a_{k,2}(t) \frac{\partial \omega}{\partial \bar{z}}(t)\} = \gamma_k(t), \quad t \in \Gamma, \quad k = 1, 2. \tag{20}$$

Along with the problems formulated above, let us consider the following boundary value problems.

**Problem  $R_0(\overline{p})$ .** Given an integer  $p$ , it is required to find a function  $\phi_0(z)$  holomorphic in the domain  $G$ , continuously extendable to  $\overline{G}$  and satisfying the boundary condition

$$\operatorname{Re}\{\alpha(t)\phi_0(t)\} = \gamma(t), \quad t \in \Gamma, \tag{21}$$

where  $\alpha(t) = a(t)t^{2-p} \exp\left\{\frac{\lambda_1 \bar{t}}{t^2}\right\}$ .

**Problem  $Q'_0(p)$ .** Given an integer  $p$ , it is required to find a vector function  $(\phi_0(z), \psi_0(z))$  holomorphic in the domain  $G$ , continuously extendable to  $\overline{G}$  and satisfying the boundary condition

$$\operatorname{Re}\{\alpha_{k,1}(t)\phi_0(t) + \alpha_{k,2}(t)\psi_0(t)\} = \gamma_k(t), \quad t \in \Gamma, \quad k = 1, 2, \tag{22}$$

where

$$\alpha_{k,m}(t) = \left[ a_{k,1}(t)t^{2-p} + \frac{\lambda_m a_{k,2}(t)}{t^p} \right] \exp \left\{ \frac{\lambda_m \bar{t}}{t^2} \right\}, \quad k, m = 1, 2.$$

**Problem  $Q_0''(p)$ .** Given an integer  $p$ , it is required to find a vector function  $(\phi_0(z), \psi_0(z))$  holomorphic in the domain  $G$ , continuously extendable to  $\overline{G}$  and satisfying the boundary condition

$$\operatorname{Re}\{\beta_{k,1}(t)\phi_0(t) + \beta_{k,2}(t)\psi_0(t)\} = \gamma_k(t), \quad t \in \Gamma, \quad k = 1, 2, \quad (23)$$

where

$$\begin{aligned} \beta_{k,1}(t) &= \left[ \frac{a_{k,1}(t)}{t^p} |t|^2 + a_{k,2}(t) \left( t^{1-p} + \frac{\lambda_0 \bar{t}}{t^{2+p}} \right) \right] \exp \left\{ \frac{\lambda_0 \bar{t}}{t^2} \right\} \\ \beta_{k,2}(t) &= \left[ a_{k,1}(t) t^{2-p} + \frac{\lambda_0}{t^p} a_{k,2}(t) \right] \exp \left\{ \frac{\lambda_0 \bar{t}}{t^2} \right\}. \end{aligned}$$

On the basis of the following obvious relations

$$\alpha(t) \neq 0, \quad t \in \Gamma,$$

$$\det \|\beta_{k,m}(t)\| = -t^{3-2p} \det \|a_{k,m}(t)\| e^{\frac{2\lambda_0 \bar{t}}{t^2}} = 0, \quad t \in \Gamma,$$

$$\det \|\alpha_{k,m}(t)\| = (\lambda_2 - \lambda_1) t^{2-2p} \det \|a_{k,m}(t)\| e^{\frac{\lambda_1 + \lambda_2 \bar{t}}{t^2}} \neq 0, \quad t \in \Gamma,$$

if only  $\lambda_1 \neq \lambda_2$ , we conclude that for every integer  $p$  the problems  $R_0(p)$ ,  $Q_0'(p)$ ,  $Q_0''(p)$  refer to those boundary value problems which are well-studied (see, e.g., [4], [5]). In particular, it is known that the corresponding homogeneous problems ( $\gamma(t) \equiv \gamma_1(t) \equiv \gamma_2(t) \equiv 0$ ) have finite numbers of linearly independent solutions<sup>1</sup> (and, as it is not difficult to see, these numbers become arbitrarily large as  $p \rightarrow +\infty$ ). Also formulas for index calculation and criteria for the solvability of the problems are available.

6<sup>0</sup>. We have the following

**Theorem 2.** *Let  $|\lambda_1| < |\lambda_2|$ . Then the boundary value problems  $R(|\lambda_1|, 1)$  and  $R_0(0)$  are simultaneously solvable (unsolvable), and in case of their solvability the relation*

$$\omega(z) = z^2 \phi_0(z) \exp \left\{ \frac{\lambda_1 \bar{z}}{z^2} \right\}, \quad z \in G \setminus \{0\}, \quad (24)$$

*establishes a bijective relation between the solutions of these problems.*

*Proof.* First we have to find a general representation of solutions of the equation (17) which are continuously extendable to  $\overline{G} \setminus \{0\}$  and satisfy the condition (18), where  $\delta = |\lambda_1|$ ,  $\sigma = 1$ . Towards this end, we use the equalities (10) and find that the functions  $\phi(z)$  and  $\psi(z)$ , holomorphic in the domain  $G \setminus \{0\}$ , satisfy the conditions

$$\begin{aligned} \phi(z) &= 0 \left( \exp \left\{ \frac{|\lambda_1|}{|z|} \left[ 1 - \cos(\psi_1 - 3 \arg z) \right] \right\} \right), \quad z \rightarrow 0, \\ \psi(z) &= 0 \left( \exp \left\{ \frac{|\lambda_1|}{|z|} \left[ 1 - \frac{|\lambda_2|}{|\lambda_1|} \cos(\psi_2 - 3 \arg z) \right] \right\} \right), \quad z \rightarrow 0, \\ \psi_k &= \arg \lambda_k, \quad k = 1, 2. \end{aligned}$$

<sup>1</sup> Here and everywhere below, the linear independence is understood over the field of real numbers.



The first of the above conditions on the basis of Lemma 2 shows that  $z = 0$  is a removable singular point for the function  $\phi(z)$ . Next, if we take into account the inequality  $\left| \frac{\lambda_2}{\lambda_1} \right| > 1$ , then by virtue of Lemma 1 the second condition shows that the function  $\psi(z) \equiv 0$ . This immediately implies that the relation

$$\frac{\partial \omega}{\partial \bar{z}} = \frac{\lambda_1}{z^2} \phi(z) \exp \left\{ \frac{\lambda_1 \bar{z}}{z^2} \right\}$$

is valid. Consequently,

$$\left| \frac{\lambda_1}{z^2} \right| |\phi(z)| = 0 \left( \exp \left\{ \frac{|\lambda_1|}{|z|} (1 - \cos(\psi_1 - 3 \arg z)) \right\} \right), \quad z \rightarrow 0. \quad (25)$$

In turn, (25) yields

$$\left| \frac{\lambda_1}{z^2} \right| |\phi(z)| = 0(1), \quad z \rightarrow 0, \quad \arg z = \frac{\psi_1}{3}. \quad (26)$$

Considering the Taylor series expansion of the holomorphic function  $\lambda_1 \phi(z)$

$$\lambda_1 \phi(z) = a_0 + a_1 z + a_2 z^2 + \dots,$$

and substituting this expansion in (26), we obtain

$$\left| \frac{a_0 + a_1 z}{z^2} \right| = 0(1), \quad z \rightarrow 0, \quad \arg z = \frac{\psi_1}{3},$$

and hence  $a_0 = a_1 = 0$ . From the above-said it follows that

$$\omega(z) = z^2 \phi_0(z) \exp \left\{ \frac{\lambda_1 \bar{z}}{z^2} \right\}, \quad z \in G \setminus \{0\}, \quad (27)$$

where  $\phi_0(z)$  is a function holomorphic in the domain  $G$ . Further, if the solution  $\omega(z)$  is continuously extendable to  $\bar{G} \setminus \{0\}$ , then the function  $\phi_0(z)$  is likewise continuously extendable to  $\bar{G}$ .

Conversely, it is obvious that any function of the type (27) provides us with a solution of the equation (17), which is continuously extendable to  $\bar{G} \setminus \{0\}$  and satisfies the condition (18), where  $\delta = |\lambda_1|$ ,  $\sigma = 1$ .

It remains to take into account the boundary conditions (19) and (21) (where  $p = 0$ ) which immediately leads us to the validity of the theorem.  $\square$

Since any linearly independent system of functions  $\phi_0(z)$  by means of the relation (24) transforms into that of the functions  $\omega(z)$  (and conversely), on the basis of the above proven Theorem 2 it is possible to carry out the complete investigation of the boundary value problem  $R(|\lambda_1|, 1)$  under the assumption  $|\lambda_1| < |\lambda_2|$ .

We have the following

**Theorem 3.** *Let at least one of the relations*

$$\delta = |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| < |\lambda_2|, \quad (28)$$

*be violated. Then either the homogeneous problem  $R(\delta, \sigma)$  has an infinite set of linearly independent solutions, or the inhomogeneous problem is unsolvable for any right-hand side  $\gamma(t) \not\equiv 0$ .*

*Proof.* By the inequality (4), violation at least of one of the relations (28) means the fulfilment of one of the following conditions:

$$\delta \neq |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| < |\lambda_2|, \quad (29)$$

or

$$\sigma \neq 1 \quad (\sigma \text{ is arbitrary}), \quad |\lambda_1| < |\lambda_2|, \quad (30)$$

or

$$\delta = |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| = |\lambda_2|, \quad (31)$$

or

$$\delta \neq |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| = |\lambda_2|, \quad (32)$$

or

$$\sigma \neq 1 \quad (\delta \text{ is arbitrary}) \quad |\lambda_1| = |\lambda_2|. \quad (33)$$

We consider these cases separately. Let (29) be fulfilled. In its turn, this case splits into the following two cases: either

$$\delta < |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| < |\lambda_2|, \quad (29^*)$$

or

$$\delta > |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| < |\lambda_2|. \quad (29^{**})$$

Let the case (29<sup>\*</sup>) be fulfilled, and let  $\omega(z)$  be a solution of the equation (17) satisfying the condition (18). Since  $\nu = 2$ , the number  $\beta$  given by the formula (8) is equal to zero. On the basis of Theorem 1, this implies that the solution  $\omega(z) \equiv 0$ , and hence the inhomogeneous boundary value problem  $R(\delta, 1)$  is unsolvable for any right-hand side  $\gamma(t) \not\equiv 0$ .

Let now the condition (29<sup>\*\*</sup>) be fulfilled. We call an arbitrary real number  $N$  and prove that the number of linearly independent solutions of the homogeneous boundary value problem  $R(\delta, 1)$  is greater than  $N$ . Indeed, we select a natural number  $p$  so large that the number of linearly independent solutions of the homogeneous boundary value problem  $R_0(p)$  be greater than  $N$ . Denote these solutions by  $\phi_0^{(1)}(z), \phi_0^{(2)}(z), \dots, \phi_0^{(m)}(z)$ , ( $m > N$ ) and introduce the functions

$$\omega_k(z) = z^{2-p} \phi_0^{(k)} \exp \left\{ \frac{\lambda_1 \bar{z}}{z^2} \right\}, \quad k = 1, 2, \dots, m. \quad (34)$$

It is clear that the system of functions (34) is likewise independent.

By the representation (5), every function from (34) is a continuously extendable to  $\bar{G} \setminus \{0\}$  solution of the equation (17) which by virtue of (21) satisfies the homogeneous boundary condition (19). Further, since the condition (29<sup>\*\*</sup>) is fulfilled, on the basis of the obvious relation

$$\frac{\partial \omega_k}{\partial \bar{z}} = \frac{\lambda_1}{z^p} \phi_0^{(k)}(z) \exp \left\{ \frac{\lambda_1 \bar{z}}{z^2} \right\} = 0 \left( \exp \left\{ \frac{\delta}{|z|} \right\} \right), \quad z \rightarrow 0,$$

we immediately can conclude that every function of the system (34) satisfies the asymptotic condition (18), and hence the homogeneous boundary value problem  $R(\delta, 1)$  has infinitely many linearly independent solutions.

Let now the condition (30) be fulfilled. This case in its turn falls into two cases: either

$$\sigma < 1 \quad (\delta \text{ is arbitrary}), \quad |\lambda_1| < |\lambda_2|, \tag{30*}$$

or

$$\sigma > 1 \quad (\delta \text{ is arbitrary}), \quad |\lambda_1| < |\lambda_2|. \tag{30**}$$

It is evident that in the case (30\*) (analogously to the case (29\*)) the inhomogeneous boundary value problem  $R(\delta, \sigma)$  is unsolvable for any right-hand side  $\gamma(t) \not\equiv 0$ , and in the case (30\*\*) (analogously to the case (29\*\*)) the homogeneous boundary value problem  $R(\delta, \sigma)$  has infinitely many linearly independent solutions.

Let now the condition (31) be fulfilled. This case in its turn splits into two cases: either

$$\delta = |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| = |\lambda_2|, \quad \lambda_1 \neq \lambda_2, \tag{31*}$$

or

$$\delta = |\lambda_1|, \quad \sigma = 1, \quad \lambda_1 = \lambda_2. \tag{31**}$$

Let us prove that in both cases (31\*) and (31\*\*) the homogeneous boundary value problem  $R(\delta, 1)$  has infinitely many linearly independent solutions. We start with the case (31\*). Evidently, every function of the type

$$\omega(z) = z^2 \phi_0(z) e^{\frac{\lambda_1 \bar{z}}{z^2}} + z^2 \psi_0(z) e^{\frac{\lambda_2 \bar{z}}{z^2}}, \quad z \in G \setminus \{0\} \tag{35}$$

(where  $\phi_0(z), \psi_0(z)$  are holomorphic in the domain  $G$  functions) is a solution of the equation (17) satisfying the condition (18), where  $\sigma = |\lambda_1|, \sigma = 1$  (in proving Theorem 5 below we will show that the converse statement is valid, i.e., every solution of the equation (17) satisfying the condition (18) with  $\delta = |\lambda_1|, \sigma = 1$  has the form (35)). Next, if the holomorphic functions  $\phi_0(z)$  and  $\psi_0(z)$  are continuously extendable to  $\bar{G}$ , then the solution  $\omega(z)$  is likewise continuously extendable to  $\bar{G} \setminus \{0\}$ . Consider the following problem: find two functions  $\phi_0(z)$  and  $\psi_0(z)$ , holomorphic in the domain  $G$  and continuously extendable to  $G$  by the boundary condition

$$\operatorname{Re} \left\{ a(t) t^2 \phi_0(t) e^{\frac{\lambda_1 \bar{t}}{t^2}} + a(t) t^2 \psi_0(t) e^{\frac{\lambda_2 \bar{t}}{t^2}} \right\} = 0, \quad t \in \Gamma. \tag{36}$$

It follows from the above said that every solution of the problem (36) provides us by the formula (35) with a solution of the homogeneous boundary value problem  $R(|\lambda_1|, 1)$ .

On the other hand, the problem (36) has infinitely many linearly independent solutions. Indeed, let

$$\phi_1^*(z), \phi_2^*(z), \dots, \phi_l^*(z)$$

be a complete system of solutions of the conjugate boundary value problem: given a real Hölder continuous function  $\beta(t)$ , find the function  $\phi_0(z)$  holomorphic in the domain  $G$  and continuously extendable to  $\bar{G}$  by the boundary condition

$$\operatorname{Re} [\alpha(t) \phi_0(t)] = \beta(t), \quad t \in \Gamma, \tag{37}$$

where

$$\alpha(t) = a(t)t^2 \exp \left\{ \frac{\lambda_1 \bar{t}}{t^2} \right\}.$$

Take an arbitrary natural number  $N_0$  and consider a natural number  $N$  such that

$$N + 1 - 2l > N_0.$$

Introduce now the polynomial

$$\psi_0(z) = C_0 + C_1 z + \dots + C_n z^N, \quad (38)$$

where  $C_j$ ,  $j = 0, 1, \dots, N$ , are yet undefined real coefficients. Further, taking the right-hand side of the problem (37) in the form

$$\beta(t) = -\operatorname{Re} \left[ a(t)t^2 \exp \left\{ \frac{\lambda_2 \bar{t}}{t^2} \right\} \psi_0(t) \right], \quad t \in \Gamma,$$

we obtain a boundary value problem which will certainly be solvable if

$$\int_{\Gamma} \alpha(t)\beta(t)\phi_k^*(t)dt = 0, \quad 1 \leq k \leq l.$$

Thus if real constants  $C_j$  are chosen such that

$$\sum_{j=0}^N D_{kj}C_j = 0, \quad k = 1, 2, \dots, l, \quad (39)$$

where

$$D_{kj} = \int_{\Gamma} \alpha(t)\phi_k^*(t) \operatorname{Re} \left[ a(t)t^{2+j} e^{\frac{\lambda_2 \bar{t}}{t^2}} \right] dt,$$

then the problem (37) is solvable. In turn, the conditions (39) form a system consisting of  $2l$  linear algebraic homogeneous equations with  $N + 1$  real unknowns, of which at least  $N + 1 - 2l$  we can take arbitrarily. This means that in the decomposition (38) we can take  $N + 1 - 2l$  real coefficients. Substituting this decomposition in the boundary condition (36), we can find the function  $\phi_0(z)$ . It is obvious that the problem (36) has an infinite number of linearly independent solutions.

If the condition (31\*\*) is fulfilled, then any function of the type

$$\omega(z) = (z\bar{z}\phi_0(z) + z^2\psi_0(z))e^{\frac{\lambda_1 \bar{z}}{z^2}}, \quad z \in G \setminus \{0\} \quad (40)$$

(where  $\phi_0(z)$  and  $\psi_0(z)$  are functions holomorphic in  $G$ ), is a solution of the equation (17) satisfying the condition (18), where  $\delta = |\lambda_1|$ ,  $\sigma = 1$  (in proving Theorem 6 below, we will establish the validity of the converse statement, i.e., any solution of the equation (17) satisfying the condition (18), where  $\delta = |\lambda_1|$ ,  $\sigma = 1$ , has the form (40)). Moreover, if the holomorphic functions  $\phi_0(z)$  and  $\psi_0(z)$  are continuously extendable to  $\bar{G}$ , then the solution  $\omega(z)$  is likewise continuously extendable to  $\bar{G} \setminus \{0\}$ .

Let us consider the following boundary value problem. Find two functions  $\phi_0(z)$  and  $\psi_0(z)$ , holomorphic in the domain  $G$  and continuously extendable to  $\bar{G}$  by the boundary condition

$$\operatorname{Re} \left[ a(t)(t\bar{t}\phi_0(t) + t^2\psi_0(t))e^{\frac{\lambda_1\bar{t}}{t^2}} \right] = 0, \quad t \in \Gamma. \tag{41}$$

Any solution of the problem (41) provides us by the formula (40) with a solution of the boundary value problem  $R(|\lambda_1|, 1)$ . But the problem (41), just as the problem (36), has an infinite number of linearly independent solutions. Hence the homogeneous problem  $R(|\lambda_1|, 1)$  has an infinite number of linearly independent solutions.

The case (32) splits into the following two cases: either

$$\delta < |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| = |\lambda_2|, \tag{32*}$$

or

$$\delta > |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| = |\lambda_2|. \tag{32**}$$

In the case (32\*), just as in the case (29\*), on the basis of Theorem 1 we immediately find that the equation (17) has no non-trivial solution satisfying the condition (18), and hence the inhomogeneous boundary value problem  $R(\delta, 1)$  is unsolvable for any right-hand side  $\gamma(t) \not\equiv 0$ .

In the case (32\*) it is obvious that any solution of the boundary value problem  $R(|\lambda_1|, 1)$  is also a solution of the problem  $R(\delta, 1)$ . But the homogeneous boundary value problem  $R(|\lambda_1|, 1)$  has an infinite number of linearly independent solutions (see the case (31) above), consequently the homogeneous problem  $R(\delta, 1)$  has an infinite number of linearly independent solutions, as well.

The case (33) splits into the following two cases: either

$$\sigma < 1 \quad (\delta \text{ is arbitrary}), \quad |\lambda_1| = |\lambda_2| \tag{33*}$$

or

$$\sigma > 1 \quad (\delta \text{ is arbitrary}), \quad |\lambda_1| = |\lambda_2|. \tag{33**}$$

In the case (33\*), just as in the case (32\*), on the basis of Theorem 1 we immediately find that the inhomogeneous boundary value problem  $R(\delta, \sigma)$  is unsolvable for any right-hand side  $\gamma(t) \not\equiv 0, t \in \Gamma$ , and in the case (33\*\*) (just as in the case (32\*\*)) the homogeneous boundary value problem  $R(\delta, \sigma)$  has an infinite number of linearly independent solutions.  $\square$

On the basis of the above proven Theorems 2 and 3 we have

**Theorem 4.** *The boundary value problem  $R(\delta, \sigma)$  is Noetherian if and only if the relations (28) are fulfilled.*

7<sup>0</sup>. In the foregoing section we have investigated the boundary value problem  $R(\delta, \sigma)$ . As we have found out, this problem is correct only under the condition (28). The last of those relations allows one to exclude from the consideration a wide class of equations of the type (17).

In the present section, not mentioning it specially, we assume that

$$|\lambda_1| = |\lambda_2|,$$

and for equations of the type (17) we give the correct statement and investigation of the boundary value problems.

Everywhere below, by  $\delta_0$  we denote the number  $\delta_0 = |\lambda_1|$ . We have the following

**Theorem 5.** *If*

$$\arg \lambda_1 \neq \arg \lambda_2,$$

*then the boundary value problems  $Q(\delta_0, 1)$  and  $Q'_0(0)$  are simultaneously solvable (unsolvable), and in case they are solvable, the relation (35) allows us to establish a bijective correspondence between the solutions of these problems.*

*Proof.* First we have to find a general representation of those solutions of the equation (17) which (together with its derivative with respect to  $\bar{z}$ ) are continuously extendable to  $G \setminus \{0\}$  and satisfy the condition (18), where  $\delta = \delta_0$ ,  $\sigma = 1$ . To this end, we again use the equalities (10) and find that the functions  $\phi(z)$  and  $\psi(z)$ , holomorphic in the domain  $G \setminus \{0\}$ , satisfy the conditions

$$\phi(z) = 0 \left( \exp \left\{ \frac{\delta_0}{|z|} [1 - \cos(\psi_1 - 3 \arg z)] \right\} \right), \quad z \rightarrow 0,$$

$$\psi(z) = 0 \left( \exp \left\{ \frac{\delta_0}{|z|} [1 - \cos(\psi_2 - 3 \arg z)] \right\} \right), \quad z \rightarrow 0,$$

$$\psi_k = \arg \lambda_k, \quad k = 1, 2.$$

Thus on the basis of Lemma 2 we conclude that  $z = 0$  is a removable singular point for the functions  $\phi(z)$  and  $\psi(z)$ . Further, it is obvious that

$$\begin{aligned} \frac{\partial \omega}{\partial \bar{z}} &= \frac{\lambda_1 \phi(z)}{z^2} \exp \left\{ \frac{\lambda_1 \bar{z}}{z^2} \right\} + \frac{\lambda_2 \psi(z)}{z^2} \exp \left\{ \frac{\lambda_2 \bar{z}}{z^2} \right\} = \\ &= 0 \left( \exp \left\{ \frac{\delta_0}{|z|} \right\} \right), \quad z \rightarrow 0. \end{aligned}$$

Hence we obtain the following two relations:

$$\begin{aligned} \frac{\delta_0}{r^2} \left| \phi \left( r \exp \left\{ \frac{i\psi_1}{3} \right\} \right) \right| &\leq \text{const} + \\ &+ \frac{\delta_0}{r^2} \left| \psi \left( r \exp \left\{ \frac{i\psi_1}{3} \right\} \right) \right| \exp \left\{ \frac{\delta_0}{r} [\cos(\psi_2 - \psi_1) - 1] \right\}, \\ \frac{\delta_0}{r^2} \left| \psi \left( r \exp \left\{ \frac{i\psi_2}{3} \right\} \right) \right| &\leq \text{const} + \\ &+ \frac{\delta_0}{r^2} \left| \phi \left( r \exp \left\{ \frac{i\psi_1}{3} \right\} \right) \right| \exp \left\{ \frac{\delta_0}{r} [\cos(\psi_2 - \psi_1) - 1] \right\}, \end{aligned}$$

whence it respectively follow

$$\left| \frac{\phi(z)}{z^2} \right| = 0(1), \quad z \rightarrow 0, \quad \arg z = \frac{\psi_1}{3},$$

and

$$\left| \frac{\psi(z)}{z^2} \right| = 0(1), \quad z \rightarrow 0, \quad \arg z = \frac{\psi_2}{3}.$$

This implies that the functions  $\phi(z)$  and  $\psi(z)$  admit the representations

$$\phi(z) = z^2 \phi_0(z), \quad \psi(z) = z^2 \psi_0(z),$$

where  $\phi_0(z)$  and  $\psi_0(z)$  are functions holomorphic in the domain  $G$ .

Consequently, any solution of the equation (17) satisfying the condition (18) ( $\delta = \delta_0, \sigma = 1$ ) is representable in the form

$$\omega(z) = z^2 \phi_0(z) \exp \left\{ \frac{\lambda_1 \bar{z}}{z^2} \right\} + z^2 \psi_0(z) \exp \left\{ \frac{\lambda_2 \bar{z}}{z^2} \right\}, \quad (42)$$

and hence

$$\frac{\partial \omega}{\partial \bar{z}} = \lambda_1 \phi_0(z) \exp \left\{ \frac{\lambda_1 \bar{z}}{z^2} \right\} + \lambda_2 \psi_0(z) \exp \left\{ \frac{\lambda_2 \bar{z}}{z^2} \right\}. \quad (43)$$

Next, if the solution (42) (together with its derivative (43)) is continuously extendable to  $\bar{G} \setminus \{0\}$ , then we find that the functions  $\phi_0(z)$  and  $\psi_0(z)$  are likewise continuously extendable to  $\bar{G}$ .

Conversely, it is evident that any function of the type (42) provides us with a continuously extendable (together with its derivative  $\frac{\partial \omega}{\partial \bar{z}}$ ) solution of the equation (17), satisfying the condition (18), where  $\delta = \delta_0, \sigma = 1$ . It remains to take into account the boundary conditions (20) and (22) (where  $p = 0$ ) which directly leads to the conclusion of our theorem.

On the basis of the above proven Theorem 5 in particular it follows that the number of linearly independent solutions of the homogeneous boundary value problem  $Q(\sigma_0, 1)$  is finite. This number coincides with that of the linearly independent solutions of the homogeneous boundary value problem  $Q'_0(0)$ , because any linearly independent system of holomorphic vector functions

$$(\phi_k(z), \psi_k(z)), \quad 1 \leq k \leq m, \quad (44)$$

transforms by the relation

$$\omega_k(z) = \phi_k(z) \exp \left\{ \frac{\lambda_1 \bar{z}}{z^2} \right\} + \psi_k(z) \exp \left\{ \frac{\lambda_2 \bar{z}}{z^2} \right\}, \quad k = 1, 2, \dots, m, \quad (45)$$

into a linearly independent system of functions  $\omega_k(z), k = 1, 2, \dots, m$ , and vice versa. Indeed, let the system of holomorphic vector functions (44) be independent, and

$$\sum_{k=1}^m C_k \omega_k(z) \equiv 0,$$

where  $C_k$  are complex (in particular, real) coefficients. Then

$$\sum_{k=1}^m C_k \phi_k(z) \equiv -e^{\frac{\lambda_2 - \lambda_1}{z^2} \bar{z}} \sum_{k=1}^m C_k \psi_k(z). \quad (46)$$

Differentiating both parts of the equality (46) with respect to  $\bar{z}$ , we obtain

$$\frac{\lambda_2 - \lambda_1}{z^2} e^{\frac{\lambda_2 - \lambda_1}{z^2} \bar{z}} \sum_{k=1}^m C_k \psi_k(z) \equiv 0.$$

Hence (since  $\lambda_2 \neq \lambda_1$ )

$$\sum_{k=1}^m C_k \psi_k(z) \equiv 0. \quad (47)$$

It follows from (46) and (47) that

$$\sum_{k=1}^m C_k \phi_k(z) \equiv 0, \quad (48)$$

while (48) and (47), by virtue of the fact that the system (44) is linearly independent, yield  $C_k = 0$ ,  $k = 1, 2, \dots, m$ .  $\square$

The converse statement is obvious because the linear dependence of the system of vector functions (44) immediately implies that of the system of functions (45).

We have the following

**Theorem 6.** *If*

$$\psi_1 \equiv \arg \lambda_1 = \arg \lambda_2,$$

*then the boundary value problems  $Q(\delta_0, 1)$  and  $Q_0''(0)$  are simultaneously solvable (unsolvable), and if they are solvable, then the relation (40) allows us to establish the bijective correspondence between the solutions of these problems.*

*Proof.* First of all, just as in the proof of Theorems 2 and 5, we have to find a general representation of those solutions of the equation (17) which (together with the derivative  $\frac{\partial \omega}{\partial \bar{z}}$ ) are continuously extendable to  $\overline{G} \setminus \{0\}$  and satisfy the condition (18), where  $\delta = \delta_0$ ,  $\sigma = 1$ . Towards this end, we use the equalities (13) and find that the functions  $\phi(z)$  and  $\psi(z)$ , holomorphic in  $G \setminus \{0\}$ , satisfy the conditions

$$z^2 \phi(z) = 0(g(z)), \quad z^2 \psi(z) = 0(g(z)), \quad z \rightarrow 0, \quad (49)$$

where

$$g(z) = \exp \left\{ \frac{\delta_0}{|z|} (1 - \cos(\psi_1 - 3 \arg z)) \right\}.$$

By virtue of the relations (49) and Lemma 2, we obtain that  $z = 0$  is a removable singular point for the functions  $z^2 \phi$  and  $z^2 \psi$ , i.e., the solution  $\omega$  is representable in the form

$$\omega(z) = H(z) \exp \left\{ \frac{\lambda_1 \bar{z}}{z^2} \right\}, \quad z \in G \setminus \{0\}, \quad (50)$$

where

$$H(z) = \bar{z} \frac{\tilde{\phi}(z)}{z^2} + \frac{\tilde{\psi}(z)}{z^2},$$



and  $\tilde{\phi}$  and  $\tilde{\psi}$  are functions holomorphic in  $G$ . In turn, from the representation (50) it follows

$$\frac{\partial \omega}{\partial \bar{z}} = H_1(z) \exp \left\{ \frac{\lambda_1 \bar{z}}{z^2} \right\}, \quad z \in G \setminus \{0\},$$

where

$$H_1(z) = \frac{\tilde{\phi}(z)}{z^2} \left( 1 + \frac{\lambda_1 \bar{z}}{z^2} \right) + \frac{\lambda_1}{z^4} \tilde{\psi}(z).$$

Further, taking into account the condition (18), we get

$$H(z) = 0(1), \quad z \rightarrow 0, \arg z = \frac{1}{3}(\psi_1 + 2\pi k), \tag{51}$$

$$H_1(z) = 0(1), \quad z \rightarrow 0, \arg z = \frac{1}{3}(\psi_1 + 2\pi k), \tag{52}$$

$$k = 0, 1, 2, \dots$$

Expanding the holomorphic functions  $\tilde{\phi}$  and  $\tilde{\psi}$  into their Taylor series

$$\tilde{\phi}(z) = a_0 + a_1 z + a_2 z^2 + \dots, \tag{53}$$

$$\tilde{\psi}(z) = b_0 + b_1 z + b_2 z^2 + \dots,$$

and substituting them in (51), we have

$$\frac{a_0 \bar{z} + b_1 z + b_0}{z^2} = 0(1), \quad \arg z = \frac{\psi_1 + 2\pi k}{3}, \tag{54}$$

where the coefficient  $b_0 = 0$ . Taking this into account and using the relation (54) for the coefficients  $a_0$  and  $b_1$ , we obtain the following equalities

$$a_0 e^{-2i\varphi_0} + b_1 = 0, \quad \varphi_0 = \frac{\psi_1}{3},$$

$$a_0 e^{-2i\varphi_0} + b_1 = 0, \quad \varphi_1 = \frac{\psi_1 + 2\pi}{3},$$

which (with regard for  $e^{-2i\varphi_0} - e^{-2i\varphi_1} \neq 0$ ) show that the coefficients  $a_0 = b_1 = 0$ .

Substituting now the expansions (53) and (52), we have

$$\begin{aligned} & \frac{1}{r^3} [\lambda_1 a_1 e^{-4i\varphi_k} + \lambda_1 b_2 r e^{-i\varphi_k} + r^2 (a_1 + \lambda_1 a_2 e^{-2i\varphi_k} + \\ & + \lambda_1 b_3)] = 0(1), \quad r \rightarrow 0, \quad \varphi_k = \frac{\psi_1 + 2\pi k}{3}, \quad k = 0, 1, 2, \dots, \end{aligned}$$

which immediately give us  $a_1 = 0$ . Taking this fact into account, we obtain

$$\frac{1}{r^2} [\lambda_1 b_2 e^{-i\varphi_k} + \lambda_1 r (a_2 e^{-2i\varphi_k} + b_3)] = O(1), \quad r \rightarrow 0,$$

and therefore  $b_2 = 0$ . In its turn, we have

$$a_2 e^{-2i\varphi_0} + b_3 = 0, \quad \varphi_0 = \frac{\psi_1}{3},$$

$$a_2 e^{-2i\varphi_1} + b_3 = 0, \quad \varphi_1 = \frac{\psi_1 + 2\pi}{3},$$

by virtue of which  $a_2 = b_3 = 0$ .

Thus the holomorphic functions  $\tilde{\phi}$  and  $\tilde{\psi}$  have the form

$$\tilde{\phi}(z) = z^3 \phi_0(z), \quad \tilde{\psi}(z) = z^4 \psi_0(z), \quad (55)$$

where the functions  $\phi_0$  and  $\psi_0$  are holomorphic in the domain  $G$ . Substituting (55) and (50), we obtain the representation (40). Next, if the solution (40) together with its derivative

$$\frac{\partial \omega}{\partial \bar{z}} = \left[ \phi_0(z) \left( z + \frac{\lambda_1 \bar{z}}{z^2} \right) + \lambda_1 \psi_0(z) \right] e^{\frac{\lambda_1 \bar{z}}{z^2}} \quad (56)$$

is continuously extendable to  $\overline{G} \setminus \{0\}$ , we will find that the holomorphic functions  $\phi_0$  and  $\psi_0$  are continuously extendable to  $\overline{G}$ .

Conversely, any function of the type (40) provides us with a continuously extendable (together with its derivative (56)) to  $\overline{G} \setminus \{0\}$  solution of the equation (17), satisfying the condition (18) with  $\delta = \delta_0$ ,  $\sigma = 1$ . It remains to take into account the boundary conditions (20) and (23) (with  $p = 0$ ) which immediately leads us to the conclusion of our theorem.  $\square$

It is not difficult to see that any linearly independent system of holomorphic vector functions (44) transforms by the relation

$$\omega_k(z) = \left( z \bar{z} \phi_k(z) + z^2 \psi_k(z) \exp \left\{ \frac{\lambda_1 \bar{z}}{z^2} \right\} \right), \quad z \in G \setminus \{0\}$$

(analogously to the relation (45)), into a linearly independent system of functions  $\omega_k(z)$ ,  $k = 1, 2, \dots, m$ , and vice versa. Therefore the numbers of linearly independent solutions of homogeneous boundary problems  $Q(\delta_0, 1)$  and  $Q''_0(0)$  coincide.

We have the following

**Theorem 7.** *Let at least one of the equalities*

$$\delta = \delta_0, \quad \sigma = 1, \quad (57)$$

*be violated. Then either the homogeneous boundary value problem  $Q(\delta, \sigma)$  has an infinite number of linearly independent solutions, or the inhomogeneous problem is unsolvable for any right-hand side  $(\gamma_1(t), \gamma_2(t)) \not\equiv 0$ .*

*Proof.* The violation of at least one of the equalities (57) implies that one of the following conditions is fulfilled:

$$\delta < \delta_0, \quad \sigma = 1, \quad (58)$$

or

$$\delta > \delta_0, \quad \sigma = 1, \quad (59)$$

or

$$\sigma < 1 \quad (\sigma \text{ is arbitrary}), \quad (60)$$

or

$$\sigma > 1 \quad (\sigma \text{ is arbitrary}). \quad (61)$$

Under the condition (58) (and under the condition (60)), on the basis of Theorem 1 it immediately follows that the equation (17) has no

non-trivial solution satisfying the condition (18), and hence the inhomogeneous boundary value problem  $Q(\delta, \sigma)$  is unsolvable for any right-hand side  $(\gamma_1(t), \gamma_2(t)) \neq 0$ .

Let us prove that under the condition (59) the homogeneous boundary value problem  $Q(\delta, 1)$  has an infinite number of linearly independent solutions. Indeed, let the condition (59) be fulfilled and, moreover,  $\arg \lambda_1 \neq \arg \lambda_2$ . We take an arbitrary natural number  $N$  and choose a natural number  $p$  so large that the number of linearly independent solutions of the homogeneous boundary value problem  $Q'_0(p)$  be greater than  $N$ . We denote these solutions by

$$(\phi_0^{(k)}(z), \psi_0^{(k)}(z)), \quad k = 1, 2, \dots, m, \quad m > N. \quad (62)$$

It is not difficult to see that the system of functions (62) transforms by the relation

$$\begin{aligned} \omega_k(z) = & z^2 \phi_0^{(k)}(z) \exp \left\{ \frac{\lambda_1 \bar{z}}{z^2} \right\} + \\ & + z^2 \psi_0^{(k)}(z) \exp \left\{ \frac{\lambda_2 \bar{z}}{z^2} \right\}, \quad z \in G \setminus \{0\}, \end{aligned}$$

into a linearly independent system of solutions of the homogeneous boundary value problem  $Q(\delta, \sigma)$ . Therefore this problem has an infinite number of linearly independent solutions.

Let now the condition (59) be fulfilled, and  $\arg \lambda_1 = \arg \lambda_2$ . We take an arbitrary natural number  $N$  and choose a natural number  $p$  so large that the number of linearly independent solutions of the homogeneous boundary value problem  $Q''_0(p)$  be greater than  $N$ . We denote again these solutions by (62). It is not difficult to see that the system of functions (62) transforms by the relation

$$\begin{aligned} \omega_k(z) = & (z \bar{z} \phi_0^{(k)}(z) + z^2 \psi_0^{(k)}(z)) \exp \left\{ \frac{\lambda_1 \bar{z}}{z^2} \right\}, \\ & z \in G \setminus \{0\}, \quad k = 1, 2, \dots, m, \end{aligned}$$

into a linearly independent system of solutions of the homogeneous boundary value problem  $Q(\delta, \sigma)$ . Therefore this problem has an infinite number of linearly independent solutions.

It remains to consider the case (61). But any solution of the homogeneous boundary value problem  $Q(\delta, 1)$  (for  $\delta > \delta_0$ ) is likewise a solution of the homogeneous boundary value problem  $Q(\delta, \sigma)$  (for  $\sigma > 1$ ). Therefore the latter problem has an infinite number of linearly independent solutions.  $\square$

On the basis of the above-proven Theorems 6 and 7 we have the following

**Theorem 8.** *The boundary value problem  $Q(\delta, \sigma)$  is Noetherian if and only if the condition (57) is fulfilled.*

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