Solutions of Two-Term Fractional Differential Equations on the Half-Line Via Lower and Upper Solutions Method

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We are interested in the fractional initial value problem

$$ {}^c\!D^{\alpha}u(t) = q(t,u(t))\,{}^c\!D^{\beta}u(t) + f(t,u(t)), \tag{1} $$

$$ u(0) = a, \tag{2} $$

where $0 < \beta < \alpha \le 1$, $q, f \in C(\mathbb{R}_0 \times \mathbb{R})$, $\mathbb{R}_0 = [0,\infty)$, $a \in \mathbb{R}$ and ${}^c\!D$ denotes the Caputo fractional derivative.

We recall that the Caputo fractional derivative ${}^c\!D^{\gamma}x$ of order $\gamma > 0$, $\gamma \notin \mathbb{N}$, of a function $x\colon \mathbb{R}_0 \to \mathbb{R}$ is given as [1,2]

$$ {}^c\!D^{\gamma}x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int\limits_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)} \Big(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!}\, s^k \Big)\, \mathrm{d}s, $$

where Γ is the Euler gamma function, $n = [\gamma] + 1$, $[\gamma]$ means the integral part of the fractional number γ. If $\gamma \in \mathbb{N}$, then ${}^c\!D^{\gamma}x(t) = x^{(\gamma)}(t)$.

In particular,

$$ {}^c\!D^{\gamma}x(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\, (x(s) - x(0))\, \mathrm{d}s, \quad \gamma \in (0,1). $$

If $\alpha = 1$, then (1) has the form

$$ u'(t) = q(t,u(t))\,{}^c\!D^{\beta}u(t) + f(t,u(t)). $$

This equation is called the generalized Basset fractional differential equation [1–4].

Definition 1. We say that $u : \mathbb{R}_0 \to \mathbb{R}$ is *a solution of equation* (1) if $u, {}^c\!D^{\alpha}u \in C(\mathbb{R}_0)$ and (1) holds for $t \in \mathbb{R}_0$. A solution u of (1) satisfying the initial condition (2) is called *a solution of problem* (1), (2).

The existence, uniqueness and the structure of solutions to problem (1), (2) is proved by the lower and upper solutions method combined with the extension method, the Schauder fixed point theorem and the maximum principle for the Caputo fractional derivative [5].

Definition 2. A function $\sigma : \mathbb{R}_0 \to \mathbb{R}$ is called *a lower solution of* (1) if $\sigma, {}^c\!D^{\alpha}\sigma \in C(\mathbb{R}_0)$ and

$$ {}^c\!D^{\alpha}\sigma(t) \le q(t,\sigma(t))\,{}^c\!D^{\beta}\sigma(t) + f(t,\sigma(t)) \ \text{ for } \ t \in \mathbb{R}_0. $$

If the inequality is reversed, then σ is called *an upper solution of* (1).

Theorem 1. *Let*

(H_1) *$q(t,x) \le 0$ for $(t,x) \in \mathbb{R}_0 \times \mathbb{R}$ if $\alpha < 1$ and $q(t,x) < 0$ for $(t,x) \in \mathbb{R}_0 \times \mathbb{R}$ if $\alpha = 1$;*

(H_2) *there are a lower solution φ and an upper solution ρ of (1) such that*

$$\varphi(t) < \rho(t) \ \text{ for } \ t \in \mathbb{R}_0.$$

Then for $a \in (\varphi(0), \rho(0))$ there exists at least one solution of problem $(1),(2)$ and its solutions u satisfy

$$\varphi(t) < u(t) < \rho(t) \ \text{ for } \ t \in \mathbb{R}_0.$$

Example 1. Let $\mu > 0$, $q \in C(\mathbb{R}_0 \times \mathbb{R})$ satisfy (H_1) and $r \in C(\mathbb{R}_0)$, $0 \le r(t) \le (1+t)^\mu$ for $t \in \mathbb{R}_0$. Then $\varphi(t) = 0$ and $\rho(t) = 1+t$ for $t \in \mathbb{R}_0$ are lower and upper solutions of the equation

$$^{c}\!D^{\alpha} u(t) = q(t,u(t))\,^{c}\!D^{\beta} u(t) + r(t) - |u(t)|^{\mu}. \tag{3}$$

Theorem 1 ensures that for $a \in (0,1)$ solutions u of problem $(3),(2)$ satisfy $0 < u(t) < 1+t$ for $t \in \mathbb{R}_0$.

Corollary 1. *Let (H_1) hold and let there exist $A, B \in \mathbb{R}$, $A < B$, such that*

$$f(t,A) \ge 0, \ \ f(t,B) \le 0 \ \text{ for } \ t \in \mathbb{R}_0.$$

Then for $a \in (A,B)$ there exists at least one solution of problem $(1),(2)$ and its solutions u satisfies $A < u(t) < B$ for $t \in \mathbb{R}_0$.

Corollary 2. *Let (H_1) and (H_2) hold and let $\lim\limits_{t\to\infty} \varphi(t) = 0$, $\lim\limits_{t\to\infty} \rho(t) = 0$. Then for $a \in (\varphi(0), \rho(0))$ there exists at least one solution of problem $(1),(2)$ and its solutions u satisfy $\lim\limits_{t\to\infty} u(t) = 0$ and $\varphi(t) < u(t) < \rho(t)$ for $t \in \mathbb{R}_0$.*

Example 2. Let $r, w \in C(\mathbb{R}_0)$ and $0 < r(t) \le M$, $|w(t)| < 2+M$ for $t \in \mathbb{R}_0$, where M is a positive constant. Let $\mu > 0$ and $S \ge 4(2+M)$. Then $\varphi(t) = -e^{-t}$ and $\rho(t) = e^{t}$ are lower and upper solutions of the equation

$$^{c}\!D^{\alpha} u(t) = -\big(r(t) + |u(t)|^{\mu}\big)\,^{c}\!D^{\beta} u(t) + w(t) - Se^{t} u(t). \tag{4}$$

By Corollary 2, for $a \in (-1,1)$ solutions u of problem $(4),(2)$ satisfy $\lim\limits_{t\to\infty} u(t) = 0$ and $-e^{-t} < u(t) < e^{-t}$ for $t \in \mathbb{R}_0$.

Now, we consider the equation

$$^{c}\!D^{\alpha} u(t) = b(t)\,^{c}\!D^{\beta} u(t) + f(t,u(t)), \tag{5}$$

that is the special case of (1). The separation property of solutions to equation (5) is given in the following theorem.

Theorem 2. *Let (H_2) and*

(H_1^*) *$b \in C(\mathbb{R}_0)$, $b \le 0$ on $\mathbb{R}_0$ if $\alpha < 1$ and $b < 0$ on $\mathbb{R}_0$ if $\alpha = 1$*

hold. Let u, v be solutions of equation (5) such that $\varphi(0) < u(0) < v(0) < \rho(0)$. Then

$$\varphi(t) < u(t) < v(t) < \rho(t) \ \text{ for } \ t \in \mathbb{R}_0.$$

The following result gives the existence of a unique solution of problem $(5),(2)$.

Theorem 3. *Let (H_1^*) and (H_2) hold. Let for each $t \in \mathbb{R}_0$ the function $f(t,x)$ is decreasing in the variable x on the interval $[\varphi(t), \rho(t)]$. Then for $a \in (\varphi(0), \rho(0))$ there exists a unique solution u of problem $(5), (2)$ and $\varphi(t) < u(t) < \rho(t)$ for $\mathbb{R}_0$.*

Under the conditions of Theorem 3, for each $a \in (\varphi(0), \rho(0))$ there exists a unique solution of problem $(5), (2)$. We denote u_a this solution. By Theorem 2,

$$\varphi(t) < u_{a_1}(t) < u_{a_2}(t) < \rho(t) \ \text{ for } \ t \in \mathbb{R}_0, \ \ \varphi(0) < a_1 < a_2 < \rho(0).$$

For $a \in \{\varphi(0), \rho(0)\}$, we have the following result.

Lemma 1. *Let the conditions of Theorem 3 be satisfied. Then for $a \in \{\varphi(0), \rho(0)\}$ there exists a unique solution u_a of problem $(5), (2)$ satisfying $\varphi(t) \le u_a(t) \le \rho(t)$ on $\mathbb{R}_0$.*

We denote by $u_{\varphi(0)}$ and $u_{\rho(0}$ the unique solution of $(5), (2)$ for $a = \varphi(0)$ and $a = \rho(0)$, respectively.

The following result says that the set $\mathcal{Z} = \{(t,x) \in \mathbb{R}^2 \colon t \in \mathbb{R}_0, \ u_{\varphi(0)}(t) \le x \le u_{\rho(0)}(t)\}$ is covered by graphs of solutions u of equation (5) with $u(0) \in [\varphi(0), \rho(0)]$.

Theorem 4. *Let the conditions of Theorem 3 hold. Then for $(T, x_0) \in \mathbb{R}_0 \times [u_{\varphi(0)}(T), u_{\rho(0)}(T)]$ there exists a unique solution u of equation (5) such that $u(T) = x_0$ and $\varphi(t) \le u(t) \le \rho(t)$ on $\mathbb{R}_0$.*

Example 3. Let $\mu > 0$, $w \in C(\mathbb{R}_0)$, $0 \le w \le M$ on $\mathbb{R}_0$, where M is a positive constant, and let b satisfy (H_1^*). Then the constant functions $\varphi = 0$ and $\rho = \sqrt[\mu]{M}$ are lower and upper solutions of the fractional differential equation

$$^{c}D^{\alpha} u(t) = b(t) {}^{c}D^{\beta} u(t) + w(t) - |u(t)|^{\mu}. \tag{6}$$

By Theorem 3 and Lemma 1, for $a \in [0, \sqrt[\mu]{M}]$ there exists a unique solution of problem $(6), (2)$. Theorem 4 guarantees that for $T \in \mathbb{R}_0$ and $x_0 \in [0, \sqrt[\mu]{M}]$ there exists a unique solution u of (6) satisfying $u(T) = x_0$ and $0 \le u(t) \le \sqrt[\mu]{M}$ on $\mathbb{R}_0$.

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