Positive Solutions of a One-Dimensional Superlinear Indefinite Capillarity-Type Problem

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In this paper we are interested in the existence of positive solutions of the quasilinear Neumann problem

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1+(u')^2}}\Big)' = a(x)f(u) \ \text{ in } (0,1),\\ u'(0)=0, \quad u'(1)=0, \end{cases} \tag{1}$$

where $a \in L^1(0,1)$ changes sign and $f : [0,+\infty) \to [0,+\infty)$ is a continuous function having superlinear growth.

Problem (1) is a particular, one-dimensional, version of the elliptic problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = g(x,u) & \text{in } \ \Omega,\\ -\dfrac{\nabla u\cdot\nu}{\sqrt{1+|\nabla u|^2}} = \sigma & \text{on } \ \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, with outward pointing normal ν, and $g : \Omega\times\mathbb{R} \to \mathbb{R}$ and $\sigma : \partial\Omega \to \mathbb{R}$ are given functions. This problem plays a relevant role in the mathematical analysis of a number of physical or geometrical issues such as capillarity phenomena for incompressible fluids, reaction-diffusion processes where the flux features saturation at high regimes, or prescribed mean curvature problems for cartesian surfaces in the Euclidean space.

Although there is a large amount of literature devoted to the existence of positive solutions for semilinear elliptic problems with superlinear indefinite nonlinearities, no result is available for the problem (2), even in the one-dimensional case (1), in spite of the interest that this topic may have both mathematically and from the point of view of the applications.

As it will become clear later, according to Proposition below, the existence of a positive solution for the homogeneous Neumann problem (1) forces the right hand side of the equation to change sign, thus ruling out the possibility, if f is non-negative, that the sign of the weight function a be constant. Hence, the absence of any previous result in the existing literature might be attributable to the fact that superlinear indefinite weighted problems are fraught with a number of technical difficulties which do not arise in dealing with purely sublinear or superlinear problems, even in the most classical semilinear case, not to talk about the degenerate quasilinear problem dealt with in this paper. In addition, as an effect of the spatial heterogeneities incorporated into the formulation of the problem the complexity of the structure of the solution sets might be quite intricate, even in the semilinear case.

When the homogeneous Neumann boundary conditions are replaced in (1) by Dirichlet conditions, the existence of positive solutions is compatible with the right hand side of the equation having constant sign. As in this case technicalities are partially reduced, there are various results about existence, non-existence and multiplicity of positive solutions, even in higher dimension, assuming that both the functions a and f are non-negative.

Our aim here is therefore to begin the analysis of the effects of spatial heterogeneities in the simplest one-dimensional prototype problem (1). Although part of our discussion has slightly been inspired by some available results in the context of semilinear elliptic problems, it must be stressed that the specific structure of the mean curvature operator,

$$u \longmapsto \Big(\frac{u'}{\sqrt{1+(u')^2}}\Big)',$$

makes the analysis much more delicate and sophisticated, as it may determine the occurrence of discontinuous solutions.

Since problem (1) has a variational structure, it is natural to look for its solutions as critical points of an associated action functional such as

$$\mathcal{H}(v)=\int_0^1\Big(\sqrt{1+(v')^2}-1\Big)\,dx-\int_0^1 a\,F(v)\,dx,$$

with

$$F(s)=\int_0^s f(\xi)\,d\xi. \tag{3}$$

As the functional $\mathcal{H}$ grows linearly with respect to the gradient v', it is well-defined in the Sobolev space $W^{1,1}(0,1)$ of all absolutely continuous functions in $(0,1)$. Yet, this space, which might be an obvious candidate where to settle the study of $\mathcal{H}$, is not a favorable framework to deal with critical point theory. Therefore, we replace the space $W^{1,1}(0,1)$ with the space $\mathrm{BV}(0,1)$ of all bounded variation functions in $(0,1)$, and the functional $\mathcal{H}$ with its relaxation $\mathcal{I}$ to $\mathrm{BV}(0,1)$. Namely, we introduce the functional $\mathcal{J}:\mathrm{BV}(0,1)\to\mathbb{R}$ defined by

$$\mathcal{J}(v)=\int_0^1\Big(\sqrt{1+|Dv|^2}-1\Big)\,dx, \tag{4}$$

where, for $v\in\mathrm{BV}(0,1)$,

$$\int_0^1\sqrt{1+|Dv|^2}\,dx=\sup_{\substack{w_1,w_2\in C_0^1(0,1)\\ \|w_1^2+w_2^2\|_{L^\infty}\le 1}}\int_0^1(vw_1+w_2)\,dx,$$

Then, we denote by $\mathcal{I}:\mathrm{BV}(0,1)\to\mathbb{R}$ the functional defined by

$$\mathcal{I}(v)=\mathcal{J}(v)-\mathcal{F}(v), \tag{5}$$

where, for $v\in\mathrm{BV}(0,1)$,

$$\mathcal{F}(v)=\int_0^1 a\,F(v)\,dx.$$

The relaxed functional $\mathcal{I}$ is not differentiable in $\mathrm{BV}(0,1)$, at least in the usual sense, yet it is the sum of the convex (Lipschitz) continuous functional $\mathcal{J}$ and of the continuously differentiable functional $\mathcal{F}$. Hence we say that a critical point of $\mathcal{I}$ is a function $u \in \mathrm{BV}(0,1)$ such that

$$\mathcal{F}'(u) \in \partial \mathcal{J}(u),$$

where $\partial \mathcal{J}(u)$ denotes the subdifferential of $\mathcal{J}$ at the point u in the sense of convex analysis, or, equivalently, such that the variational inequality

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \int_0^1 af(u)(v-u)\,dx \tag{6}$$

holds for all $v \in \mathrm{BV}(0,1)$. Accordingly, the concept of solution used in this work is fixed by the next definition.

Definition 1. A solution of problem (1) is a function $u \in \mathrm{BV}(0,1)$ such that (6) holds for all $v \in \mathrm{BV}(0,1)$. In addition, a solution u of (1) is said to be positive if $\operatorname{ess\,inf} u \geq 0$ and $\operatorname{ess\,sup} u > 0$, and strictly positive if $\operatorname{ess\,inf} u > 0$.

The notion of solution for problem (1) introduced by Definition 1 has already been used and discussed in various papers. We just stress here its relevance because it allows to consider bounded variation solutions which arise as critical points of a different nature than minimizers of the associated action functional. However, here we will go further in the investigation of the regularity properties of the bounded variation solutions we will find, by proving that they are actually $W^{2,1}_{loc}$, and therefore classically satisfy the equation, on each open interval where the weight function a has a constant sign. Consequently, the discontinuities of the solutions that we construct may occur only in the nodal set of a, and we show that such discontinuity points must be 'vertical'ones.

In order to better motivate the hypotheses we are going to impose on the coefficients a and f, we first observe that if a positive solution u of (1) exists, then the function $a\,f(u)$ must change sign, unless it vanishes a.e. in $[0,1]$. Indeed, by choosing $v = u \pm 1$ as test functions in (6), we get

$$\int_0^1 a\,f(u)\,dx = 0. \tag{7}$$

Thus, if f has a constant sign, the function $a(x)$ must change sign in $[0,1]$. However, in the frame of (1) a stronger property holds if f is assumed to be increasing, as expressed by the following result. As usual, we write

$$a^+ = \max\{a,0\} \ \text{ and } \ a^- = -\min\{a,0\}.$$

Proposition. *Assume that*

(a_1) $a \in L^1(0,1)$ *and* $a \neq 0$,

and

(f_1) $f \in C^1[0,+\infty)$ *is such that* $f(0) \geq 0$ *and* $f'(s) > 0$ *for all* $s > 0$.

Suppose that problem (1) *has a strictly positive solution. Then the following holds*

(a_2) $a^+ \neq 0$ *and* $\int_0^1 a\,dx < 0$.

Remark. Even when $a \in L^1(0,1)$ satisfies (a_2), the condition (f_1) is not in general sufficient for guaranteeing the existence of a positive solution of (1). Indeed, suppose that there is an interval $[x_1, x_2] \subset (0,1)$ such that $a(x) > 0$ a.e. in $[x_1, x_2]$. Let ϕ_1 be a positive eigenfunction associated with the principal eigenvalue of $-d^2/dx^2$ in $H_0^1(x_1, x_2)$ and define

$$\phi(x) = \begin{cases} \phi_1(x) & \text{if } x \in [x_1, x_2], \\ 0 & \text{if } x \in [0,1] \setminus (x_1, x_2). \end{cases}$$

Suppose that (1) admits a positive solution u. Then, taking $u + \phi$ as a test function in (6) and using (f_1), we are driven to

$$\|\phi_1'\|_{L^1} \geq \int\limits_{x_1}^{x_2} a\, f(u)\, \phi_1\, dx \geq f(\operatorname{ess\,inf} u) \int\limits_{x_1}^{x_2} a\, \phi_1\, dx,$$

which clearly imposes a restriction on the size of f on the range of u, or on the amplitude of a in (x_1, x_2). This shows that some additional control on f, or on a, is needed.

Based on the observation that the mean curvature operator $(u'/\sqrt{1+(u')^2}\,)'$ behaves like the Laplace operator u'' at 0 and like the 1-Laplace operator $(u'/|u'|)'$ at infinity, and hence the functional $\mathcal{J}(u)$, defined in (4), behaves like $\frac{1}{2}\int\limits_0^1 |u'|^2\, dx$ at 0 and like $\int\limits_0^1 |u'|\, dx$ at infinity, we are led to impose on the potential F, defined in (3), some superquadraticity conditions at 0 and some superlinearity conditions at $+\infty$.

Theorem 1. *Assume that*

(a_3) $a \in L^1(0,1)$ *is such that* $\int\limits_0^1 a\, dx < 0$ *and* $a(x) > 0$ *a.e. on an interval* $K \subset [0,1]$,

(f_2) $f \in C^0[0,+\infty)$ *is such that* $f(s) \geq 0$ *for* $s \geq 0$,

(f_3) *there exist* $p > 2$ *and* $L > 0$ *such that*

$$\lim_{s \to 0^+} \frac{F(s)}{s^p} = L,$$

(f_4) *there exist* $q > 1$ *and* $M > 0$ *such that*

$$\lim_{s \to +\infty} \frac{F(s)}{s^q} = M,$$

(f_5) *there exists* $\vartheta > 1$ *such that*

$$\lim_{s \to +\infty} \frac{\vartheta F(s) - f(s)\, s}{s} = 0,$$

with F *defined in* (3)*. Then problem* (1) *has at least one positive solution* u*, with* $\mathcal{I}(u) > 0$*. In addition,*

$$u \in W^{2,1}_{loc}(\alpha, \beta) \cap W^{1,1}(\alpha, \beta)$$

for each interval $(\alpha, \beta) \subset (0,1)$ *such that* $a(x) \geq 0$ *a.e. in* (α, β)*, or* $a(x) \leq 0$ *a.e. in* (α, β)*. Moreover,* $u \in W^{2,1}_{loc}[0, \beta)$*, with* $u'(0) = 0$*, if* $\alpha = 0$*, while* $u \in W^{2,1}_{loc}(\alpha, 1]$*, with* $u'(1) = 0$*, if* $\beta = 1$*. Finally,* u *satisfies the equation*

$$-\Big(\frac{u'}{\sqrt{1+(u')^2}}\Big)' = a(x) f(u), \tag{8}$$

a.e. in each of such intervals.

Suppose further that

(f_6) *f is locally Lipschitz in $[0,+\infty)$.*

Then for every pair of adjacent intervals, (α,β), $(\beta,\gamma)\subset(0,1)$ with $a(x)\geq 0$ a.e. in (α,β) and $a(x)\leq 0$ a.e. in (β,γ) (respectively, $a(x)\leq 0$ a.e. in (α,β) and $a(x)\geq 0$ a.e. in (β,γ)), either

$$u\in W^{2,1}_{loc}(\alpha,\gamma),$$

or

$$u(\beta^-)\geq u(\beta^+)\ \text{ and }\ u'(\beta^-)=-\infty=u'(\beta^+)$$
$$\big(\text{respectively, }\ u(\beta^-)\leq u(\beta^+)\ \text{ and }\ u'(\beta^-)=+\infty=u'(\beta^+)\big),$$

where $u'(\beta^-)$, $u'(\beta^+)$ are, respectively, the left and the right Dini derivatives at β.

Assume further that

(a_4) *the function a changes sign finitely many times in $(0,1)$, in the sense that there is a decomposition*

$$[0,1]=\bigcup_{i=1}^{k}[\alpha_i,\beta_i],\ \text{ with }\ \alpha_i<\beta_i=\alpha_{i+1}<\beta_{i+1},\ \text{ for }\ i=1,\dots,k-1,$$

such that

$$(-1)^i a(x)\geq 0\ \text{ a.e. in }\ (\alpha_i,\beta_i),\ \text{ for }\ i=1,\dots,k,$$

or

$$(-1)^i a(x)\leq 0\ \text{ a.e. in }\ (\alpha_i,\beta_i),\ \text{ for }\ i=1,\dots,k.$$

Then u is a strictly positive (special) function of bounded variation.

The proof Theorem 1 relies on a perturbation argument. The approximating problems are solved by using a minimax technique. Then, the obtention of $W^{1,1}$-estimates allow us to pass to the limit in the approximation scheme to get a bounded variation solution of the original problem. A further concavity/convexity argument, combined with ordinary differential equations techniques, finally permits to conclude the partial regularity of the obtained bounded variation solutions. We refer to [1, 2] for further results and complete proofs.

References

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[2] J. López-Gómez, P. Omari and S. Rivetti, Bifurcation of positive solutions for a one-dimensional indefinite quasilinear Neumann problem. *Nonlinear Anal.* **155** (2017), 1–51.