On a Class of Linear Functional Differential Systems Under Integral Control

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1 Introduction

In the classical control problem for the differential system

$$(\mathcal{L}x)(t) \equiv \dot{x}(t) + A(t)x(t) = B(t)u(t) + f(t), \quad t \in [0,T],$$

one needs to find a control u taking the system from a given initial state $x(0) = \alpha$ to a prescribed terminal position $x(T) = \beta$. In the case with no constraints according to control, any terminal state β is attainable if the matrix

$$V = \int_0^T Y(t)B(t)B^\top(t)Y^\top(t)\,dt$$

is nonsingular, where $Y(t)$ is the inverse to the fundamental matrix of the system, $\cdot^\top$ stands for transposition. If the control $u(t)$ is constrained, say by $v(t) \le u(t) \le V(t)$, $t \in [0,T]$, there arises the question about the attainability set, i.e. the set of all terminal positions that are attainable by the use of controls such that the constraints are fulfilled. After the fundamental work by N. N. Krasovskiĭ [6] the questions of attainability are studied systematically for various classes of systems with continuous and discrete times (see, for instance, [1, 5, 8] and the references therein).

We consider a quite broad class of functional differential systems under control implemented by an integral operator in the case that the goal of control takes into account a collection of terminal and previous states of the system under control. Such problems find practical use, in particular, in economic dynamics [10, 12].

First we descript in detail a class of functional differential equations with linear Volterra operators and appropriate spaces where those are considered. Next the setting of the control problem (CP) is given and discussed, and some conditions for the solvability of CP are recalled. Those are obtained for the case of unconstrained control as applied to various classes of control (see, for instance, in [10, 11]). A theorem is formulated which gives a description of the attainability set for CP under consideration. Two illustrative example of application of the theorem are presented.

2 Control problem

We follow the notation and basic statements of the general functional differential theory in the part concerning linear systems with aftereffect [2, 4, 9].

Let $L^n = L^n[0,T]$ be the Lebesgue space of all summable functions $z : [0,T] \to R^n$ defined on a finite segment $[0,T]$ with the norm $\|z\|_{L^n} = \int_0^T |z(t)|\,dt$, where $|\cdot|$ is a norm in R^n. Denote by

$AC^n = AC^n[0,T]$ the space of absolutely continuous functions $x : [0,T] \to R^n$ with the norm

$$\|x\|_{AC^n} = |x(0)| + \|\dot{x}\|_{L^n}.$$

In what follows we will use some results from [2,4,9].

To give a description of the controlled functional differential system with aftereffect, introduce the linear operator $\mathcal{L}$:

$$(\mathcal{L}x)(t) = \dot{x}(t) - \int\limits_0^t K(t,s)\dot{x}(s)\,ds - A(t)x(0), \ \ t \in [0,T]. \tag{2.1}$$

Here the elements $k_{ij}(t,s)$ of the kernel $K(t,s)$ are measurable on the set $\Delta = \{(t,s) : \ 0 \leqslant s \leqslant t \leqslant T\}$ and such that the estimates $|k_{ij}(t,s)| \leqslant \kappa(t)$, $i,j = 1,\dots,n$, hold on Δ with a function κ summable on $[0,T]$. The elements of $(n \times n)$-matrix $A(t)$ are summable on $[0,T]$. The operator $\mathcal{L} : AC^n \to L^n$ is bounded. The functional differential system $\mathcal{L}x = f$ covers linear differential equations with concentrated and/or distributed delay and Volterra integro-differential systems (see, for instance, [9]). For a particular case of $(\mathcal{L}x)(t) = \dot{x}(t) - \int\limits_0^t d_s\, R(t,s)x(s)$, where without loss of generality we can put $R(t,t) = 0$, we have $K(t,s) = R(t,s)$, $A(t) = R(t,0)$.

Under the listed conditions the linear operator $Q : L^n \to L^n$, $(Qz)(t) = z(t) - \int\limits_0^t K(t,s)z(s)(s)\,ds$ has the bounded inverse operator $(Q^{-1}f)(t) = f(t) + \int\limits_0^t \mathcal{R}(t,s)f(s)(s)\,ds$, where $\mathcal{R}(t,s)$ is the resolvent kernel with respect to $K(t,s)$. The matrix $C(t,s) = E + \int\limits_s^t \mathcal{R}(\xi,s)\,d\xi$, where E is the identity $(n \times n)$-matrix, is called the Cauchy matrix. The general solution of the equation $\mathcal{L}x = f$ has the form

$$x(t) = X(t)\alpha + \int\limits_0^t C(t,s)f(s)\,ds,$$

where $X(t)$ is the fundamental matrix to the homogeneous equation $\mathcal{L}x = 0$. The properties of the Cauchy matrix used below are studied in detail in [9].

The system under control is described by the equation

$$(\mathcal{L}x)(t) = (Bu)(t), \ \ t \in [0,T]. \tag{2.2}$$

Here

$$(Bu)(t) = \int\limits_0^t B(t,s)u(s)\,ds,$$

the elements $b_{ij}(t,s)$ of the $(n \times r)$-matrix $B(t,s)$ are measurable on the set $\Delta = \{(t,s) : \ 0 \leqslant s \leqslant t \leqslant T\}$ and such that the estimates $|b_{ij}(t,s)| \leqslant b(t)$, $i,j = 1,\dots,n$, hold on Δ with a function b summable on $[0,T]$.

The initial state of system (2.2) is fixed:

$$x(0) = 0. \tag{2.3}$$

To define the on-target vector-functional, let us fix a collection of points $\{t_i\}$ in $[0,T]$: $0 < t_1 \leqslant \cdots \leqslant t_{m-1} \leqslant t_m = T$. The aim of control is prescribed with a given linear bounded vector-functional $\ell : AC^n \to R^N$:

$$\ell x \equiv \sum_{i=1}^m P_i x(t_i) = \beta, \tag{2.4}$$

where P_i, $i = 1, \dots, m$, are given $(N \times n)$-matrices. For the case of unconstrained control from the space of square summable functions $u : [0, T] \to R^r$, a condition of the solvability of (2.2)–(2.4) is given by the theorem appearing below, which is a corollary from Theorem 3.1 [10]. To formulate it, we introduce $(N \times n)$-matrix Φ:

$$\Phi(s) = \sum_{i=1}^{m} P_i \chi_i(s) C(t_i, s),$$

where $\chi_i(s)$ is the characteristic function of the segment $[0, t_i]$.

Theorem 2.1. *Problem* (2.2)–(2.4) *is solvable iff the* $(N \times N)$*-matrix*

$$W = \int_0^T \bigg\{ \int_\tau^T \Phi(s) B(s, \tau)\, ds \cdot \int_\tau^T B^\top(s, \tau) \Phi^\top(s)\, ds \bigg\}\, d\tau$$

is nonsingular.

Now we introduce the constraints with respect to the control $u(t)$:

$$Gu(t) \leqslant g, \ \ t \in [0, T], \tag{2.5}$$

where G is a given $(N_1 \times r)$-matrix, $g \in R^{N_1}$. It is assumed that the set of all solutions to $Gv \leqslant g$ (i.e. the set $\mathcal{V}$ of admissible values of control $u(t)$) is nonempty and bounded in R^r.

Definition. We say that a set $\Xi \subset R^N$ is the ℓ-attainability set of (2.2) with (2.3) under constraints (2.5) iff problem (2.2)–(2.4) is solvable for any $\beta \in \Xi$.

Define $(N \times r)$-matrix $M(s)$ by the equality

$$M(s) = \int_s^T \Phi(\tau) B(\tau, s)\, d\tau.$$

Due to the Cauchy matrix we reduce problem (2.2)–(2.5) to the moment problem [6]

$$\int_0^T M(s) u(s)\, ds = \beta, \ \ Gu(t) \leqslant g, \ \ t \in [0, T],$$

and, employing Theorem 7.1 [7], obtain

Theorem 2.2. *Let* $B(\tau, \cdot)$ *be continuous on* $[0, \tau]$ *for almost all* $\tau \in [0, T]$*, and for any fixed* $\lambda \in R^N$ *the linear programming problem*

$$z = \lambda^\top M(s) v \longrightarrow \max, \ \ Gv \leqslant g \tag{2.6}$$

be uniquely solvable for almost all $s \in [0, T]$*. Then problem* (2.2)–(2.5) *is solvable iff for any fixed* $\lambda \in R^N$ *the inequality*

$$\lambda^\top \beta \leqslant \int_0^T \lambda^\top M(s) u(s, \lambda)\, ds$$

holds, where $u(s, \lambda)$ *is a solution to* (2.6).

Example 1. Let us consider the system under control

$$\left.\begin{aligned} \dot{x}_1(t) &= x_2(t-1) + \int\limits_0^t u_1(s)\,ds, \\ \dot{x}_2(t) &= -x_2(t) + \int\limits_0^t u_2(s)\,ds, \end{aligned}\right. \quad t \in [0,3], \tag{2.7}$$

where $x_2(s) = 0$ if $s < 0$, with the initial conditions

$$x_1(0) = 0, \quad x_2(0) = 0, \tag{2.8}$$

and on-target conditions as follows:

$$x_1(3) = \beta_1, \quad x_2(3) = \beta_2, \quad x_2(2) = \beta_3, \tag{2.9}$$

under the control constrained by the inequalities

$$0 \leqslant u_i(t) \leq 1, \ \ i = 1, 2. \tag{2.10}$$

Here we have

$$(Bu)(t) = \operatorname{col}\Big(\int\limits_0^t u_1(s)\,ds, \int\limits_0^t u_2(s)\,ds \Big),$$

$$C(t,s) = \begin{pmatrix} 1 & \int\limits_s^t \chi_{[1,3]}(\tau)\chi_{[0,\tau-1]}(s)\exp(1-\tau+s)\,d\tau \\ 0 & \exp(s-t) \end{pmatrix},$$

$$\ell x = \operatorname{col}\big(x_1(3), x_2(3), x_2(2)\big).$$

Now, to study the ℓ-attainability set, after calculations we obtain the system $\int\limits_0^3 M(s)u(s)\,ds = \beta$, where $\beta = \operatorname{col}(\beta_1, \beta_2, \beta_3)$, $u(s) = \operatorname{col}(u_1(s), u_2(s))$, $M(s) = \{\mu_{ij}(s)\}_{i=1,2,3;\, j=1,2}$,

$$\mu_{11}(s) = 3 - s, \quad \mu_{12}(s) = \begin{cases} 1 - s + e^{s-2} & \text{if } \ s \in [0,2], \\ 0 & \text{otherwise}, \end{cases} \quad \mu_{21}(s) = 0,$$

$$\mu_{22}(s) = 1 - e^{s-3}, \quad \mu_{31}(s) = 0, \quad \mu_{32}(s) = \begin{cases} 1 - e^{s-2} & \text{if } \ s \in [0,2], \\ 0 & \text{otherwise}. \end{cases}$$

By Theorem 2.2 it can be shown that the set $\mathcal{A} = \{(\beta_1, \beta_2, \beta_3) : \ \beta_1 \in [0, 1.5], \ \beta_2 \in [0, 0.35], \ 0 \leq \beta_3 \leq \beta_2\}$ is a subset of the ℓ-attainability set to problem (2.7)–(2.10).

Example 2. Consider system (2.7) with the initial conditions (2.8) and the on-target conditions

$$x_1(3) = \beta_1, \quad x_2(3) + x_2(2) = \beta_2, \tag{2.11}$$

under the control constrained by the inequalities

$$u_i(t) \geqslant 0, \ \ i = 1, 2; \ \ u_1 + u_2 \leqslant 1. \tag{2.12}$$

For this case, we have $\ell x = \operatorname{col}(x_1(3), x_2(3) + x_2(2))$. By Theorem 2.2 it can be shown that the union of the triangle with corner points $(0; 0)$, $(4.0; 0)$, $(0.9; 2.7)$ and the rectangle with corner points $(0; 0)$, $(4.0; 0)$, $(0; -0.5)$, $(4; -0.5)$ is a subset of the ℓ-attainability set to problem (2.7), (2.8), (2.11), (2.12).

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