Oscillation Criteria for Second-Order Linear Advanced Differential Equations

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On the half-line $\mathbb{R}_+ = [0, +\infty[$, we consider the second-order linear differential equation with argument deviation

$$u''(t) + p(t)u(\sigma(t)) = 0, \tag{1}$$

where $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a locally Lebesgue integrable function and $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that

$$\sigma(t) \geq t \ \text{ for } \ t \geq 0.$$

Oscillation theory for linear ordinary differential equations is a widely studied and well-developed topic of the general theory of differential equations. We mention some results which are closely related to those of this paper, in particular, works of E. Hille, E. Müller-Pfeiffer, and A. Wintner (see, e.g., [1–3, 6]). We should note that oscillation properties for the linear differential equation with deviating argument (1), but in the case when $\sigma(t)$ is a delay, were studied in [4, 5]

Solutions to equation (1) can be defined in various ways. Since we are interested in properties of solutions in a neighbourhood of $+\infty$, we introduce the following commonly used definitions.

Definition 1. Let $t_0 \in \mathbb{R}_+$. A continuous function $u\colon [t_0, +\infty[\to \mathbb{R}$ is said to be a *solution to equation* (1) *on the interval* $[t_0, +\infty[$ if it is absolutely continuous together with its first derivative on every compact interval contained in $[t_0, +\infty[$ and satisfies equality (1) almost everywhere in $[t_0, +\infty[$.

Definition 2. A solution to equation (1) is said to be *oscillatory* if it has a zero in any neighbourhood of infinity, and *non-oscillatory* otherwise.

Firstly, we remind that if $\int\limits_0^{+\infty} sp(s)\,\mathrm{d}s < +\infty$, then (1) has a proper non-oscillatory solution (see [4, Proposition 2.1]). Therefore, we assume throughout the paper that

$$\int\limits_0^{+\infty} sp(s)\,\mathrm{d}s = +\infty.$$

Let us put

$$F_* = \liminf_{t\to+\infty} t \int\limits_t^{+\infty} p(s)\,\mathrm{d}s, \quad F^* = \limsup_{t\to+\infty} t \int\limits_t^{+\infty} p(s)\,\mathrm{d}s. \tag{2}$$

We prove our main results by using lemma on a priori estimate of non-oscillatory solutions. If we have non-oscillatory solution, then we need to find a suitable a priori lower bound of the quantity $u(\sigma(t))/u(t)$. It is not difficult to verify that

$$1 \leq \frac{u(\sigma(t))}{u(t)} \ \text{ for large } t.$$

However, we succeeded in finding a more precise estimate in Lemma 1, which allow us to establish more efficient results.

Lemma 1. *Let u be a solution to equation* (1) *on the interval $[t_u,+\infty[$ satisfying the inequality*

$$u(t)>0 \ \ for \ \ t\geq t_u.$$

Then

$$F^*\leq 1$$

and, moreover, for any $\varepsilon\in[0,1[$, there exists $t_0(\varepsilon)\geq t_u$ such that

$$\Big(\frac{\sigma(t)}{t}\Big)^{\varepsilon F_*}\leq\frac{u(\sigma(t))}{u(t)} \ \ for \ \ \sigma(t)\geq t\geq t_0(\varepsilon),$$

where the numbers F_* and F^* are given by relations (2).

One can see that from Lemma 1 we obtain the following proposition.

Proposition. *Let*

$$F^*>1.$$

Then every proper solution to equation (1) *is oscillatory.*

Hence, it is natural to suppose that

$$F_*\leq 1. \tag{3}$$

Now we formulate main results. The first one contains Wintner type oscillation criterion.

Theorem 1. *Let condition* (3) *be fulfilled and let there exist $\lambda\in[0,1[$ and $\varepsilon\in[0,1[$ such that*

$$\int\limits_0^{+\infty}s^\lambda\Big(\frac{\sigma(s)}{s}\Big)^{\varepsilon F_*}p(s)\,\mathrm{d}s=+\infty. \tag{4}$$

Then every proper solution to equation (1) *is oscillatory.*

Next criterion generalizes a result of E. Müller-Pfeiffer proved for ordinary differential equations in [3].

Theorem 2. *Let conditions* (3) *hold and there exist $\varepsilon\in[0,1[$ such that*

$$\limsup_{t\to+\infty}\frac{1}{\ln t}\int\limits_0^t\Big(\frac{\sigma(s)}{s}\Big)^{\varepsilon F_*}p(s)\,\mathrm{d}s>\frac{1}{4}\,.$$

Then every proper solution to equation (1) *is oscillatory.*

In view of Theorem 1, we can assume that

$$\int\limits_0^{+\infty}s^\lambda\Big(\frac{\sigma(s)}{s}\Big)^{\varepsilon F_*}p(s)\,\mathrm{d}s<+\infty \ \text{ for all } \ \lambda\in[0,1[\,, \ \ \varepsilon\in[0,1[\,.$$

It allows one to define, for any $\varepsilon\in[0,1[$, the function

$$Q(t;\varepsilon):=t\int\limits_t^{+\infty}\Big(\frac{\sigma(s)}{s}\Big)^{\varepsilon F_*}p(s)\,\mathrm{d}s \ \text{ for } \ t>0. \tag{5}$$

By using the lower and upper limits

$$Q_*(\varepsilon)=\liminf_{t\to+\infty} Q(t;\varepsilon), \quad Q^*(\varepsilon)=\limsup_{t\to+\infty} Q(t;\varepsilon), \tag{6}$$

we establish new Hille type oscillation criteria, which coincide with some well-known results in the case of ordinary differential equations (see, [2]).

Theorem 3. *Let conditions* (3) *hold and there exist* $\varepsilon\in[0,1[$ *such that*

$$Q^*(\varepsilon)>1.$$

Then every proper solution to equation (1) *is oscillatory.*

Theorem 4. *Let conditions* (3) *hold and there exist* $\varepsilon\in[0,1[$ *such that*

$$Q_*(\varepsilon)>\frac{1}{4}. \tag{7}$$

Then every proper solution to equation (1) *is oscillatory.*

Finally, we show two examples, where we can apply oscillatory criteria from Theorems 1 and 3 succesfully.

Example 1. Let us consider the following equation

$$u''(t)+\frac{1}{(t+1)^2}\,u\big((t+1)^2\big)=0 \ \text{ for } \ t\geq 0. \tag{8}$$

One can see that

$$F_*=\liminf_{t\to+\infty} t\int\limits_t^{+\infty}\frac{1}{(s+1)^2}\,\mathrm{d}s=\liminf_{t\to+\infty}\frac{t}{t+1}=1,$$

i.e. condition (3) is fulfilled.

On the other hand, if we put $\lambda=\varepsilon=\frac{1}{2}$, then we obtain

$$\int\limits_0^{+\infty} s^{\lambda}\Big(\frac{\sigma(s)}{s}\Big)^{\varepsilon F_*}p(s)\,\mathrm{d}s=\int\limits_0^{+\infty}\frac{1}{s+1}\,\mathrm{d}s=+\infty.$$

Consequently, condition (4) is satisfied and according to Theorem 1 every proper solution to equation (8) is oscillatory.

Example 2. Let us consider the equation

$$u''(t)+\frac{2+\sin(\ln t)+\cos(\ln t)}{t^2}\,u(4t)=0 \ \text{ for } \ t>0. \tag{9}$$

One can show that

$$F_*=\liminf_{t\to+\infty} t\int\limits_t^{+\infty}\frac{2+\sin(\ln s)+\cos(\ln s)}{s^2}\,\mathrm{d}s=\liminf_{t\to+\infty}(2+\cos(\ln t))=1,$$

i.e. condition (3) is fulfilled.

On the other hand, if we put $\varepsilon = \frac{1}{2}$, then from notation (5) and (6) we obtain

$$Q_*\Big(\frac{1}{2}\Big) = \liminf_{t\to+\infty} Q\Big(t;\frac{1}{2}\Big)$$
$$= \liminf_{t\to+\infty} t\int\limits_t^{+\infty} 2\,\frac{2+\sin(\ln s)+\cos(\ln s)}{s^2}\,\mathrm{d}s = \liminf_{t\to+\infty}(4+2\cos(\ln t)) = 2.$$

Consequently, condition (7) is satisfied and according to Theorem 3 every proper solution to equation (9) is oscillatory.

References

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