The Analogue of the Floquet's–Lyapunov's Theorem for Linear Differential Systems with Slowly Varying Parameters

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Let

$$G(\varepsilon_0) = \big\{t, \varepsilon :\ 0 < \varepsilon < \varepsilon_0,\ -L\varepsilon^{-1} \le t \le L\varepsilon^{-1},\ 0 < L < +\infty\big\}.$$

Definition 1. We say that a function $p(t, \varepsilon)$ belongs to the class $S(m; \varepsilon_0)$ ($m \in \mathbf{N} \cup \{0\}$) if

- $p : G(\varepsilon_0) \to \mathbf{C}$;
- $p(t, \varepsilon) \in C^m(G(\varepsilon_0))$ with respect to t;
- $d^k p(t, \varepsilon)/dt^k = \varepsilon^k p_k^*(t, \varepsilon)$ $(0 \le k \le m)$,

$$\|p\|_{S(m;\varepsilon_0)} \overset{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |p_k^*(t, \varepsilon)| < +\infty.$$

Under a slowly varying function we mean a function of class $S(m; \varepsilon_0)$.

Definition 2. We say that a function $f(t, \varepsilon, \theta(t, \varepsilon))$ belongs to the class $F(m; \varepsilon_0; \theta)$ ($m \in \mathbf{N} \cup \{0\}$) if this function can be represented as

$$f(t, \varepsilon, \theta(t, \varepsilon)) = \sum_{n=-\infty}^{\infty} f_n(t, \varepsilon) \exp(in\,\theta(t, \varepsilon)),$$

and

- $f_n(t, \varepsilon) \in S(m; \varepsilon_0)$;
- $$\|f\|_{F(m;\varepsilon_0;\theta)} \overset{def}{=} \sum_{n=-\infty}^{\infty} \|f_n\|_{S(m;\varepsilon_0)} < +\infty,$$
- $\theta(t, \varepsilon) = \int\limits_0^t \varphi(\tau, \varepsilon)\, d\tau$, $\varphi(t, \varepsilon) \in \mathbf{R}^+$, $\varphi(t, \varepsilon) \in S(m; \varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)} \varphi(t, \varepsilon) = \varphi_0 > 0$.

We consider the next system of differential equations

$$\frac{dx_j}{dt} = \lambda_j(t, \varepsilon) x_j + \mu \sum_{k=1}^{N} p_{jk}(t, \varepsilon, \theta) x_k, \quad j = \overline{1, N}, \tag{1}$$

where $\lambda_j(t, \varepsilon) \in S(m; \varepsilon_0)$, $p_{jk}(t, \varepsilon, \theta) \in F(m; \varepsilon_0; \theta)$ $(j, k = \overline{1, N})$, $\mu \in (0, \mu_0) \subset \mathbf{R}^+$.

We study the problem about the structure of fundamental system of solutions $x_{jk}(t, \varepsilon, \mu)$ $(j, k = \overline{1, N})$ of system (1).

Lemma 1. *Let the function*

$$f(t,\varepsilon,\theta(t,\varepsilon))=\sum_{\substack{n=-\infty\\(n\neq0)}}^{\infty}f_n(t,\varepsilon)\exp(in\,\theta(t,\varepsilon))$$

belong to the class $F(m-1;\varepsilon_0;\theta)$*. Then the function*

$$x(t,\varepsilon,\theta(t,\varepsilon))=\varepsilon\int\limits_0^t f(\tau,\varepsilon,\theta(\tau,\varepsilon))\,d\tau$$

belongs to the class $F(m-1;\varepsilon_0;\theta)$ *also, and there exists* $K_1\in(0,+\infty)$*, that does not depend on the function* f *such that*

$$\|x(t,\varepsilon,\theta)\|_{F(m-1;\varepsilon_0;\theta)}\leq K_1\|f(t,\varepsilon,\theta)\|_{F(m-1;\varepsilon_0;\theta)}.$$

Lemma 2. *Let we have the linear nonhomogeneous first-order differential equation:*

$$\frac{dx}{dt}=\lambda(t,\varepsilon)x+\varepsilon u(t,\varepsilon,\theta(t,\varepsilon)),\tag{2}$$

where $\lambda(t,\varepsilon)\in S(m;\varepsilon_0)$*,* $u(t,\varepsilon,\theta)\in F(m-1;\varepsilon_0;\theta)$*. Let the condition* $|\operatorname{Re}\lambda(t,\varepsilon)|\geq\gamma_0>0$ *hold. Then equation* (2) *has a particularly solution* $x(t,\varepsilon,\theta(t,\varepsilon))\in F(m-1;\varepsilon_0;\theta)$*, and there exists* $K_2\in(0,+\infty)$*, that does not depend on the function* $u(t,\varepsilon,\theta)$ *such that*

$$\|x(t,\varepsilon,\theta)\|_{F(m-1;\varepsilon_0;\theta)}\leq\frac{K_2}{\gamma_0}\,\|u(t,\varepsilon,\theta)\|_{F(m-1;\varepsilon_0;\theta)}.\tag{3}$$

Lemma 3. *Let the system* (1) *be such that*

$$\big|\operatorname{Re}(\lambda_j(t,\varepsilon)-\lambda_k(t,\varepsilon))\big|\geq\gamma_1>0\ \ (j\neq k).\tag{4}$$

Then there exists $\mu_1\in(0,\mu_0)$ *such that for all* $\mu\in(0,\mu_1)$ *there exists the Lyapunov's transformation of kind*

$$x_j=y_j+\mu\sum_{k=1}^N\psi_{jk}(t,\varepsilon,\theta,\mu)y_k,\ \ j=\overline{1,N},\tag{5}$$

where $\psi_{jk}\in F(m-1;\varepsilon_0;\theta)$*, reducing the system* (1) *to*

$$\frac{dy_j}{dt}=\big(\lambda_j(t,\varepsilon)+\mu u_j(t,\varepsilon,\mu)\big)y_j+\mu\varepsilon\sum_{k=1}^N v_{jk}(t,\varepsilon,\theta,\mu)y_k,\ \ j=\overline{1,N},\tag{6}$$

where $u_j\in S(m;\varepsilon_0)$*,* $v_{jk}\in F(m-1;\varepsilon_0;\theta)$ $(j,k=\overline{1,N})$*.*

Lemma 4. *Let the condition* (4) *hold. Then there exists* $\mu_2\in(0,\mu_1)$ *(*μ_i *are defined in Lemma* 3*) such that for all* $\mu\in(0,\mu_2)$ *there exists the Lyapunov's transformation of kind*

$$y_j=z_j+\mu\sum_{k=1}^N q_{jk}(t,\varepsilon,\theta,\mu)z_k,\ \ j=\overline{1,N},\tag{7}$$

where $q_{jk}\in F(m-1;\varepsilon_0;\theta)$*, reducing the system* (6) *to the pure diagonal form*

$$\frac{dz_j}{dt}=d_j(t,\varepsilon,\theta,\mu)z_j,\ \ j=\overline{1,N},\tag{8}$$

where

$$d_j = \lambda_j(t,\varepsilon) + \mu u_j(t,\varepsilon,\mu) + \mu\varepsilon v_{jj}(t,\varepsilon,\theta,\mu) + \mu\varepsilon \sum_{\substack{k=1 \\ (k\neq j)}}^{N} v_{jk}(t,\varepsilon,\theta,\mu) q_{kj}(t,\varepsilon,\theta,\mu), \ \ j=\overline{1,N}. \tag{9}$$

Theorem. *Let for the system* (1) *the condition* (4) *holds. Then there exists* $\mu_3 \in (0,\mu_0)$ *such that for all* $\mu \in (0,\mu_3)$ *the system* (3) *has a fundamental system of solutions of kind:*

$$x_{jk} = r_{jk}(t,\varepsilon,\theta,\mu) \exp\Big(\int_0^t \sigma_j(s,\varepsilon,\mu)\,ds\Big), \ \ j,k=\overline{1,N} \tag{10}$$

j – the number of solution, k – the number of component, where $r_{jk}(t,\varepsilon,\theta,\mu) \in F(m-1;\varepsilon_0;\theta)$, $\sigma_j(t,\varepsilon,\mu) \in S(m-1;\varepsilon_0)$.

Proof. The fundamental system of solutions (FSS) of the system (8) has a kind:

$$z_{jk} = \delta_j^k \exp\Big(\int_0^t d_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds\Big), \ \ j,k=\overline{1,N}$$

j – the number of solution, k – the number of component, δ_j^k – the symbol of Kronecker. By virtue (7) FSS of system (6) has a kind:

$$y_{jk} = \widetilde{q}_{jk}(t,\varepsilon,\theta,\mu) \exp\Big(\int_0^t d_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds\Big), \ \ k=\overline{1,N},$$

where $\widetilde{q}_{jk} = \delta_j^k + (1-\delta_j^k)\mu q_{jk}$ (j – the number of solution, k – the number of component). By virtue (5) FSS of system (1) has a kind:

$$x_{jk} = \Big(\sum_{l=1}^{N} \widetilde{\psi}_{kl}(t,\varepsilon,\theta,\mu)\widetilde{q}_{lj}(t,\varepsilon,\theta,\mu)\Big) \exp\Big(\int_0^t d_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds\Big), \ \ j,k=\overline{1,N}, \tag{11}$$

where $\widetilde{\psi}_{jk} = \delta_j^k + \mu\psi_{jk}$($\psi_{jk}$ are defined in Lemma 3).

Consider

$$\int_0^t d_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds = \int_0^t \big(\lambda_j(s,\varepsilon) + \mu_j(s,\varepsilon,\mu)\big)\,ds + \mu\varepsilon \int_0^t w_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds,$$

where $w_j = v_{jj} + \sum\limits_{k=1}^{N} \psi_{jk} q_{kj} \in F(m-1;\varepsilon_0;\theta)$. We represent the functions w_j as $w_j = w_j^*(t,\varepsilon,\mu) + \widetilde{w}_j(t,\varepsilon,\theta,\mu)$, where

$$w_j^*(t,\varepsilon,\mu) = \overline{w_j(t,\varepsilon,\theta,\mu)} = \frac{1}{2\pi}\int_0^{2\pi} w_j(t,\varepsilon,\theta,\mu)\,d\theta \in S(m-1;\varepsilon_0).$$

Accordingly, $\widetilde{w}_j \in F(m-1;\varepsilon_0;\theta)$, and $\overline{\widetilde{w}_j(t,\varepsilon,\theta,\mu)} \equiv 0$. Then

$$\begin{aligned}\exp\bigg(\int\limits_0^t d_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds\bigg)&\\ =\exp\bigg(\int\limits_0^t \big(\lambda_j(s,\varepsilon)+\mu u_j(s,\varepsilon,\mu)+\mu\varepsilon w_j^*(s,\varepsilon,\mu)\big)\bigg)&\exp\bigg(\mu\varepsilon\int\limits_0^t \widetilde{w}_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds\bigg). \qquad (12)\end{aligned}$$

By virtue Lemma 1, we conclude

$$\varepsilon\int\limits_0^t \widetilde{w}_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds \in F(m-1;\varepsilon_0;\theta) \ \ (j=\overline{1,N}).$$

It follows by virtue of the properties of functions from class $F(m;\varepsilon_0;\theta)$ that

$$g_j(t,\varepsilon,\theta,\mu)=\exp\bigg(\mu\varepsilon\int\limits_0^t \widetilde{w}_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds\bigg) \in F(m-1;\varepsilon_0;\theta) \ \ (j=\overline{1,N}). \qquad (13)$$

By virtue of (11)–(13) we obtain the statement of the theorem. □

Obviously, the formula (10) is an analogue of Floquet's–Lyapunov's theorem for the systems of kind (1).