The Vinograd–Millionshchikov Central Exponents and their Simplified Variants

I. N. Sergeev

Lomonosov Moscow State University, Moscow, Russia
E-mail: igniserg@gmail.com

1 The original central exponents

For the Euclidean space $\mathbb{R}^n$, $n > 1$, we denote by $\mathcal{M}^n$ the set of bounded and piecewise continuous operator-functions $A : \mathbb{R}^+ \to \operatorname{End}\mathbb{R}^n$ generating systems of the form

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}^+ \equiv [0, \infty).$$

The *Lyapunov* and *Perron exponents* play an important role in the investigation of solutions of differential equations and systems for Lyapunov and Poisson stability. From this point of view, the so-called *central exponents* are no less useful, being responsible for the stability of all slightly perturbed systems with any sufficiently close linearization (see, e.g., the monograph [1]).

Definition 1.1. The *Vinograd–Millionshchikov central* exponents of a system $A \in \mathcal{M}^n$ are given by the formulas: two *upper* (*up-limit* with a hat, and *low-limit* with a tick) ones

$$\widehat{\Omega}(A) \equiv \inf_{T>0} \varlimsup_{m\to\infty} \frac{1}{Tm} \sum_{i=1}^{m} \ln \big\| X_A(Ti, T(i-1)) \big\|,$$
$$\check{\Omega}(A) \equiv \inf_{T>0} \varliminf_{m\to\infty} \frac{1}{Tm} \sum_{i=1}^{m} \ln \big\| X_A(Ti, T(i-1)) \big\|,$$

and, respectively, two *lower* ones

$$\widehat{\omega}(A) \equiv \sup_{T>0} \varlimsup_{m\to\infty} \frac{1}{Tm} \sum_{i=1}^{m} \ln \big| X_A(Ti, T(i-1)) \big|,$$
$$\check{\omega}(A) \equiv \sup_{T>0} \varliminf_{m\to\infty} \frac{1}{Tm} \sum_{i=1}^{m} \ln \big| X_A(Ti, T(i-1)) \big|,$$

where $\|X\| \equiv \sup\limits_{|x|=1} |Xx|$, $|X| \equiv \|X^{-1}\|^{-1}$, and X_A is the Cauchy operator of the system A.

The first and the last of these four exponents are introduced by R. E. Vinograd in 1957, and the penultimate one is suggested by V. M. Millionshchikov in 1969. The upper (lower) central exponents coincide with the upper (lower) limits of the highest (lowest) Lyapunov and Perron exponents at the point A in the *uniform topology* of the space $\mathcal{M}^n$.

2 The various time scales

Non-uniform scales were employed for the first time by N. A. Izobov in 1982.

Definition 2.1. Let $\mathcal{T}$ be the set of all time *scales*, i.e. strictly increasing unbounded sequences of the form

$$\tau \equiv (\tau_k)_{k\in\mathbb{N}}, \ \ \tau_k \in \mathbb{R}^+ \ \ (\tau_0 \equiv 0).$$

We say that the scale $\tau \in \mathcal{T}$ is:

1) *uniform* (with the *difference* $T > 0$), if $\tau = \tau(T) \equiv (Tk)_{k\in\mathbb{N}}$;

2) *slowly increasing* (denoted by $\tau \in \mathcal{T}^1$), if $\overline{\lim\limits_{k\to\infty}}\, \tau_k/\tau_{k-1} = 1$.

3) *dense* (denoted by $\tau \in \mathcal{T}^0$), if $\|\tau\| \equiv \overline{\lim\limits_{k\to\infty}}\, (\tau_k - \tau_{k-1}) < \infty$;

4) *expanding* (denoted by $\tau \in \mathcal{T}_\infty$), if $|\tau| \equiv \underline{\lim\limits_{k\to\infty}}\, (\tau_k - \tau_{k-1}) = \infty$;

5) *rarefied* (denoted by $\tau \in \mathcal{T}^\infty$), if $|\tau| < \infty = \|\tau\|$;

6) *slowly expanding* (denoted by $\tau \in \mathcal{T}^1_\infty \equiv \mathcal{T}^1 \cap \mathcal{T}_\infty$), if it is both slowly increasing and expanding.

The distribution of scales in three types (dense, expanding and rarefied ones) defines their complete classification, i.e. the representation $\mathcal{T} = \mathcal{T}^0 \sqcup \mathcal{T}^\infty \sqcup \mathcal{T}_\infty$ holds.

3 The simplified central exponents

As we saw above, the formulas for calculating the central exponents in terms of the Cauchy operators of the original linear system are objectively rather complex. We try to simplify those formulas by an appropriate choice of a non-uniform scale.

Definition 3.1. For each scale $\tau \in \mathcal{T}$ we define four *simplified central* exponents of a system $A \in \mathcal{M}^n$: two *upper* (*up-limit* and *low-limit*) ones

$$\widehat{\Delta}_\tau(A) \equiv \overline{\lim_{m\to\infty}}\, \frac{1}{\tau_m} \sum_{i=1}^{m} \ln \|X_A(\tau_i, \tau_{i-1})\|, \quad \check{\Delta}_\tau(A) \equiv \underline{\lim_{m\to\infty}}\, \frac{1}{\tau_m} \sum_{i=1}^{m} \ln \|X_A(\tau_i, \tau_{i-1})\|, \tag{3.1}$$

and, respectively, two *lower* ones

$$\widehat{\delta}_\tau(A) \equiv \overline{\lim_{m\to\infty}}\, \frac{1}{\tau_m} \sum_{i=1}^{m} \ln |X_A(\tau_i, \tau_{i-1})|, \quad \check{\delta}_\tau(A) \equiv \underline{\lim_{m\to\infty}}\, \frac{1}{\tau_m} \sum_{i=1}^{m} \ln |X_A(\tau_i, \tau_{i-1})|. \tag{3.2}$$

The Vinograd–Millionshchikov central exponents (non-simplified) of a system $A \in \mathcal{M}^n$ can now be given by the following formulas (with an additional exact bound): *upper* ones

$$\widehat{\Omega}(A) \equiv \inf_{T>0} \widehat{\Delta}_{\tau(T)}(A), \quad \check{\Omega}(A) \equiv \inf_{T>0} \check{\Delta}_{\tau(T)}(A), \tag{3.3}$$

and, respectively, *lower* ones

$$\widehat{\omega}(A) \equiv \sup_{T>0} \widehat{\delta}_{\tau(T)}(A), \quad \check{\omega}(A) \equiv \sup_{T>0} \check{\delta}_{\tau(T)}(A). \tag{3.4}$$

4 The central exponents in dense scales

The properties of simplified central exponents in dense scales are similar to those of the corresponding ones in uniform scales. Thus, estimates of the central exponents, which in the special case of a uniform scale follow directly from Definition 1.1, are also valid for all dense scales.

Theorem 4.1. *For any system $A \in \mathcal{M}^n$ and any dense scale $\tau \in \mathcal{T}^0$ the following inequalities hold:*

$$\widehat{\Delta}_\tau(A) \geqslant \widehat{\Omega}(A), \quad \check{\Delta}_\tau(A) \geqslant \check{\Omega}(A), \quad \widehat{\delta}_\tau(A) \leqslant \widehat{\omega}(A), \quad \check{\delta}_\tau(A) \leqslant \check{\omega}(A). \tag{4.1}$$

The estimates (4.1) cannot be improved even for uniform scales, let alone for all dense ones.

Theorem 4.2. *For any system $A \in \mathcal{M}^n$ the equalities hold*

$$\widehat{\Omega}(A) = \inf_{\tau\in\mathcal{T}^0} \widehat{\Delta}_\tau(A), \quad \check{\Omega}(A) = \inf_{\tau\in\mathcal{T}^0} \check{\Delta}_\tau(A), \quad \widehat{\omega}(A) = \sup_{\tau\in\mathcal{T}^0} \widehat{\delta}_\tau(A), \quad \check{\omega}(A) = \sup_{\tau\in\mathcal{T}^0} \check{\delta}_\tau(A). \tag{4.2}$$

The exact bounds in the equalities (4.2) (just as in the equalities (3.3) and (3.4)) can be replaced by the limits as $|\tau| \to \infty$.

Theorem 4.3. *For any system $A \in \mathcal{M}^n$ the equalities hold*

$$\widehat{\Omega}(A) = \lim_{\tau\in\mathcal{T}^0,\, |\tau|\to\infty} \widehat{\Delta}_\tau(A), \quad \check{\Omega}(A) = \lim_{\tau\in\mathcal{T}^0,\, |\tau|\to\infty} \check{\Delta}_\tau(A),$$
$$\widehat{\omega}(A) = \lim_{\tau\in\mathcal{T}^0,\, |\tau|\to\infty} \widehat{\delta}_\tau(A), \quad \check{\omega}(A) = \lim_{\tau\in\mathcal{T}^0,\, |\tau|\to\infty} \check{\delta}_\tau(A).$$

The exact bounds in the equalities (4.2), generally speaking, are not attained and, moreover, for some system all four bounds are not attained at once.

Theorem 4.4. *There exists a diagonal system $A \in \mathcal{M}^2$ such that for each dense scale $\tau \in \mathcal{T}^0$ all the inequalities* (4.1) *are strict.*

The assertion of Theorem 4.4 extends to systems of an arbitrary order $n > 1$. Theorems 5.2 and 5.5–6.2 below admit a similar generalization.

5 The central exponents in expanding scales

Simplified central exponents in slowly expanding scales are also estimated by the corresponding central ones, but from the other side, opposite to the estimates (4.1).

Theorem 5.1. *For any system $A \in \mathcal{M}^n$ and any expanding scale $\tau \in \mathcal{T}_\infty$ the inequalities*

$$\widehat{\Delta}_\tau(A) \leqslant \widehat{\Omega}(A), \quad \check{\delta}_\tau(A) \geqslant \check{\omega}(A) \tag{5.1}$$

hold and for any slowly expanding scale $\tau \in \mathcal{T}^1_\infty$ additional inequalities hold:

$$\check{\Delta}_\tau(A) \leqslant \check{\Omega}(A), \quad \widehat{\delta}_\tau(A) \geqslant \widehat{\omega}(A). \tag{5.2}$$

The requirement that an expanding scale be slowly increasing is not superfluous for the validity of the inequalities (5.2) in Theorem 5.1.

Theorem 5.2. *For any not slowly increasing scale $\tau \in \mathcal{T} \setminus \mathcal{T}^1$ there exists a diagonal system $A \in \mathcal{M}^2$ such that both the inequalities* (5.2) *are not true.*

The estimates (5.1) and (5.2) in Theorem 5.1 cannot be improved. Moreover, for none of them there is an assertion analogous to Theorem 4.4 (for dense scales).

Theorem 5.3. *For any system $A \in \mathcal{M}^n$ there exists a slowly expanding scale $\tau \in \mathcal{T}_\infty^1$ such that the following equalities hold:*

$$\widehat{\Delta}_\tau(A) = \widehat{\Omega}(A), \quad \check{\Delta}_\tau(A) = \check{\Omega}(A), \quad \widehat{\delta}_\tau(A) = \widehat{\omega}(A), \quad \check{\delta}_\tau(A) = \check{\omega}(A). \tag{5.3}$$

Thus, one more view is possible on the central exponents (3.3) and (3.4).

Theorem 5.4. *For any system $A \in \mathcal{M}^n$ the following equalities hold:*

$$\max_{\tau \in \mathcal{T}_\infty} \widehat{\Delta}_\tau(A) = \widehat{\Omega}(A) = \max_{\tau \in \mathcal{T}_\infty^1} \widehat{\Delta}_\tau(A), \quad \check{\Omega}(A) = \max_{\tau \in \mathcal{T}_\infty^1} \check{\Delta}_\tau(A),$$
$$\widehat{\omega}(A) = \min_{\tau \in \mathcal{T}_\infty^1} \widehat{\delta}_\tau(A), \quad \min_{\tau \in \mathcal{T}_\infty} \check{\delta}_\tau(A) = \check{\omega}(A) = \min_{\tau \in \mathcal{T}_\infty^1} \check{\delta}_\tau(A).$$

The inequalities (5.1) and (5.2) under the conditions of Theorem 5.1 do not, in general, turn into equalities for any fixed expanding scale.

Theorem 5.5. *For any expanding scale $\tau \in \mathcal{T}_\infty$ there exists a diagonal system $A \in \mathcal{M}^2$ such that all the inequalities* (5.1) *and* (5.2) *are strict.*

6 The central exponents in rarefied scales

Rarefied scales, occupying, by definition, an intermediate position between dense and expanding ones, may possess the properties of both the types of scales. In particular, none of Theorems 4.1 and 5.1 applies to all rarefied scales.

Theorem 6.1. *For some rarefied scale $\tau \in \mathcal{T}^\infty$ there exist two diagonal systems $A', A'' \in \mathcal{M}^2$ such that for $A = A'$ all the inequalities* (4.1) *are strict and for $A = A''$ all the inequalities* (5.1) *and* (5.2) *are strict.*

No rarefied scale ensures any of the equalities (5.3) for all systems at once.

Theorem 6.2. *For any rarefied scale $\tau \in \mathcal{T}^\infty$ there exists a diagonal system $A \in \mathcal{M}^2$ such that either all the inequalities* (4.1) *are strict or all the inequalities* (5.1) *and* (5.2) *are strict.*

7 Universal scales on a subset

We consider the possibility of completely removing exact bounds or limits appearing explicitly in the formulas (3.3), (3.4) for central exponents, by replacing them with simplified ones (3.1), (3.2) with a well-chosen scale.

Definition 7.1. The scale $\tau \in \mathcal{T}$ will be called *universal on a subset* $\mathcal{M} \subset \mathcal{M}^n$ if for any system $A \in \mathcal{M}$ it satisfies all the equalities (5.3).

On the one hand, Theorems 4.4, 5.5 and 6.2 imply that there is no universal scale on the entire set $\mathcal{M}^n$. On the other hand, such scales still exist on certain standard subsets of it, as the following three theorems say.

Theorem 7.1. *Each scale $\tau \in \mathcal{T}$ is universal on the subset of autonomous diagonal systems $A \in \mathcal{M}^n$.*

Theorem 7.2. *Each expanding scale $\tau \in \mathcal{T}_\infty$ is universal on the subset of autonomous systems $A \in \mathcal{M}^n$.*

Theorem 7.3. *Each slowly expanding scale $\tau \in \mathcal{T}^1_\infty$ is universal on the subset of exponentially separated systems $A \in \mathcal{M}^n$.*

We note that the subset of exponentially separated systems is open and everywhere dense in the topological space $\mathcal{M}^n$ (the last assertion was proved by V. M. Millionshchikov in 1969).

Finally, according to Theorem 5.3, on each *one-point* subset, regardless of the properties of the corresponding system, there is an *individual* universal slowly expanding scale. This statement extends to arbitrary *compact* subsets.

Theorem 7.4. *For any compact set $\mathcal{K} \subset \mathcal{M}^n$ there is slowly expanding scale $\tau \in \mathcal{T}^1_\infty$ that is universal on $\mathcal{K}$.*

References

[1] N. A. Izobov, *Lyapunov Exponents and Stability.* Cambridge Scientific Publishers, 2012.