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ON THE QUESTION OF STABILITY OF LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Consider a linear homogeneous system of generalized ordinary differential equations

$$dx(t) = dA(t) \cdot x(t), \tag{1}$$

where $A : [0, +\infty[\to \mathbb{R}^{n\times n}$ is a real matrix-function with locally bounded variation components.

In this paper we give some sufficient conditions imposed on the components of matrix-function A, wich guarantee the stability of the system (1) in the Liapunov sense with respect to small perturbations. This conditions are differed from those given in [1]. Analogous conditions for ordinary differential equations are given in [2].

The following notations and definitions will be used in the paper:

$$\mathbb{R} =]-\infty, +\infty[, \quad \mathbb{R}_+ = [0, +\infty[, \quad [a, b] \quad \text{and} \quad]a, b[\quad (a, b \in \mathbb{R})$$

are, respectively, a closed and open intervals;

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|;$$

$O_{n\times m}$ (or O) is zero $n \times m$-matrix;

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$;

If $X \in \mathbb{R}^{n\times n}$, then X^{-1} and $\det(X)$ are, respectively, the matrix inverse to X and the determinant of X; I_n is the identity $n \times n$-matrix;

$V_0^{+\infty}(X) = \sup_{b\in\mathbb{R}_+} V_0^b(X)$, where $V_0^b(X)$ is the sum of total variations on $[0, b]$ of the compnents x_{ij} $(i = 1, \dots, n; j = 1, \dots, m)$ of the matrix-function $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$; $V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(0) = 0$ and $v(x_{ij})(t) = V_0^t(x_{ij})$ for $0 < t < +\infty$ $(i = 1, \dots, n; j = 1, \dots, m)$.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ at the point t; $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$;

$BV_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times m})$ is the set of all matrix-functions of bounded variations on every closed interval from $\mathbb{R}_+$.

$s_0 : BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}) \to BV_{\rm loc}(\mathbb{R}_+, \mathbb{R})$ is an operator defined by

$$s_0(x)(t) \equiv x(t) - \sum_{0<\tau\le t} d_1x(\tau) - \sum_{0\le\tau<t} d_2x(\tau).$$

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If $g:\mathbb{R}_+\to\mathbb{R}$ is a nondecreasing function, $x:\mathbb{R}_+\to\mathbb{R}$ and $0\le s<t<+\infty$, then

$$\int\limits_s^t x(\tau)dg(\tau)=\int\limits_{]s,t[} x(\tau)dg_1(\tau)-$$

$$-\int\limits_{]s,t[} x(\tau)dg_2(\tau)+\sum_{s<\tau\le t} x(\tau)d_1g(\tau)-\sum_{s\le\tau<t} x(\tau)d_2g(\tau),$$

where $g_j:\mathbb{R}_+\to\mathbb{R}$ $(j=1,2)$ are continuous nondecreasing functions, such that $g_1(t)-g_2(t)\equiv s_0(g)(t)$, and $\int_{]s,t[} x(\tau)dg_j(\tau)$ is Lebesgue-Stieltjes integral over the open interval $]s,t[$ with respect to the measure corresponding to the function g_j $(j=1,2)$ (if $s=t$, then $\int_s^t x(\tau)dg(\tau)=0$);

A matrix-function is said to be nondecreasing if each of its component is such.

If $G=(g_{ik})_{i,k=1}^{l,n}:\mathbb{R}_+\to\mathbb{R}^{l\times n}$ is a nondecreasing matrix-function, $X=(x_{ik})_{i,k=1}^{n,m}:\mathbb{R}_+\to\mathbb{R}^{n\times m}$, then

$$\int\limits_s^t dG(\tau)\cdot X(\tau)=\Big(\sum_{k=1}^n\int\limits_s^t x_{kj}(\tau)dg_{ik}(\tau)\Big)_{i,j=1}^{l,m}\quad\text{for}\quad 0\le s\le t<+\infty.$$

If $G_j:\mathbb{R}_+\to\mathbb{R}^{l\times n}$ $(j=1,2)$ are nondecreasing matrix-functions, $G(t)\equiv G_1(t)-G_2(t)$ and $X:\mathbb{R}_+\to\mathbb{R}^{n\times m}$, then

$$\int\limits_s^t dG(\tau)\cdot X(\tau)=\int\limits_s^t dG_1(\tau)\cdot X(\tau)-\int\limits_s^t dG_2(\tau)\cdot X(\tau)\quad\text{for}\quad 0\le s\le t<+\infty.$$

$r(H)$ is the spectral radius of the matrix $H\in\mathbb{R}^{n\times n}$.

Under a solution of the system (1) we understand a vector function $x\in BV_{\rm loc}(\mathbb{R}_+,\mathbb{R}^n)$ such that

$$x(t)=x(s)+\int\limits_s^t dA(\tau)\cdot x(\tau)\quad(0\le s\le t<+\infty).$$

We will assume that $A=(a_{ik})_{i,k=1}^n\in BV_{\rm loc}(\mathbb{R}_+,\mathbb{R}^{n\times n})$, $A(0)=O_{n\times n}$ and

$$\det(I_n+(-1)^jd_jA(t))\ne 0\quad\text{for}\quad t\in\mathbb{R}_+\ (j=1,2).$$

Let $x_0\in BV_{\rm loc}(\mathbb{R}_+,\mathbb{R}^n)$ be a solution of the system (1).

Definition 1. Let $\xi:\mathbb{R}_+\to\mathbb{R}_+$ be a nondecreasing function such that

$$\lim_{t\to+\infty}\xi(t)=+\infty.\tag{2}$$

The solution x_0 of the system (1) is called ξ-exponentially asymptotically stable, if there exists a positive number η such that for every $\varepsilon>0$ there exists a positive number $\delta=\delta(\varepsilon)$ such that an arbitrary solution x of the system (1), satisfying the inequality

$$\|x(t_0)-x_0(t_0)\|<\delta$$

for some $t_0\in\mathbb{R}_+$, admits the estimate

$$\|x(t)-x_0(t)\|<\varepsilon\exp(-\eta(\xi(t)-\xi(t_0)))\quad\text{for}\quad t\ge t_0.$$

Stability, uniformly stability and asymptotically stability of the solution x_0 are defined analogously as for systems of ordinary differential equations (see [2]), i.e. in case when

$A(t)$ is the diagonal matrix-function with diagonal elements equal to t. Note that exponentially asymptotically stability ([2]) is particular case of ξ-exponentially asymptotically stability if we assume $\xi(t) \equiv t$.

Definition 2. The system (1) is called stable (uniformly stable, asymptotically stable or ξ-exponentially asymptotically stable) if every solution of this system is stable (uniformly stable, asymptotically stable or ξ-exponentially asymptotically stable).

Definition 3. The matrix-function A is called stable (uniformly stable, asymptotically stable or ξ-exponentially asymptotically stable) if the system (1) is stable (uniformly stable, asymptotically stable or ξ-exponentially asymptotically stable).

If $X \in BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, then $\mathcal{A}(X,\cdot) : BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times m}) \to BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times m})$ is an operator defined by

$$\mathcal{A}(X,Y)(t) = Y(t) + \sum_{0<\tau\le t} d_1X(\tau)\cdot(I_n - d_1X(\tau))^{-1}\cdot d_1Y(\tau) - \\ - \sum_{0\le\tau<t} d_2X(\tau)\cdot(I_n + d_2X(\tau))^{-1}\cdot d_2Y(\tau) \quad \text{for} \quad t\in\mathbb{R}_+;$$

If $a \in BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}_+)$ and $1 + (-1)^j d_j a(t) \neq 0$ for $t \in \mathbb{R}_+$ $(j = 1, 2)$, then $J : BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}_+) \to BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}_+)$ is an operator defined by

$$J(a)(t) = \sum_{0<s\le t} (d_1a(s) + \ln|1 - d_1a(s)|) + \sum_{0\le s<t} (d_2a(s) - \ln|1 + d_2a(s)|) \quad \text{for} \quad t\in\mathbb{R}_+.$$

Theorem 1. *Let the components a_{ik} $(i,k = 1,\dots,n)$ of the matrix-function A satisfy the conditions*

$$1 + (-1)^j d_j a_{ii}(t) \neq 0 \quad \textit{for} \quad t \ge t^* \quad (j = 1,2; i = 1,\dots,n), \tag{3}$$

$$\int_{t^*}^{t} \exp(a_{ii}(t) - J(a_{ii})(t) - a_{ii}(\tau) + J(a_{ii}((\tau))dv(b_{ik})(\tau) \le h_{ik} \tag{4}$$

$$\textit{for} \quad t \ge t^* \quad (i \neq k; i,k = 1,\dots,n)$$

and

$$\sup\{a_{ii}(t) - J(a_{ii})(t) : t \in \mathbb{R}_+\} < +\infty \quad (i = 1,\dots,n),$$

where $b_{ik}(t) \equiv \mathcal{A}(a_{ii}, a_{ik})(t)$ $(i,k = 1,\dots,n)$, t^* and $h_{ik} \in \mathbb{R}_+$ $(i \neq k; i,k = 1,\dots,n)$. Let, moreover, the matrix $H = (h_{ik})_{i,k=1}^n$, where $h_{ii} = 0$ $(i = 1,\dots,n)$, be such that

$$r(H) < 1. \tag{5}$$

Then the matrix-function A is stable.

Theorem 2. *Let the components a_{ik} $(i,k = 1,\dots,n)$ of the matrix-function A satisfy the conditions* (3), (4) *and*

$$\sup\{a_{ii}(t) - J(a_{ii})(t) - a_{ii}(\tau) + J(a_{ii})(\tau) : t \ge \tau \ge 0\} < +\infty,$$

where $t^* \in \mathbb{R}_+$, and $h_{ik} \in \mathbb{R}_+$ $(i \neq k; i,k = 1,\dots,n)$ are such that the matrix $H = (h_{ik})_{i,k=1}^n$, where $h_{ii} = 0$ $(i = 1,\dots,n)$, satisfies the condotion (5). *Then the matrix-function A is uniformly stable.*

Corollary 1. *Let the components* a_{ik} $(i,k=1,\dots,n)$ *of the matrix-function* A *satisfy the conditions* (3) *and*

$$V_\tau^t b_{ik} \le -h_{ik}(b_{ii}(t)-b_{ii}(\tau)) \quad \text{for} \quad t\ge\tau\ge t^* \quad (i\ne k; i,k=1,\dots,n), \tag{6}$$

where $t_*\in\mathbb{R}_+$, $b_{ik}(t)\equiv\mathcal{A}(a_{ii},a_{ik})(t)$ $(i,k=1,\dots,n)$, b_{ii} $(i=1,\dots,n)$ *are nonincreasing functions, and* $h_{ik}\in\mathbb{R}_+$ $(i\ne k; i,k=1,\dots,n)$ *are such that the matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *satisfies the condition* (5). *Then the matrix-function* A *is uniformly stable.*

Theorem 3. *Let the components* a_{ik} $(i,k=1,\dots,n)$ *of the matrix-function* A *satisfy the conditions* (3),

$$a_{ii}(t)-J(a_{ii})(t)-a_{ii}(t^*)+J(a_{ii})(t^*)\le-\xi(t)+\xi(t^*) \quad \text{for} \quad t\ge t^* \quad (i=1,\dots,n)$$

and

$$\int_{t^*}^{t}\exp(\xi(t))-\xi(\tau)+a_{ii}(t)-J(a_{ii})(t)-a_{ii}(\tau)+J(a_{ii})(\tau))dv(b_{ik})(\tau)\le$$
$$\le h_{ik} \quad \text{for} \quad t\ge t^* \quad (i\ne k; i,k=1,\dots,n),$$

where t^* *and* $h_{ik}\in\mathbb{R}_+$ $(i\ne k; i,k=1,\dots,n)$, $b_{ik}(t)\equiv\mathcal{A}(a_{ii},a_{ik})(t)$ $(i,k=1,\dots,n)$. *Let, moreover, the matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *satisfy the condition* (5), *and the function* $\xi\in BV_{\rm loc}(\mathbb{R}_+,\mathbb{R}_+)$ *satisfies the condition* (2). *Then the matrix-function* A *is asymptotically stable.*

Corollary 2. *Let the components* a_{ik} $(i,k=1,\dots,n)$ *of the matrix-function* A *satisfy the conditions* (3) *and* (6), *where* $t_*\in\mathbb{R}_+$, $b_{ik}(t)\equiv\mathcal{A}(a_{ii},a_{ik})(t)$ $(i,k=1,\dots,n)$, b_{ii} $(i=1,\dots,n)$ *are nonincreasing functions, and* $h_{ik}\in\mathbb{R}_+$ $(i\ne k; i,k=1,\dots,n)$ *are such that the matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *satisfies the condition* (5). *Let, moreover,*

$$\lim_{t\to+\infty}a_0(t)=+\infty,$$

where

$$a_0(t)=\min\{|a_{ii}(t)-J(a_{ii})(t)-a_{ii}(t^*)+J(a_{ii})(t^*)|:i=1,\dots,n\} \quad (t\ge t^*).$$

Then the matrix-function A *is uniformly and asymptotically stable.*

Corollary 3. *Let the components* a_{ik} $(i,k=1,\dots,n)$ *of the matrix-function* A *satisfy the conditions* (3),

$$a_{ii}(t)-J(a_{ii})(t)-a_{ii}(t^*)+J(a_{ii})(t^*)\le-\gamma(t-t^*) \quad \text{for} \quad t\ge t^* \quad (i=1,\dots,n) \tag{7}$$

and

$$\int_{t^*}^{t}\exp(\gamma(t-\tau)+a_{ii}(t)-J(a_{ii})(t)-a_{ii}(\tau)+J(a_{ii})(\tau))dv(b_{ik})(\tau)\le h_{ik}$$
$$\text{for} \quad t\ge t^* \quad (i\ne k; i,k=1,\dots,n),$$

where $\gamma>0$, t^* *and* $h_{ik}\in\mathbb{R}_+$ $(i\ne k; i,k=1,\dots,n)$, $b_{ik}(t)\equiv\mathcal{A}(a_{ii},a_{ik})(t)$ $(i,k=1,\dots,n)$. *Let, moreover, the matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *satisfy the condition* (5). *Then* A *is exponentially asymptotically stable.*

Corollary 4. *Let the components* a_{ik} $(i,k=1,\dots,n)$ *of the matrix-function* A *satisfy the conditions* (3), (6) *and* (7), *where* $\gamma>0$, t^* *and* $h_{ik}\in\mathbb{R}_+$ $(i\neq k; i,k=1,\dots,n)$, $b_{ik}(t)\equiv\mathcal{A}(a_{ii},a_{ik})(t)$ $(i,k=1,\dots,n)$. *Let, moreover, the matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *satisfy the condition* (5). *Then* A *is exponentially asymptotically stable.*

Theorem 4. *Let* $\overline{A}=(\overline{a}_{ik})\in BV_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^{n\times n})$ *be a matrix-function such that*

$$\|d_j\overline{A}(t)\|<1 \quad \textit{for} \quad t\geq 0,$$
$$s_0(a_{ii})(t)-s_0(a_{ii})(s)\leq s_0(\overline{a}_{ii})(t)-s_0(\overline{a}_{ii})(s)$$
$$\textit{for} \quad t>s\geq 0; \quad (i=1,\dots,n),$$
$$|s_0(a_{ik})(t)-s_0(a_{ik})(s)|\leq s_0(\overline{a}_{ik})(t)-s_0(\overline{a}_{ik})(s)$$
$$\textit{for} \quad t>s\geq 0; \quad (i\neq k; i=1,\dots,n)$$

and

$$|d_ja_{ik}(t)|\leq d_j\overline{a}_{ik}(t) \;\; \textit{for} \;\; t\geq 0 \;\; (j=1,2; i,k=1,\dots,n).$$

Let, moreover, $\overline{a}_{ik}$ $(i\neq k; i,k=1,\dots,n)$ are nondecreasing functions, $\overline{A}$ be stable (uniformly stable, asymptotically stable or ξ-exponentially asymptotically stable). Then A will be stable (uniformly stable, asymptotically stable or ξ-exponentially asymptotically stable), too.

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