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LIDSTONE CONTINUOUS AND DISCRETE BOUNDARY VALUE PROBLEMS

Abstract. Nonsingular continuous and discrete Lidstone boundary value problems are discussed in this paper. Existence criteria for one or more solutions are presented.

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1. Introduction

In this paper we discuss the existence of one and more solutions to Lidstone continuous and discrete boundary value problems. Problems of this type have become quite popular and many articles have appeared in the literature [4–8, 9, 13–16]. The results presented here extend, complement and improve those in the literature.

Our paper will be divided into two main sections. In section 2 we discuss the Lidstone continuous problem

$$\begin{cases} (-1)^n\, y^{(2n)}(t) = \phi(t)\, f(t, y(t)), \;\; 0 < t < 1, \\ y^{(2i)}(0) = y^{(2i)}(1) = 0, \;\; 0 \le i \le n-1, \end{cases} \tag{1.1}$$

where $n \ge 1$. We begin Section 2 by presenting an existence principle for (1.1). This principle together with Krasnosel'skiĭ's fixed point theorem in a cone will enable us to establish the existence of one or more solutions to (1.1). Throughout Section 2 we will let $G_n(t,s)$ denote Green's function for the boundary value problem

$$\begin{cases} y^{(2n)} = 0 \quad \text{on} \quad (0,1), \\ y^{(2i)}(0) = y^{(2i)}(1) = 0, \;\; 0 \le i \le n-1. \end{cases} \tag{1.2}$$

Now $G_n(t,s)$ can be expressed as [5]

$$G_n(t,s) = \int_0^1 G(t,u)\, G_{n-1}(u,s)\, du,$$

where

$$G_1(t,s) = G(t,s) = \begin{cases} t\,(s-1), \;\; 0 \le t \le s, \\ s\,(t-1), \;\; s \le t \le 1. \end{cases}$$

The following inequalities have appeared in the literature [13, 14]

$$0 \le (-1)^n\, G_n(t,s) \le \frac{1}{6^{n-1}}\, s\,(1-s) \quad \text{for} \quad (t,s) \in [0,1] \times [0,1], \tag{1.3}$$

and for $\delta \in (0, \frac{1}{2})$ fixed,

$$(-1)^n\, G_n(t,s) \ge \theta_n\, s\,(1-s) \quad \text{for} \quad (t,s) \in [\delta, 1-\delta] \times [0,1], \tag{1.4}$$

where $0 < \theta_n < \frac{1}{6^{n-1}}$ is given by

$$\theta_n = \delta^n \Big(\frac{4\,\delta^3 - 6\,\delta^2 + 1}{6} \Big)^{n-1}. \tag{1.5}$$

In Section 3 we discuss the Lidstone discrete problem

$$\begin{cases} (-1)^m \triangle^{2m} y(k) = f(k, y(k)) \quad \text{for} \quad k \in I_N, \\ \triangle^{2i} y(0) = \triangle^{2i} y(N + 2m - 2i) = 0, \;\; 0 \le i \le m - 1. \end{cases} \tag{1.6}$$

Here $N \in \{1, 2, \ldots\}$, $m \ge 1$, $I_N = \{0, 1, \ldots, N\}$ and $y : I_{N+2m} = \{0, 1, \ldots, N + 2m\} \to \mathbf{R}$. Existence of one or more solutions to (1.6) is established in Section 3. Throughout Section 3 we will let $G^1_m(k, l)$ denote Green's function for

$$\begin{cases} \triangle^{2m} y = 0 \quad \text{on} \quad I_N, \\ \triangle^{2i} y(0) = \triangle^{2i} y(N + 2m - 2i) = 0, \;\; 0 \le i \le m - 1. \end{cases} \tag{1.7}$$

Now G^1_m can be expressed as [1]

$$G^1_m(k, l) = \sum_{i=0}^{N+2m-2} G_m(k, i)\, G^1_{m-1}(i, l),$$

where

$$G_m(k, l) = \begin{cases} -\dfrac{(N + 2m - k)(l + 1)}{N + 2m}, & l \in \{0, 1, \ldots, k - 2\} \\ -\dfrac{k\,(N + 2m - 1 - l)}{N + 2m}, & l \in \{k - 1, \ldots, N + 2m - 2\} \end{cases}$$

and

$$G^1_1(k, l) = G_1(k, l).$$

The following inequalities have appeared in the literature [15, 16]:

$$0 \le (-1)^m G^1_m(k, l) \le a_m\,(l + 1)(N + 1 - l) \quad \text{for} \quad (k, l) \in I_{N+2m} \times I_N \tag{1.8}$$

with

$$a_m = \left[\prod_{i=1}^{m} (N + 2i)\right]^{-1} \prod_{i=1}^{m-1} s_{2i}, \tag{1.9}$$

where for $j \ge 1$,

$$s_j = \sum_{i=0}^{N+j} (i + 1)\,(N + j + 1 - i) = \frac{1}{6}\,(N + j + 3)^{(3)},$$

and

$$(-1)^m G^1_m(k, l) \ge b_m \min\{l + 1\,,\, N + 1 - l\} \quad \text{for} \quad (k, l) \in J_N \times I_N, \tag{1.10}$$

where $J_n = \{1, \ldots, N + 2m - 1\}$,

$$b_m = \left[\prod_{i=1}^{m} (N + 2i)\right]^{-1} \prod_{i=1}^{m-1} T_{2i-1} \tag{1.11}$$

with

$$T_j = \sum_{i=1}^{N+j} \min\{i+1, N+j+2-i\} =$$

$$= \begin{cases} \dfrac{(N+j)^2 + 6\,(N+j) + 1}{4} & \text{if } N+j \text{ odd,} \\ \dfrac{(N+j)(N+j+6)}{4} & \text{if } N+j \text{ even} \end{cases}$$

for $j \geq 1$, Finally we state Krasnosel'skiĭ's Fixed Point Theorem in a cone.

Theorem 1.1. *Let $E = (E, \|\,.\,\|)$ be a Banach space and let $K \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$ and let $A : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$ be continuous and completely continuous. In addition suppose either*

$$\|A\,u\| \leq \|u\| \;\; \textit{for} \;\; u \in K \cap \partial\Omega_1 \;\; \textit{and} \;\; \|A\,u\| \geq \|u\| \;\; \textit{for} \;\; u \in K \cap \partial\Omega_2$$

or

$$\|A\,u\| \geq \|u\| \;\; \textit{for} \;\; u \in K \cap \partial\Omega_1 \;\; \textit{and} \;\; \|A\,u\| \leq \|u\| \;\; \textit{for} \;\; u \in K \cap \partial\Omega_2$$

hold. Then A has a fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

2. Continuous Problem

In this section we present existence criteria for one or more solutions to (1.1). Our theory will rely on the following existence principle.

Theorem 2.1. *Assume that*

$$f : [0,1] \times \mathbf{R} \to \mathbf{R} \;\; \textit{is continuous,} \tag{2.1}$$

$$\phi \in C(0,1) \;\; \textit{with} \;\; \phi > 0 \;\; \textit{on} \;\; (0,1) \;\; \textit{and} \;\; \int_0^1 t\,(1-t)\,\phi(t)\,dt < \infty \tag{2.2}$$

and

$$\begin{cases} \lim\limits_{t\to 0^+} t^2\,(1-t)\,\phi(t) = 0 \;\; \textit{if} \;\; \displaystyle\int_0^1 (1-t)\,\phi(t)\,dt = \infty \\ \textit{and} \;\; \lim\limits_{t\to 1^-} t\,(1-t)^2\,\phi(t) = 0 \;\; \textit{if} \;\; \displaystyle\int_0^1 t\,\phi(t)\,dt = \infty \end{cases} \tag{2.3}$$

hold. Suppose there is a constant $M > 0$ with

$$|y|_0 = \sup_{[0,1]} |y(t)| \neq M$$

for any solution $y \in C^{2n-2}[0,1] \cap C^{2n}(0,1)$ *to*

$$\begin{cases} (-1)^n\, y^{(2n)}(t) = \lambda\, \phi(t)\, f(t, y(t)), \quad 0 < t < 1, \\ y^{(2i)}(0) = y^{(2i)}(1) = 0, \quad 0 \le i \le n-1 \end{cases} \tag{2.4$_\lambda$}$$

for each $\lambda \in (0,1)$. *Then* (1.1) *has a solution* $y \in C^{2n-2}[0,1] \cap C^{2n}(0,1)$ *with* $|y|_0 \le M$.

Proof. Solving $(2.4)_\lambda$ is equivalent to finding a solution $y \in C[0,1]$ to

$$y(t) = \lambda \int_0^1 (-1)^n\, G_n(t,s)\, \phi(s)\, f(s, y(s))\, ds, \tag{2.5$_\lambda$}$$

where $G_n(t,s)$ is as in Section 1. $\square$

Remark 2.1. From (1.3) we can see that

$$\int_0^1 (-1)^n\, G_n(t,s)\, \phi(s)\, ds \le \frac{1}{6^{n-1}} \int_0^1 s\,(1-s)\, \phi(s)\, ds.$$

Remark 2.2. Showing the equivalence of $(2.4)_\lambda$ and $(2.5)_\lambda$ is just a matter of modifying slightly the argument in [11, 12] using the ideas in [5 p. 3]. It is enough for us to note that if $y \in C[0,1]$ and (2.1)–(2.3) are satisfied, then

$$\begin{aligned} r_1(t) &= \int_0^t (1-t)\, s\, \phi(s)\, f(s, y(s))\, ds + \int_t^1 t\,(1-s)\, \phi(s)\, f(s, y(s))\, ds = \\ &= \int_0^1 (-1)\, G_1(t,s)\, \phi(s)\, f(s, y(s))\, ds \;\in\; C[0,1] \end{aligned}$$

with $r_1(0) = r_1(1) = 0$ and $-r_1''(t) = \phi(t)\, f(t, y(t))$ for $t \in (0,1)$. Next note that

$$\begin{aligned} r_2(t) &= \int_0^1 (-1)^2\, G_2(t,s)\, \phi(s)\, f(s, y(s))\, ds = \\ &= \int_0^1 G_1(t,x) \left[\int_0^1 G_1(x,s)\, \phi(s)\, f(s, y(s))\, ds \right] dx \;\in\; C^2[0,1] \end{aligned}$$

with $r_2(0) = r_2''(0) = r_2(1) = r_2''(0) = 0$ and $r_2''(t) = -r_1(t)$ so $r_2^{(4)}(t) = \phi(t)\, f(t, y(t))$ for $t \in (0,1)$. In general,

$$r_n(t) = \int_0^1 (-1)^n\, G_n(t,s)\, \phi(s)\, f(s, y(s))\, ds \in C^{2n-2}[0,1]$$

with $r_n^{(2i)}(0) = r_n^{(2i)}(1) = 0$ for $0 \le i \le n-1$ and $r_n^{(2n)}(t) = (-1)^n \phi(t) \times f(t, y(t))$ for $t \in (0,1)$.

Let $N : C[0,1] \to C[0,1]$ be given by

$$N\, y(t) = \int_0^1 (-1)^n\, G_n(t,s)\, \phi(s)\, f(s, y(s))\, ds.$$

We now show that $N : C[0,1] \to C[0,1]$ is continuous and completely continuous. The continuity follows immediately from the Lebesgue dominated convergence theorem since

$$|N\, y_m(t) - N\, y(t)| \le \frac{1}{6^{n-1}} \int_0^1 s\,(1-s)\,\phi(s)\, |f(s, y_m(s)) - f(s, y(s))|\, ds$$

for $y_m,\, y \in C[0,1]$. To show the complete continuity, we will use the Arzela–Ascoli theorem. To see this, let $\Omega \subseteq C[0,1]$ be bounded, i.e., suppose that there exists $r_0 > 0$ with $|u|_0 \le r_0$ for each $u \in \Omega$. Also there exists a constant K_0 with $|f(s, u(s))| \le K_0$ for $s \in [0,1]$ and for all $u \in \Omega$. Now if $u \in \Omega$ and $t \in [0,1]$, we have

$$|N\, u(t)| \le \frac{K}{6^{n-1}} \int_0^1 s\,(1-s)\,\phi(s)\, ds \tag{2.6}$$

with

$$|(N\, u)'(t)| \le L_n \int_0^t x\, dx + L_n \int_t^1 (1-x)\, dx \equiv \eta_n(t) \quad \text{if } n > 1 \tag{2.7}$$

and

$$|(N\, u)'(t)| \le K_0 \int_0^t s\phi(s)\, ds + K_0 \int_t^1 (1-s)\phi(s)\, ds \equiv \eta_1(t) \quad \text{if } n = 1; \tag{2.8}$$

here

$$L_n = K_0 \sup_{x \in [0,1]} \int_0^1 (-1)^n\, G_{n-1}(x,s)\, \phi(s)\, ds \quad \text{if } n > 1.$$

Remark 2.3. Note that (2.7) is immediate since if $n > 1$,

$$N\,u(t) = \int_0^1 G(t,x)\Big(\int_0^1 (-1)^n\, G_{n-1}(x,s)\,\phi(s)\,f(s,u(s))\,ds\Big)\,dx =$$

$$= (1-t)\int_0^t x\,\Big(\int_0^1 (-1)^n\, G_{n-1}(x,s)\,\phi(s)\,f(s,u(s))\,ds\Big)\,dx +$$

$$+\,t\int_t^1 (1-x)\,\Big(\int_0^1 (-1)^n\, G_{n-1}(x,s)\,\phi(s)\,f(s,u(s))\,ds\Big)\,dx$$

and (2.8) is immediate since if $n = 1$,

$$N\,u(t) = (1-t)\int_0^t s\,\phi(s)\,f(s,u(s))\,ds + t\int_t^1 (1-s)\,\phi(s)\,f(s,u(s))\,ds.$$

Note that $\eta_n \in L^1[0,1]$ for $n \ge 1$. Now (2.6) together with (2.7) and (2.8) imply that $N\,\Omega$ is a bounded, equicontinuous family on $[0,1]$, so the Arzela–Ascoli theorem guarantees that $N : C[0,1] \to C[0,1]$ is completely continuous. Let

$$U = \{u \in C[0,1] :\ |u|_0 < M\}.$$

The nonlinear alternative of Leray–Schauder [3, 12] guarantees that N has a fixed point in $\overline{U}$, i.e., (1.1) has a solution $y \in C^{2n-2}[0,1] \cap C^{2n}(0,1)$ with $|y|_0 \le M$. $\Box$

We are now in a position to establish the existence of one or more nonnegative solutions to (1.1). First we present two results which guarantee the existence of at least one solution.

Theorem 2.2. *Suppose the following conditions are satisfied:*

$$\begin{cases} f : [0,1]\times[0,\infty) \to [0,\infty) \ \ \textit{is continuous with} \\ f(t,u) > 0 \ \ \textit{for} \ \ (t,u) \in [0,1]\times(0,\infty), \end{cases} \tag{2.9}$$

$$\phi \in C(0,1) \ \ \textit{with} \ \ \phi > 0 \ \ \textit{on} \ \ (0,1) \ \ \textit{and} \ \ \int_0^1 t\,(1-t)\,\phi(t)\,dt < \infty, \tag{2.10}$$

$$\begin{cases} \lim\limits_{t\to 0^+} t^2\,(1-t)\,\phi(t) = 0 \ \ \textit{if} \ \ \displaystyle\int_0^1 (1-t)\,\phi(t)\,dt = \infty \\ \textit{and} \ \ \lim\limits_{t\to 1^-} t\,(1-t)^2\,\phi(t) = 0 \ \ \textit{if} \ \ \displaystyle\int_0^1 t\,\phi(t)\,dt = \infty, \end{cases} \tag{2.11}$$

$$\begin{cases} f(t,u) \le w(u) \quad on \quad [0,1]\times[0,\infty) \quad with \quad w \ge 0 \\ continuous\ and\ nondecreasing\ on \quad [0,\infty) \end{cases} \tag{2.12}$$

and

$$\exists\, r > 0 \quad with \quad \frac{r}{w(r)\, \sup_{t\in[0,1]} \int_0^1 (-1)^n\, G_n(t,s)\, \phi(s)\, ds} > 1. \tag{2.13}$$

Then (1.1) *has a solution* $y_1 \in C^{2n-2}[0,1] \cap C^{2n}(0,1)$ *with* $y_1 \ge 0$ *on* $[0,1]$ *and* $|y_1|_0 < r$.

Proof. We will use Theorem 2.1. The idea is to look at the boundary value problem

$$\begin{cases} (-1)^n\, y^{(2n)}(t) = \lambda\, \phi(t)\, f^\star(t, y(t)), \quad 0 < t < 1, \\ y^{(2i)}(0) = y^{(2i)}(1) = 0, \quad 0 \le i \le n-1 \end{cases} \tag{$2.14)_\lambda$}$$

for $0 < \lambda < 1$; here

$$f^\star(t,u) = \begin{cases} f(t,u), & u \ge 0, \\ f(t,0), & u < 0. \end{cases}$$

Let y be any solution of $(2.14)_\lambda$. Then $y(t) \ge 0$ for $t \in [0,1]$ and

$$\begin{aligned} y(t) &= \lambda \int_0^1 (-1)^n\, G_n(t,s)\, \phi(s)\, f^\star(s, y(s))\, ds \le \\ &\le w(|y|_0) \sup_{t\in[0,1]} \int_0^1 (-1)^n\, G_n(t,s)\, \phi(s)\, ds \end{aligned}$$

for $t \in [0,1]$. Consequently

$$\frac{|y|_0}{w(|y|_0)\, \sup_{t\in[0,1]} \int_0^1 (-1)^n\, G_n(t,s)\, \phi(s)\, ds} \le 1. \tag{2.15}$$

Now (2.13) and (2.15) imply $|y|_0 \ne r$. Thus Theorem 2.1 guarantees that $(2.14)_1$ has a solution y_1 with $|y_1|_0 < r$ (note that $|y_1|_0 \le r$ by Theorem 2.1 but $|y_1|_0 \ne r$ by an argument similar to the one above). In fact, $0 \le y_1(t) \le r$ for $t \in [0,1]$ and so y_1 is a solution of (1.1). $\square$

In Theorem 2.2 note that it is possible to have y_1 with $|y_1|_0 = 0$ in some application. We remove this situation in the next theorem.

Theorem 2.3. *Suppose* (2.9)–(2.13) *are satisfied. In addition assume that the following conditions hold:*

$$\begin{cases} \textit{there exists } \ \delta \in \left(0, \frac{1}{2}\right) \ \ \textit{(choose and fix it) and } \ \tau \in C[\delta, 1-\delta] \\ \textit{with } \ \tau > 0 \ \ \textit{on} \ \ [\delta, 1-\delta] \ \ \textit{and with} \ \ \phi(t)\, f(t,u) \geq \tau(t)\, w(u) \\ \textit{on} \ \ [\delta, 1-\delta] \times (0, \infty) \end{cases} \tag{2.16}$$

and

$$\exists\ R > r \ \ \textit{with} \ \ \frac{R}{w\,(6^{n-1}\,\theta_n\, R)} \leq \int_{\delta}^{1-\delta} (-1)^n\, G_n(\sigma, s)\, \tau(s)\, ds; \tag{2.17}$$

here $0 \leq \sigma \leq 1$ *is such that*

$$\int_{\delta}^{1-\delta} (-1)^n\, G_n(\sigma, s)\, \tau(s)\, ds = \sup_{t \in [0,1]} \int_{\delta}^{1-\delta} (-1)^n\, G_n(t, s)\, \tau(s)\, ds \tag{2.18}$$

and $0 < \theta_n < \frac{1}{6^{n-1}}$ *is as in* (1.5). *Then* (1.1) *has a solution* $y_2 \in C^{2n-2}[0,1] \cap C^{2n}(0,1)$ *with* $y_2 \geq 0$ *on* $[0,1]$, $y_2(t) > 0$ *for* $t \in [\delta, 1-\delta]$ *and* $r < |y_2|_0 \leq R$.

Proof. To show the existence of y_2, we use Theorem 1.1. Let $E = (C[0,1], |\,.\,|_0)$ and

$$K = \left\{ u \in C[0,1] : \ u(t) \geq 0 \ \text{ for } \ t \in [0,1] \ \text{ and } \min_{t \in [\delta, 1-\delta]} u(t) \geq 6^{n-1} \theta_n |u|_0 \right\}.$$

Clearly K is a cone of E. Let $A : K \to C[0,1]$ be defined by

$$A\, u(t) = \int_0^1 (-1)^n\, G_n(t,s)\, \phi(s)\, f(s, u(s))\, ds.$$

The argument in Theorem 2.1 implies that $A : K \to C[0,1]$ is continuous and completely continuous. We now show that $A : K \to K$. If $u \in K$, then clearly $A\, u(t) \geq 0$ for $t \in [0,1]$. Also for $t \in [0,1]$ we have from (1.3) that

$$A\, u(t) \leq \frac{1}{6^{n-1}} \int_0^1 s\,(1-s)\, \phi(s)\, f(s, u(s))\, ds,$$

and so

$$|A\, u|_0 \leq \frac{1}{6^{n-1}} \int_0^1 s\,(1-s)\, \phi(s)\, f(s, u(s))\, ds. \tag{2.19}$$

In addition, (1.4) and (2.19) yield

$$\min_{t\in[\delta,1-\delta]} A\,u(t) = \min_{t\in[\delta,1-\delta]} \int_0^1 (-1)^n\, G_n(t,s)\,\phi(s)\,f(s,u(s))\,ds \geq$$

$$\geq \theta_n \int_0^1 s\,(1-s)\,\phi(s)\,f(s,u(s))\,ds \geq 6^{n-1}\,\theta_n\,|A\,u|_0.$$

Consequently $A\,u \in K$, so $A : K \to K$. Let

$$\Omega_1 = \{u \in C[0,1]:\ |u|_0 < r\} \quad \text{and} \quad \Omega_2 = \{u \in C[0,1]:\ |u|_0 < R\}.$$

We first show

$$|A\,u|_0 \leq |u|_0 \quad \text{for} \quad u \in K \cap \partial\Omega_1. \tag{2.20}$$

To see this, let $u \in K \cap \partial\Omega_1$, so $|u|_0 = r$. Then (2.12) and (2.13) imply for all $t \in [0,1]$ that

$$A\,u(t) \leq\ w(|u|_0) \int_0^1 (-1)^n\, G_n(t,s)\,\phi(s)\,ds \leq$$

$$\leq\ w(r) \sup_{t\in[0,1]} \int_0^1 (-1)^n\, G_n(t,s)\,\phi(s)\,ds < r = |u|_0.$$

Thus $|A\,u|_0 < |u|_0$, and so (2.20) is true. Next we show

$$|A\,u|_0 \geq |u|_0 \quad \text{for} \quad u \in K \cap \partial\Omega_2. \tag{2.21}$$

To see this, let $u \in K \cap \partial\Omega_2$, so $|u|_0 = R$ and $\min_{t\in[\delta,1-\delta]} u(t) \geq 6^{n-1}\,\theta_n\,|u|_0 = 6^{n-1}\,\theta_n\,R$ so in particular $u(t) \in [6^{n-1}\,\theta_n\,R\,,\ R]$ for $t \in [\delta, 1-\delta]$. Now with σ as defined in (2.18) we have from (2.16) and (2.17) that

$$A\,u(\sigma) = \int_0^1 (-1)^n\, G_n(\sigma,s)\,\phi(s)\,f(s,u(s))\,ds \geq$$

$$\geq \int_\delta^{1-\delta} (-1)^n\, G_n(\sigma,s)\,\phi(s)\,f(s,u(s))\,ds \geq$$

$$\geq \int_\delta^{1-\delta} (-1)^n\, G_n(\sigma,s)\,\tau(s)\,w(u(s))\,ds \geq$$

$$\geq w\left(6^{n-1}\,\theta_n\,R\right) \int_\delta^{1-\delta} (-1)^n\, G_n(\sigma,s)\,\tau(s)\,ds \geq R = |u|_0.$$

Thus $|A\,u|_0 \geq |u|_0$ and so (2.21) holds. Now Theorem 1.1 implies that A has a fixed point $y_2 \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$, i.e., $r \leq |y_2|_0 \leq R$. In fact, $r < |y_1|_0$ (argue as in the first part of the theorem). Also $y_2 \geq 0$ on $[0,1]$ and since $y_2 \in K$, we have $y_2(t) > 0$ for $t \in [\delta, 1-\delta]$ since $|y_2|_0 > r$. □

Remark 2.4. If in (2.17) we have $R < r$, then (1.1) has a solution $y \in C[0,1]$ with $R \leq |y|_0 < r$. The argument is essentially the same as that in Theorem 2.3 except here we use the other half of Theorem 1.1.

Theorem 2.4. *Suppose* (2.9)–(2.13), (2.16) *and* (2.17) *hold. Then* (1.1) *has two solutions* y_1, $y_2 \in C^{2n-2}[0,1] \cap C^{2n}(0,1)$ *with* y_1, $y_2 \geq 0$ *on* $[0,1]$, $y_2(t) > 0$ *for* $t \in [\delta, 1-\delta]$ *and* $0 \leq |y_1|_0 < r < |y_2|_0 \leq R$.

Proof. The existence of y_1 follows from Theorem 2.2 and of y_2 from Theorem 2.3. □

In Theorem 2.4 it is possible to have $|y_1|_0$ to be zero in some applications. Our next theorem guarantees the existence of two solutions y_1, $y_2 \in C^{2n-2}[0,1] \cap C^{2n}(0,1)$ with $0 < |y_1|_0 < r < |y_2|_0 \leq R$.

Theorem 2.5. *Suppose* (2.9)–(2.13), (2.16) *and* (2.17) *hold. In addition assume that*

$$\exists\, L,\ 0 < L < r \quad with \quad \frac{L}{w(6^{n-1}\theta_n L)} \leq \int_{\delta}^{1-\delta} (-1)^n G_n(\sigma, s)\tau(s)\, ds \tag{2.22}$$

is satisfied. Then (1.1) *has two solutions* y_1, $y_2 \in C^{2n-2}[0,1] \cap C^{2n}(0,1)$ *with* y_1, $y_2 \geq 0$ *on* $[0,1]$, $y_1(t) > 0$ *and* $y_2(t) > 0$ *for* $t \in [\delta, 1-\delta]$ *and* $0 < L \leq |y_1|_0 < r < |y_2|_0 \leq R$.

Proof. The existence of y_2 follows from Theorem 2.3 and of y_1 from Remark 2.4. □

Remark 2.5. It is easy to use Theorem 2.3 and Remark 2.4 to write a theorem which guarantees the existence of more than two solutions to (1.1). We leave the details to the reader.

Example. Consider the boundary value problem

$$\begin{cases} y^{(6)} + (y^{\alpha} + y^{\beta} + 1) = 0 \quad \text{on} \quad (0,1), \\ y(0) = y''(0) = y^{(4)}(0) = y(1) = y''(1) = y^{(4)}(1) = 0 \end{cases} \tag{2.23}$$

with $0 < \alpha < 1 < \beta$. Then (2.23) has two solutions y_1, $y_2 \in C^4[0,1] \cap C^6(0,1)$ (in fact in $C^6[0,1]$) with $y_1 > 0$ on $(0,1)$, $y_2 > 0$ on $(0,1)$ and $0 < |y_1|_0 < 1 < |y_2|_0$.

To show the above, we will apply Theorem 2.5 with $\phi = \tau = 1$, $n = 3$, $w(x) = x^\alpha + x^\beta + 1$, $r = 1$ and $\delta = \frac{1}{4}$. Note that (2.9), (2.10), (2.11), (2.12) and (2.16) hold. Also since

$$(-1)^3\, G_3(t,s) \le \frac{1}{36}\, s\,(1-s),$$

we have

$$\sup_{t\in[0,1]} \int_0^1 (-1)^3\, G_3(t,s)\, \phi(s)\, ds \le \frac{1}{36} \int_0^1 s\,(1-s)\, ds = \frac{1}{216}.$$

Next note that (2.13) holds with $r = 1$ since

$$\frac{r}{w(r)\, \sup_{t\in[0,1]} \int_0^1 (-1)^3\, G_3(t,s)\, \phi(s)\, ds} = \frac{216}{3} > 1.$$

Now since $\beta > 1$, we have

$$\lim_{x\to\infty} \frac{x}{w(36\,\theta_3\, x)} = \lim_{x\to\infty} \frac{x}{(36\,\theta_3\, x)^\alpha + (36\,\theta_3\, x)^\beta + 1} = 0,$$

so there exists $R > r = 1$ with (2.17) holding. Finally note that

$$\lim_{x\to 0} \frac{x}{w(36\,\theta_3\, x)} = \lim_{x\to 0} \frac{x}{(36\,\theta_3\, x)^\alpha + (36\,\theta_3\, x)^\beta + 1} = 0,$$

so there exists L, $0 < L < 1$, with (2.22) holding. Theorem 2.5 now guarantees that (2.23) has two solutions $y_1, y_2 \in C^4[0,1] \cap C^6(0,1)$ with $y_1 \ge 0$, $y_2 \ge 0$ on $[0,1]$, $y_1(t) > 0$ and $y_2(t) > 0$ for $t \in [\frac{1}{4}, \frac{3}{4}]$ and $0 < |y_1|_0 < 1 < |y_2|_0$. The extra regularity and the fact that $y_1(t) > 0$ and $y_2(t) > 0$ for $t \in (0,1)$ follows immediately from the integral representation of y_1 and y_2.

3. Discrete Problem

In this section we discuss the discrete problem (1.6). We first obtain an existence principle for (1.6). For convenience we note here that by a solution to (1.6) we mean a $w \in C(I_{N+2\,m})$ such that w satisfies the difference equation and the boundary data in (1.6). Recall that $C(I_{N+2\,m})$ denotes the class of maps w continuous on $I_{N+2\,m}$ (discrete topology) with the norm $|w|_0 = \max_{k\in I_{N+2\,m}} |w(k)|$.

Theorem 3.1. *Assume that $f : I_N \times \mathbf{R} \to \mathbf{R}$ is continuous (i.e., continuous as a map from the topological space $I_N \times \mathbf{R}$ into the topological space $\mathbf{R}$ (of course the topology on I_N is the discrete topology)). Suppose there is a constant $M > 0$ with*

$$|y|_0 = \max_{k\in I_{N+2\,m}} |y(k)| \ne M$$

for any solution $y \in C(I_{N+2m})$ *to*

$$\begin{cases}(-1)^m \triangle^{2m} y(k) = \lambda f(k, y(k)) & \text{for } k \in I_N, \\ \triangle^{2i} y(0) = \triangle^{2i} y(N+2m-2i) = 0, & 0 \le i \le m-1\end{cases} \qquad (3.1)_\lambda$$

for each $\lambda \in (0,1)$. *Then* (1.6) *has a solution* $y \in C(I_{N+2m})$ *with* $|y|_0 \le M$.

Proof. Solving $(3.1)_\lambda$ is equivalent to finding a $y \in C(I_{N+2m})$ to

$$y(k) = \lambda \sum_{l=0}^{N} (-1)^m G_m^1(k,l)\, f(l, y(l)) \quad \text{for } k \in I_{N+2m}, \qquad (3.2)_\lambda$$

where G_m^1 is as in Section 1. Define the operator $N : C(I_{N+2m}) \to C(I_{N+2m})$ by setting

$$N\, y(t) = \sum_{l=0}^{N} (-1)^m G_m^1(k,l)\, f(l, y(l)).$$

It is easy to see [2, 3] that $N : C(I_{N+2m}) \to C(I_{N+2m})$ is continuous and completely continuous. Let

$$U = \{u \in C(I_{N+2m}) : \ |u|_0 < M\} \quad \text{and} \quad E = C(I_{N+2m}).$$

The nonlinear alternative of Leray–Schauder [3, 12] guarantees that N has a fixed point in $\overline{U}$, i.e., (1.6) has a solution $y \in C(I_{N+2m})$ with $|y|_0 \le M$. $\square$

Remark 3.1. It is clear that an existence principle could also be established for

$$\begin{cases}(-1)^m \triangle^{2m} y(k) = \lambda f(k, y(k), y(k+1), \ldots, y(k+2m-1)) & \text{for } k \in I_N, \\ \triangle^{2i} y(0) = \triangle^{2i} y(N+2m-2i) = 0, \ 0 \le i \le m-1. \end{cases}$$

We leave the details to the reader.

Theorem 3.2. *Suppose the following conditions are satisfied:*

$$f : I_N \times [0,\infty) \to [0,\infty) \ \textit{is continuous with} \ f(i,u) > 0 \ \textit{for} \ (i,u) \in I_N \times (0,\infty), \qquad (3.3)$$

$$\begin{cases} f(k,u) \le q(k)\, w(u) \ \textit{on} \ I_N \times [0,\infty) \ \textit{with} \ q : I_N \to (0,\infty) \\ \textit{and} \ w \ge 0 \ \textit{continuous and nondecreasing on} \ [0,\infty) \end{cases} \qquad (3.4)$$

and

$$\exists\ r > 0 \ \textit{with} \ \frac{r}{w(r) \max_{k \in I_{N+2m}} \sum_{l=0}^{N} (-1)^m G_m^1(k,l)\, q(l)} > 1. \qquad (3.5)$$

Then (1.6) *has a solution* $y_1 \in C(I_{N+2m})$ *with* $y_1 \ge 0$ *on* I_{N+2m} *and* $|y_1|_0 < r$.

Proof. The idea is to use Theorem 3.1, so look at

$$\begin{cases} (-1)^m \triangle^{2m} y(k) = \lambda f^\star(k, y(k)) \quad \text{for} \quad k \in I_N, \\ \triangle^{2i} y(0) = \triangle^{2i} y(N + 2m - 2i) = 0, \quad 0 \le i \le m-1 \end{cases} \tag{3.6$_\lambda$}$$

for $0 < \lambda < 1$; here

$$f^\star(k, u) = \begin{cases} f(k, u), & u \ge 0, \\ f(k, 0), & u < 0. \end{cases}$$

Let y be any solution of $(3.6)_\lambda$. Then

$$y(k) = \lambda \sum_{l=0}^{N} (-1)^m G_m^1(k, l) f^\star(l, y(l)),$$

so $y(k) \ge 0$ for $k \in I_{N+2m}$ and

$$|y(k)| \le w(|y|_0) \max_{k \in I_{N+2m}} \sum_{l=0}^{N} (-1)^m G_m^1(k, l) q(l) \quad \text{for} \quad k \in I_{N+2m}.$$

Consequently

$$\frac{|y|_0}{w(|y|_0) \max_{k \in I_{N+2m}} \sum_{l=0}^{N} (-1)^m G_m^1(k, l) q(l)} \le 1. \tag{3.7}$$

Now (3.5) and (3.7) imply $|y|_0 \ne r$. Thus Theorem 3.1 guarantees that $(3.6)_1$ has a solution $y_1 \in C(I_{N+2m})$ with $|y_1|_0 < r$ (note that $|y_1|_0 \ne r$ by an argument similar to the one above). $\square$

Note that in some application $|y_1|_0$ may be zero in Theorem 3.2. We remove this situation in the next result.

Theorem 3.3. *Suppose* (3.3)–(3.5) *are satisfied. In addition assume that the following conditions hold:*

$$\begin{cases} \textit{there exists} \ \ \tau : K_N = \{1, 2, \ldots, N\} \to (0, \infty) \\ \textit{with} \ \ f(i, u) \ge \tau(i)\, w(u) \ \ \textit{on} \ \ K_N \times (0, \infty) \end{cases} \tag{3.8}$$

and

$$\exists\, R > r \quad \textit{with} \quad \frac{R}{w(\frac{b_m}{a_m} c_0 R)} \le \sum_{l=1}^{N} (-1)^m G_m^1(\sigma, l)\, \tau(l); \tag{3.9}$$

here $\sigma \in J_N = \{1, \ldots, N + 2m - 1\}$ *is such that*

$$\sum_{l=1}^{N} (-1)^m G_m^1(\sigma, l)\, \tau(l) = \max_{k \in J_N} \sum_{l=1}^{N} (-1)^m G_m^1(k, l)\, \tau(l), \tag{3.10}$$

and

$$c_0 = \min_{l \in I_N} \left[\frac{\min\{l+1\,,\; N+1-l\}}{(l+1)\,(N+1-l)} \right] \;>\; 0 \tag{3.11}$$

with a_m *as in* (1.9) *and* b_m *as in* (1.11). *Then* (1.6) *has a solution* $y_2 \in C(I_{N+2\,m})$ *with* $y_2(k) > 0$ *for* $k \in J_N$ *and* $r < |y_2|_0 \le R$.

Proof. To show the existence of y_2, we use Theorem 1.1. Let $E = (C(I_{N+2m}), |\,.\,|_0)$ and

$$K = \Big\{ u \in C(I_{N+2\,m}) : \; u(i) \ge 0 \;\text{ for }\; i \in I_{N+2\,m} \;\text{ and }\; \min_{k \in J_N} u(k) \ge \frac{b_m}{a_m}\, c_0\, |u|_0 \Big\}.$$

Let $A : K \to C(I_{N+2\,m})$ be defined by

$$A\,u(k) = \sum_{l=0}^{N} (-1)^m\, G_m^1(k,l)\, f(l, u(l)).$$

To show $A : K \to K$, let $u \in K$. Then $A\,u(k) \ge 0$ for $k \in I_{N+2\,m}$. Also (1.8) implies for $k \in I_{N+2\,m}$ that

$$A\,u(k) \le a_m \sum_{l=0}^{N} (l+1)\,(N+1-l)\, f(l, u(l))$$

and so

$$|A\,u|_0 \le a_m \sum_{l=0}^{N} (l+1)\,(N+1-l)\, f(l, u(l)). \tag{3.12}$$

In addition (1.10) and (3.12) imply

$$\begin{aligned}
\min_{k \in J_N} A\,u(k) &= \min_{k \in J_N} \sum_{l=0}^{N} (-1)^m\, G_m^1(k,l)\, f(l, u(l)) \ge \\
&\ge b_m \sum_{l=0}^{N} \min\{l+1\,,\; N+1-l\}\, f(l, u(l)) \ge \\
&\ge b_m\, c_0 \sum_{l=0}^{N} (l+1)\,(N+1-l)\, f(l, u(l)) \ge \frac{b_m}{a_m}\, c_0\, |A\,u|_0.
\end{aligned}$$

Consequently $A\,u \in K$ so $A : K \to K$. Let

$\Omega_1 = \{u \in C(I_{N+2\,m}) : \; |u|_0 < r\}$ and $\Omega_2 = \{u \in C(I_{N+2\,m}) : \; |u|_0 < R\}$.

We first show

$$|A\,u|_0 \le |u|_0 \quad \text{for} \quad u \in K \cap \partial\Omega_1. \tag{3.13}$$

Let $u \in K \cap \partial\Omega_1$, so $|u|_0 = r$. Now (3.4) and (3.5) imply for $k \in I_{N+2\,m}$ that

$$A\,u(i) \le \sum_{l=0}^{N} (-1)^m\, G_m^1(k,l)\, q(l)\, w(u(l)) \le$$
$$\le w(r) \sup_{k \in I_{N+2\,m}} \sum_{l=0}^{N} (-1)^m\, G_m^1(k,l)\, q(l)\; < r\; = |u|_0.$$

Thus $|A\,u|_0 < r = |u|_0$ and so (3.13) is true. Next we show

$$|A\,u|_0 \ge |u|_0 \quad \text{for} \quad u \in K \cap \partial\Omega_2. \tag{3.14}$$

Let $u \in K \cap \partial\Omega_2$, so $|u|_0 = R$, and $\min_{k \in J_N} u(k) \ge \frac{b_m}{a_m}\, c_0\, R$, in particular,

$$u(k) \in \Big[\frac{b_m}{a_m}\, c_0\, R\,,\, R\Big] \quad \text{for} \quad k \in J_N.$$

It is easy to see that $0 < \frac{b_m}{a_m}\, c_0\; < 1$. Now (3.8) and (3.9) (here σ is as in (3.10)) imply

$$A\,u(\sigma) = \sum_{l=0}^{N} (-1)^m\, G_m^1(\sigma,l)\, f(l,u(l)) \ge \sum_{l=1}^{N} (-1)^m\, G_m^1(\sigma,l)\, f(l,u(l)) \ge$$
$$\ge \sum_{l=1}^{N} (-1)^m\, G_m^1(\sigma,l)\, \tau(l)\, w(u(l)) \ge$$
$$\ge w\Big(\frac{b_m}{a_m}\, c_0\, R\Big) \sum_{l=1}^{N} (-1)^m\, G_m^1(\sigma,l)\, \tau(l) \ge R = |u|_0.$$

Thus $|A\,u|_0 \ge |u|_0$ and so (3.14) is true. Now Theorem 1.1 guarantees that A has a fixed point $y_2 \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$, i.e., $r \le |y_2|_0 \le R$. In fact $|y_2|_0 > r$ (argue as in the first part of the theorem). Also $y_2 \ge 0$ on I_{N+2m} and $y_2(k) > 0$ for $k \in J_N$ since $y_2 \in K$ and $|y_2|_0 > r$. $\square$

Remark 3.2. If in (3.9) we have $R < r$, then (1.6) has a solution $y_2 \in C(I_{N+2\,m})$ with $R \le |y_2|_0 < r$.

Theorem 3.4. *Suppose* (3.3)–(3.5), (3.8) *and* (3.9) *hold. Then* (1.6) *has two solutions* $y_1,\, y_2 \in C(I_{N+2\,m})$ *with* $y_1 \ge 0$ *on* $I_{N+2\,m}$, $y_2(k) > 0$ *for* $k \in J_N$ *and* $0 \le |y_1|_0 < r < |y_2|_0 \le R$.

Proof. The existence of y_1 follows from Theorem 3.2 and of y_2 from Theorem 3.3. $\square$

In Theorem 3.4 it is possible for $|y_1|_0$ to be zero.

Theorem 3.5. *Suppose* (3.3)–(3.5), (3.8) *and* (3.9) *hold. In addition assume that*

$$\exists\, L,\ 0 < L < r \quad with \quad \frac{L}{w(\frac{b_m}{a_m}\, c_0\, L)} \leq \sum_{l=1}^{N} (-1)^m\, G_m^1(\sigma, l)\, \tau(l) \quad (3.15)$$

is satisfied; here σ *is as in* (3.10), c_0 *is as in* (3.11), a_m *is as in* (1.9), *and* b_m *is as in* (1.11). *Then* (1.6) *has two solutions* $y_1,\, y_2 \in C(I_{N+2m})$ *with* $y_1(k) > 0,\ y_2(k) > 0$ *for* $k \in J_N$ *and* $0 < L \leq |y_1|_0 < r < |y_2|_0 \leq R$.

Proof. The existence of y_2 follows from Theorem 3.3 and of y_1 from Remark 3.2. □

References

1. R. P. Agarwal, Difference equations and inequalities. *Marcel Dekker, New York,* 1992.

2. R. P. Agarwal and D. O'Regan, A fixed point approach for nonlinear discrete boundary value problems. *Comput. Math. Appl.* **36(10-12)**(1998), 115–121.

3. R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, Positive solutions of Differential, Difference and Integral equations. *Kluwer Acad. Publ., Dordrecht,* 1999.

4. R. P. Agarwal and P. J. Y. Wong, On Lidstone polynomials and boundary value problems. *Comput. Math. Appl.* **17**(1989), 1397–1421.

5. R. P. Agarwal and P. J. Y. Wong, Error inequalities in polynomial interpolation and their applications. *Kluwer Acad. Publ., Dordrecht,* 1993.

6. A. Boutayeb and E. H. Twizell, Finite difference methods for twelfth order boundary value problems. *J. Comput. Appl. Math.* **35**(1991), 133–138.

7. M. M. Chawla and C. P. Katti, Finite difference methods for two point boundary value problems involving higher order differential equations. *BIT* **19**(1979), 27–33.

8. J. Davis and J. Henderson, Uniqueness implies existence for fourth order Lidstone boundary value problems. *Panamer. Math. J.* **8**(1998), (in press).

9. L. H. Erbe and H. Wang, On the existence of positive solutions of ordinary differential equations. *Proc. Amer. Math. Soc.* **120**(1994), 743–748.

10. T. H. Lamar, Existence of positive solutions in a cone for a class of $2n$-th order nonlinear boundary value problems with Lidstone boundary conditions. *Appl. Anal.* (to appear).

11. D. O'Regan, Existence principles and theory for singular Dirichlet boundary value problems. *Differential Equations Dynam. Systems* **3**(1995), 289–304.

12. D. O'Regan, Existence theory for nonlinear ordinary differential equations. *Kluwer Acad. Publ., Dordrecht,* 1997.

13. P. J. Y. Wong and R. P. Agarwal, Eigenvalues of Lidstone boundary value problems. *Appl. Math. and Comput.* (to appear).

14. P. J. Y. Wong and R. P. Agarwal, Results and estimates on multiple solutions of Lidstone boundary value problems. *Acta Math. Hungar.* (to appear).

15. P. J. Y. Wong and R. P. Agarwal, Multiple solutions of difference and partial difference equations with Lidstone conditions. *Math. Comput. Modelling,* (to appear).

16. P. J. Y. Wong and R. P. Agarwal, Characterization of eigenvalues for difference equations subject to Lidstone conditions. (*Submitted*).

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