On the Neumann Problem for Second Order Differential Equations with a Deviating Argument

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We consider the differential equation

$$u''(t)=f(t,u(\tau(t))) \tag{1}$$

with the Neumann boundary conditions

$$u'(a)=c_1,\quad u'(b)=c_2, \tag{2}$$

where $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a function satisfying the local Carathèodory conditions, and $\tau:[a,b]\to[a,b]$ is a measurable function.

For $\tau(t)\equiv t$ problem (1), (2) is investigated in detail (see, e.g., [2–7] and the references therein). However, for $\tau(t)\not\equiv t$ that problem remains practically unstudied. The exception is only the case where equation (1) is linear (see [1]).

Theorems below on the solvability and unique solvability of problem (1), (2) are analogues of the theorem by I. Kiguradze [5] for differential equations with a deviating argument.

We use the following notation.

$$\mu(t)=\left(\frac{b-a}{2}+\Big|\frac{b+a}{2}-t\Big|\right)^{\frac{1}{2}},\quad \mu_\tau(t)=\operatorname{ess\,sup}\Big\{|\tau(t)-\tau(t_0)|^{\frac{1}{2}}:\ a\le t_0\le b\Big\},$$

$$\chi_\tau(t)=\begin{cases}1 & \text{if } \tau(t)\neq t,\\ 0 & \text{if } \tau(t)=t,\end{cases}$$

$$f^*(t,x)=\max\Big\{|f(t,y)|:\ |y|\le x\Big\}\ \text{ for } t\in[a,b],\ \ x\ge 0.$$

Theorems 1 and 2 concern the cases where on the set $[a,b]\times\mathbb{R}$ either the inequality

$$f(t,x)\operatorname{sgn}(x)\ge\varphi(t,x), \tag{3}$$

or the inequality

$$f(t,x)\operatorname{sgn}(x)\le-\varphi(t,x) \tag{4}$$

is satisfied. Here the function $\varphi:[a,b]\times\mathbb{R}\to\mathbb{R}$ is such that $\varphi(\cdot,x)$ is Lebesgue integrable in the interval $[a,b]$,

$$\varphi(t,x)\ge\varphi(t,y)\ \text{ for } xy\ge 0,\ \ |x|\ge|y|, \tag{5}$$

and

$$\lim_{|x|\to+\infty}\int_a^b\varphi(t,x)\,dt>|c_1-c_2|. \tag{6}$$

Theorem 1. *If conditions* (3), (5), (6), *and*

$$\lim_{|x|\to+\infty}\left(\frac{1}{|x|}\int_a^b|t-\tau(t)|^{\frac{1}{2}}f^*(t,\mu_\tau(t)x)\,dt\right)<1$$

are fulfilled, then problem (1), (2) *has at least one solution.*

Theorem 2. *If conditions* (4)–(6), *and*

$$\limsup_{|x|\to+\infty}\bigg(\frac{1}{|x|}\int_a^b \mu(t)f^*(t,\mu_\tau(t)x)\,dt\bigg)<1$$

are fulfilled, then problem (1), (2) *has at least one solution.*

Example 1. Consider the differential equation

$$u''(t)=p(t)\frac{u(\tau(t))}{1+|u(\tau(t))|} \tag{7}$$

with the Lebesgue integrable coefficient $p:[a,b]\to\mathbb{R}$. It is clear that if $c_1\neq c_2$ and

$$\int_a^b |p(t)|\,dt\leq|c_1-c_2|,$$

then problem (7), (2) has no solution. Thus from Theorem 1 (from Theorem 2) it follows that if $c_1\neq c_2$ and

$$p(t)\geq 0 \ \text{ for } \ a\leq t\leq b \quad (p(t)\leq 0 \ \text{ for } \ a\leq t\leq b),$$

then problem (7), (2) is solvable if and only if

$$\int_a^b p(t)\,dt>|c_1-c_2| \qquad \bigg(\int_a^b p(t)\,dt<-|c_1-c_2|\bigg).$$

The above example shows that condition (6) in Theorems 1 and 2 is unimprovable and it cannot be replaced by the condition

$$\lim_{|x|\to+\infty}\int_a^b \varphi(t,x)\,dt\geq|c_1-c_2|.$$

Theorem 3. *Let on the set* $[a,b]\times\mathbb{R}$ *the conditions*

$$\big(f(t,x)-f(t,y)\big)\operatorname{sgn}(x-y)\geq p_1(t)|x-y|,$$
$$\chi_\tau(t)\big|f(t,x)-f(t,y)\big|\leq p_2(t)|x-y|$$

hold, where $p_i:[a,b]\to\mathbb{R}_+$ $(i=1,2)$ *are integrable functions such that*

$$\int_a^b p_1(t)\,dt>0,\quad \int_a^b |t-\tau(t)|^{\frac{1}{2}}\mu_\tau(t)p_2(t)\,dt<1.$$

Then problem (1), (2) *has one and only one solution.*

Theorem 4. *Let on the set* $[a,b]\times\mathbb{R}$ *the condition*

$$-p_2(t)|x-y|\leq\big(f(t,x)-f(t,y)\big)\operatorname{sgn}(x-y)\leq -p_1(t)|x-y|$$

be satisfied, where $p_i:[a,b]\to\mathbb{R}_+$ $(i=1,2)$ *are integrable functions such that*

$$\int_a^b p_1(t)\,dt>0,\quad \int_a^b \mu(t)\mu_\tau(t)p_2(t)\,dt<1.$$

Then problem (1), (2) *has one and only one solution.*

Example 2. Let $I \subset [a,b]$ and $[a,b] \setminus I$ be the sets of positive measure, and $\tau : [a,b] \to \mathbb{R}$ be the measurable function such that

$$\tau(t) = t \text{ for } t \in I, \quad \tau(t) \neq t \text{ for } t \notin I.$$

Let, moreover,

$$f(t,x) = \begin{cases} p(t)\big(\exp(|x|) - 1\big)\operatorname{sgn}(x) + q(t) & \text{if } t \in I, \\ p(t)x + q(t) & \text{if } t \in [a,b] \setminus I, \end{cases}$$

where $p : [a,b] \to (0,+\infty)$ and $q : [a,b] \to \mathbb{R}$ are integrable functions, and

$$\int_a^b |t - \tau(t)|^{\frac{1}{2}} \mu_\tau(t) p(t)\, dt < 1. \tag{8}$$

Then by Theorem 3 problem (1), (2) has one and only one solution.

Consequently, Theorem 3 covers the case, where $\tau(t) \neq t$ and for any t from some set of positive measure the function f is rapidly increasing in the phase variable, i.e.,

$$\lim_{|x| \to +\infty} \frac{f(t,x)}{x} = +\infty.$$

At the end, consider the linear differential equation

$$u''(t) = p(t)u(\tau(t)) + q(t) \tag{9}$$

with integrable coefficients $p : [a,b] \to \mathbb{R}$ and $q : [a,b] \to \mathbb{R}$.

Theorem 3 yields the following statement.

Corollary 1. *If*

$$p(t) \geq 0 \ \textit{for} \ a < t < b, \quad \int_a^b p(t)\, dt > 0, \tag{10}$$

and inequality (8) *holds, then problem* (9), (2) *has one and only one solution.*

If $\tau(t) \equiv t$, then condition (10) guarantees the unique solvability of problem (9), (2). And if $\tau(t) \not\equiv t$, then this is not so. Indeed, if, for example, $a = 0$, $b = \pi$, $\tau(t) = \pi - t$, $p(t) = 1$ for $a \leq t \leq b$, and the function q satisfies the inequality

$$\int_a^b q(t) \cos(t)\, dt \neq -c_1 - c_2, \tag{11}$$

then problem (9), (2) has no solution.

Therefore, condition (8) in Corollary 1 is essential and it cannot be omitted. However, the question on the unimprovability of that condition remains open.

Theorem 4 yields the following statement.

Corollary 2. *If*

$$p(t) \leq 0 \ \textit{for} \ a < t < b, \quad 0 < \int_a^b \mu(t)\mu_\tau(t)|p(t)|\, dt < 1, \tag{12}$$

then problem (9), (2) *has one and only one solution.*

Note that problem (9), (2) may be uniquely solvable also in the case where the differential equation

$$u''(t) = p(t)u(t) + q(t) \tag{13}$$

has no solution satisfying the boundary conditions (2). For example, if

$$p(t) = -1 \ \text{ for } \ a \le t \le b, \quad a = 0, \quad b = \pi, \tag{14}$$

and condition (11) holds, then problem (13), (2) has no solution. On the other hand, if along with (14) we have

$$\tau(t) = \begin{cases} t^3 & \text{for } \ 0 \le t \le \pi^{-1}, \\ \pi^{-3} & \text{for } \ \pi^{-1} \le t \le \pi, \end{cases}$$

then condition (12) is satisfied and, according to Corollary 2, problem (9), (2) is uniquely solvable for any c_1 and c_2.

References

[1] E. I. Bravyi, *Solvability of Boundary Value Problems for Linear Functional Differential Equations.* (Russian) Regular and chaotic dynamic, Moscow–Izhevsk, 2011.

[2] A. Cabada, P. Habets and S. Lois, Monotone method for the Neumann problem with lower and upper solutions in the reverse order. *Appl. Math. Comput.* **117** (2001), no. 1, 1–14.

[3] A. Cabada and L. Sanchez, A positive operator approach to the Neumann problem for a second order ordinary differential equation. *J. Math. Anal. Appl.* **204** (1996), no. 3, 774–785.

[4] M. Cherpion, C. De Coster and P. Habets, A constructive monotone iterative method for second-order BVP in the presence of lower and upper solutions. *Appl. Math. Comput.* **123** (2001), no. 1, 75–91.

[5] I. Kiguradze, The Neumann problem for the second order nonlinear ordinary differential equations at resonance. *Funct. Differ. Equ.* **16** (2009), no. 2, 353–371.

[6] I. T. Kiguradze and N. R. Lezhava, On the question of the solvability of nonlinear two-point boundary value problems. (Russian) *Mat. Zametki* **16** (1974), 479–490; translation in *Math. Notes* **16** (1974), 873–880.

[7] I. T. Kiguradze and N. R. Lezhava, On a nonlinear boundary value problem. *Function theoretic methods in differential equations*, pp. 259–276. Res. Notes in Math., No. 8, *Pitman, London*, 1976.