On Comparison Theorem for Neutral Stochastic Differential Equations of Reaction-Diffusion Type

A. Stanzhytskyi

Taras Shevchenko National University of Kiev, Kiev, Ukraine;
National Technical University of Ukraine, "Igor Sikorsky Kiev Polytechnic Institute", Kiev, Ukraine
E-mail: ostanzh@gmail.com

N. Marchuk

State Agrarian and Engineering University in Podilya, Kam'yanets-Podil'skyi, Ukraine
E-mail: nata.marchuk2205@gmail.com

A. Tsukanova

National Technical University of Ukraine, "Igor Sikorsky Kiev Polytechnic Institute", Kiev, Ukraine;
E-mail: shugaray@mail.ru

Abstract We prove a comparison theorem for the mild solutions (solutions) of Cauchy problems for two stochastic integro-differential equations of reaction-diffusion type with delay. According to our result, if $f_1 \geq f_2$, then $u_1 \geq u_2$ with probability one.

1 Introduction

We study the following Cauchy problems for two neutral partial stochastic integro-differential equations of reaction-diffusion type

$$d\bigg(u_i(t,x)+\int\limits_{\mathbb{R}^d} b_i(t,x,u_i(t-r,\xi),\xi)\,d\xi\bigg)=\big(\Delta_x u_i(t,x)+f_i(t,u_i(t-r,x),x)\big)\,dt$$
$$+\sigma(t,x)\,dW(t,x),\ \ 0<t\leq T,\ \ x\in\mathbb{R}^d,\ \ i\in\{1,2\}, \tag{1.1}$$
$$u_i(t,x)=\phi_i(t,x),\ \ -r\leq t\leq 0,\ \ x\in\mathbb{R}^d,\ \ r>0,\ \ i\in\{1,2\}, \tag{1.1*}$$

where $\Delta_x \equiv \sum\limits_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$, $i\in\{1,\dots,d\}$, W is $L_2(\mathbb{R}^d)$-valued Q-Wiener process, $f_i:[0,T]\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$, $i\in\{1,2\}$, $\sigma:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ and $b_i:[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$, $i\in\{1,2\}$, are some given functions to be specified later, and $\phi_i:[-r,0]\times\mathbb{R}^d\to\mathbb{R}$, $i\in\{1,2\}$, are initial-datum functions.

A problem of comparison of solutions to stochastic differential equations in finite-dimensional case has firstly arised in [8]. It has been proved in this work that under certain assumptions the solution is monotonously non-decreasing function from "drift"-coefficient f. Variations of this result have been proposed in the works [1–4, 6, 7]. In [5] the proof of comparison theorem for solutions to the Cauchy problem for stochastic differential equations with multidimensional Wiener processes in Hilbert space is given. Our aim was to prove the comparison theorem for solutions of problem

$(1.1), (1.1^*)$ using the idea from this work. This result plays an important role when studying the existence of solutions to the Cauchy problem for stochastic differential equations with non-Lipschitz conditions on "drift"-coefficients.

2 Formulation of the problem

Throughout the article $(\Omega, \mathcal{F}, \mathbf{P})$ will note a complete probability space. Let $\{e_n(x),\ n \in \{1,2,\dots\}\}$ be an orthonormal basis on $L_2(\mathbb{R}^d)$ such that $\sup\limits_{n\in\{1,2,\dots\}} \operatorname*{ess\,sup}\limits_{x\in\mathbb{R}^d} |e_n(x)| \le 1$. We define a Q-Wiener $L_2(\mathbb{R}^d)$-valued process

$$W(t,x) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n(x) \beta_n(t), \quad t \ge 0, \quad x \in \mathbb{R}^d,$$

where $\{\beta_n(t),\ n \in \{1,2,\dots\}\} \subset \mathbb{R}$ are independent standard one-dimensional Brownian motions on $t \ge 0$, $\{\lambda_n,\ n \in \{1,2,\dots\}\}$ is a sequence of positive numbers such that $\lambda = \sum\limits_{n=1}^{\infty} \lambda_n < \infty$. Let $\{\mathcal{F}_t,\ t \ge 0\}$ be a normal filtration on $\mathcal{F}$. We assume that $W(t,\,\cdot\,)$, $t \ge 0$, is $\mathcal{F}_t$-measurable and the increments $W(t+h,\,\cdot\,) - W(t,\,\cdot\,)$ are independent of $\mathcal{F}_t$ for all $h > 0$ and $t \ge 0$. Throughout the article $L_2(\mathbb{R}^d)$ will note real Hilbert space with the norm

$$\|f\|_{L_2(\mathbb{R}^d)} = \bigg(\int\limits_{\mathbb{R}^d} f^2(x)\,dx \bigg)^{\frac{1}{2}}.$$

Let the following seven assumptions be true.

(1) $f_i : [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $i \in \{1,2\}$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $b_i : [0,T] \times \mathbb{R}^d \times \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $i \in \{1,2\}$, are measurable with respect to all of their variables functions, b_i, $i \in \{1,2\}$, are continuous in the first argument;

(2) $\phi_i(t,x,\omega) : [-r,0] \times \mathbb{R}^d \times \Omega \to L_2(\mathbb{R}^d)$, $i \in \{1,2\}$, are $\mathcal{F}_0$-measurable random variables with almost surely continuous paths and such that

$$\sup_{-r \le t \le 0} \mathbf{E} \|\phi_i(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} < \infty, \quad i \in \{1,2\};$$

(3) b_i, $i \in \{1,2\}$, are uniformly continuous in the first argument and satisfy the Lipshitz condition in the third argument of the form

$$\big| b_i(t,x,u,\xi) - b_i(t,x,v,\xi) \big| \le l(t,x,\xi) |u-v|, \quad 0 \le t \le T, \quad \{x,\xi\} \subset \mathbb{R}^d, \quad \{u,v\} \subset \mathbb{R}, \quad i \in \{1,2\},$$

where $l : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is such that

$$\sup_{0 \le t \le T} \int\limits_{\mathbb{R}^d} \sqrt{\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)\,d\xi}\,dx < \infty, \quad \sup_{0 \le t \le T} \int\limits_{\mathbb{R}^d} \bigg(\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)\,d\xi \bigg)\,dx < \frac{1}{4}\,;$$

(4) there exists a function $\chi : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$, satisfying the conditions

$$\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \chi(x,\xi)\,d\xi\,dx < \infty, \quad \int\limits_{\mathbb{R}^d} \bigg(\int\limits_{\mathbb{R}^d} \chi(x,\xi)\,d\xi \bigg)^2 dx < \infty,$$

such that

$$\sup_{0 \le t \le T} |b_i(t,x,0,\xi)| \le \chi(x,\xi), \quad 0 \le t \le T, \quad x \in \mathbb{R}^d, \quad \xi \in \mathbb{R}^d, \quad i \in \{1,2\};$$

(5) there exists a function $\eta : [0,T] \times \mathbb{R}^d \to [0,\infty)$ with

$$\sup_{0\le t\le T} \int_{\mathbb{R}^d} \eta^2(t,\xi)\,d\xi < \infty,$$

such that the following linear-growth and Lipschitz conditions are valid

$$|f_i(t,u,x)| \le \eta(t,x) + L|u|, \ \ 0 \le t \le T, \ \ u \in \mathbb{R}, \ \ x \in \mathbb{R}^d \ \ i \in \{1,2\},$$
$$|f_i(t,u,x) - f_i(t,v,x)| \le L|u-v|, \ \ 0 \le t \le T, \ \ \{u,v\} \subset \mathbb{R}, \ \ x \in \mathbb{R}^d, \ \ i \in \{1,2\};$$

(6) the following condition is valid for σ

$$\sup_{0\le t\le T} \|\sigma(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} < \infty;$$

(7) for any $x \in \mathbb{R}^d$ there exist $\partial_x b$, $D_x^2 b$, and for $\nabla_x b$ and $D_x^2 b$ the following linear-growth condition with respect to the third argument is true

$$|\nabla_x b(t,x,u,\xi)| + \|D_x^2 b(t,x,u,\xi)\|_d \le \psi(t,x,\xi)\big(1+|u|\big), \ \ 0 \le t \le T, \ \ \{x,\xi\} \subset \mathbb{R}^d, \ \ u \in \mathbb{R},$$

for $D_x^2 b$ – the following Lipschitz condition

$$\big\|D_x^2 b(t,x,u,\xi) - D_x^2 b(t,x,v,\xi)\big\|_d \le \psi(t,x,\xi)|u-v|, \ \ 0 \le t \le T, \ \ \{x,\xi\} \subset \mathbb{R}^d, \ \ \{u,v\} \subset \mathbb{R},$$

where $\psi : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is such that

$$\sup_{0\le t\le T} \int_{\mathbb{R}^d} \bigg(\int_{\mathbb{R}^d} \psi(t,x,\xi)\,d\xi \bigg)^2 dx < \infty, \quad \sup_{0\le t\le T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi^2(t,x,\xi)\,d\xi\,dx < \infty,$$

and besides for any point $x_0 \in \mathbb{R}^d$ there is its neighborhood $B_\delta(x_0)$ and a nonnegative function φ such that

$$\sup_{0\le t\le T} \varphi(t,\,\cdot\,,x_0,\delta) \in L_2(\mathbb{R}^d) \cap L_1(\mathbb{R}^d), \ \ \delta > 0,$$
$$\big|\psi(t,x,\xi) - \psi(t,x_0,\xi)\big| \le \varphi(t,\xi,x_0,\delta)|x-x_0|, \ \ 0 \le t \le T, \ \ |x-x_0| < \delta, \ \ \xi \in \mathbb{R}^d.$$

Our main result is the following comparison theorem.

Theorem. *Suppose assumptions* (1)–(7) *are satisfied and*

(1) *the initial-datum functions ϕ_i, $i \in \{1,2\}$, satisfy the condition*

$$\phi_1(t,x) \ge \phi_2(t,x), \ \ 0 \le t \le T, \ \ x \in \mathbb{R}^d;$$

(2) *the functions b_i, $i \in \{1,2\}$, satisfy the conditions*

$$b_1(0,x,\phi_2(-r,\xi),\xi) = b_2(0,x,\phi_2(-r,\xi),\xi), \ \ \{x,\xi\} \subset \mathbb{R}^d,$$
$$b_1(0,x,\phi_1(-r,\xi),\xi) = b_2(0,x,\phi_1(-r,\xi),\xi), \ \ \{x,\xi\} \subset \mathbb{R}^d,$$
$$b_1(0,x,\phi_1(-r,\xi),\xi) = b_1(0,x,\phi_2(-r,\xi),\xi), \ \ \{x,\xi\} \subset \mathbb{R}^d,$$
$$b_1(t,x,u,\xi) \le b_2(t,x,u,\xi), \ \ 0 \le t \le T, \ \ \{x,\xi\} \subset \mathbb{R}^d, \ \ u \in \mathbb{R};$$

(3) *for the "drift"-functions* $f_1(t,u,x) \geq f_2(t,u,x)$, $0 \leq t \leq T$, $u \in \mathbb{R}$, $x \in \mathbb{R}^d$.

Let one of the following conditions be true:

(M1) b_1 *is monotonously non-increasing,* f_1 *is monotonously non-decreasing with respect to* u*;*

(M2) b_2 *is monotonously non-increasing,* f_2 *is monotonously non-decreasing with respect to* u*.*

Then with probability one the solutions of (1.1), (1.1*) *satisfy*

$$u_1(t,x) \geq u_2(t,x) \;\; x \in \mathbb{R}^d, \;\; 0 \leq t \leq T.$$

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