Asymptotics of Solutions of One Class of n-th Order Differential Equations with Regularly Varying Nonlinearities

K. S. Korepanova

Odessa I. I. Mechnikov National University, Odessa, Ukraine
E-mail: ye.korepanova@gmail.com

Consider the differential equation

$$y^{(n)} = \alpha p(t) \prod_{j=0}^{n-1} \varphi_j(y^{(j)}), \tag{1}$$

where $n \geq 2$, $\alpha \in \{-1, 1\}$, $p : [a, +\infty[\to]0, +\infty[$ is a continuous function, $a \in \mathbb{R}$, $\varphi_j : \Delta Y_j \to]0; +\infty[$ are continuous functions regularly varying, as $y^{(j)} \to Y_j$, of order σ_j, $j = \overline{0, n-1}$, ΔY_j is a one-sided neighborhood of the point Y_j, $Y_j \in \{0, \pm\infty\}$[1].

Among the set of monotone solutions of equation (1), defined in some neighborhood of $+\infty$, there might also be solutions for each of which there exists a number $k \in \{1, \dots, n\}$ such that

$$y^{(n-k)}(t) = c + o(1) \ \ (c \neq 0) \ \text{ as } \ t \to +\infty. \tag{2}$$

There have been obtained some results concerning the existence of solutions of type (2) in Corollaries 8.2, 8.6, 8.12 [4, Ch. II, § 8, pp. 207, 214, 223] and Corollaries 9.3, 9.7 [4, Ch. II, § 9, pp. 230, 233] of the monograph by I. T. Kiguradze and T. A. Chanturiya for the equations of general type, in Theorem 16.9 [4, Ch. IV, § 16, p. 321] for the differential equations of Emden–Fauler type. However, these results provide for a considerably strict restriction to the $(n-k+1)$-st derivative of a solution.

In the present paper, a question of performance of new results with less strict restrictions is investigated. When $k = 1, 2$, or the functions $\varphi_i(y^{(i)})$ $(i = \overline{n-k+1, n-2}\,)$ tend to the positive constants, as $y^{(i)} \to Y_i$, in the works [2] and [5] the necessary and sufficient existence conditions of solutions of type (2) of equation (1) and their asymptotic behaviour were obtained without any additional assumptions for these solutions. Otherwise, the new rather wide class of so-called $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions, $-\infty \leq \lambda_0 \leq +\infty$, of equation (1) has been assigned in the paper [3] as follows.

Definition. A solution y of the differential equation (1) is called (for $k \in \{3, \dots, n\}$) a $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution, where $-\infty \leq \lambda_0 \leq +\infty$, if it is defined on the interval $[t_0, +\infty[\subset [a, +\infty[$ and satisfies the conditions

$$\lim_{t \to +\infty} y^{(n-k)}(t) = c \ \ (c \neq 0), \quad \lim_{t \to +\infty} \frac{[y^{(n-1)}(t)]^2}{y^{(n-2)}(t) y^{(n)}(t)} = \lambda_0. \tag{3}$$

In accordance with its asymptotic properties the set of all $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions of equation (1) breaks up to the $k+1$ $(k \in \{3, \dots, n\}\,)$ disjoint subsets (see [1]) that correspond to the subsequent values of the parameter λ_0:

$$\lambda_0 \in \mathbb{R} \setminus \Big\{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\Big\}, \quad \lambda_0 = \pm\infty, \quad \lambda_0 = 1,$$
$$\lambda_0 = \frac{n-j-1}{n-j}\,, \ \ j \in \{n-k+2, \dots, n-1\}.$$

[1]For $Y_j = \pm\infty$ here and in the sequel, all numbers in the neighborhood of ΔY_j are assumed to have constant sign.

The case $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\}$ has been studied in the work [3]. The aim of the present paper is to investigate the question of existence and asymptotic behaviour of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions ($k \in \{3, \dots, n\}$) of equation (1) in special case $\lambda_0 \in \{1, \pm\infty\}$. The asymptotic, as $t \to +\infty$, formulas of their derivatives of order up to $n-1$ will be obtained too. Moreover, a question on the quantity of the studied solutions will be solved.

It is significant to note that by virtue of the results obtained by V. M. Evtukhov [1], the solutions of equation (1) satisfy the following a priori asymptotic conditions.

Lemma. *Let $k \in \{3, \dots, n\}$ and $y : [t_{0k}, +\infty[\to \mathbb{R}$ be an arbitrary $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution of equation* (1)*. Then the following, as $t \to +\infty$, assertions hold:*

- *if $\lambda_0 = \pm\infty$, then*

$$y^{(l-1)}(t) \sim \frac{t^{n-l}}{(n-l)!}\, y^{(n-1)}(t) \ \ (l = \overline{n-k+2, n-1}\,), \quad y^{(n)}(t) = o\Big(\frac{y^{(n-1)}(t)}{t}\Big);$$

- *if $\lambda_0 = 1$, then*

$$\frac{y^{(n-k+2)}(t)}{y^{(n-k+1)}(t)} \sim \frac{y^{(n-k+3)}(t)}{y^{(n-k+2)}(t)} \sim \dots \sim \frac{y^{(n)}(t)}{y^{(n-1)}(t)} \ \ \text{and} \ \lim_{t \to +\infty} \frac{t y^{(n-k+2)}(t)}{y^{(n-k+1)}(t)} = +\infty.$$

It readily follows from the form of equation (1) that $y^{(n)}(t)$ has a constant sign in some neighborhood of $+\infty$. Then $y^{(n-l)}(t)$ $(l = \overline{1, k-1}\,)$ are strictly monotone functions in the neighborhood of $+\infty$ and, by virtue of (2), can tend only to zero, as $t \to +\infty$. Therefore, it is necessary that

$$Y_{j-1} = 0 \ \text{ for } \ j = \overline{n-k+2, n}. \tag{4}$$

Let us assume here and in the sequel that the numbers μ_j $(j = \overline{0, n-1}\,)$, defined in the following way

$$\mu_j = \begin{cases} 1 & \text{if } Y_j = +\infty, \text{ or } Y_j = 0 \text{ and } \Delta Y_j \text{ is a right neighborhood of the point } 0, \\ -1 & \text{if } Y_j = -\infty, \text{ or } Y_j = 0 \text{ and } \Delta Y_j \text{ is a left neighborhood of the point } 0 \end{cases}$$

are such that

$$\mu_j \mu_{j+1} > 0 \ \text{ for } \ j = \overline{0, n-k-1}, \quad \mu_j \mu_{j+1} < 0 \ \text{ for } \ j = \overline{n-k+1, n-2}, \tag{5}$$

$$\alpha \mu_{n-1} < 0. \tag{6}$$

These conditions on μ_j $(j = \overline{0, n-1}\,)$ and α are necessary for the existence of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions of equation (1) as long as for each of them in some neighborhood of $+\infty$

$$\operatorname{sign} y^{(j)}(t) = \mu_j \ \ (j = \overline{0, n-1}\,), \ \ \operatorname{sign} y^{(n)}(t) = \alpha.$$

It is obvious that by virtue of the first relative (3), for these solutions the following representations

$$y^{(l-1)}(t) = \frac{c t^{n-l-k+1}}{(n-l-k+1)!}\, [1 + o(1)] \ \ (l = \overline{1, n-k}\,) \ \text{ as } \ t \to +\infty \tag{7}$$

hold, $c \in \Delta Y_{n-k}$ and then

$$Y_{j-1} = \begin{cases} +\infty & \text{if } \ \mu_{n-k} > 0, \\ -\infty & \text{if } \ \mu_{n-k} < 0, \end{cases} \quad \text{for } \ j = \overline{1, n-k}. \tag{8}$$

We say that a continuous function $L:\Delta Y_0\to]0,+\infty[$, slowly varying as $y\to Y_0$, satisfies the condition S_0 if
$$L\big(\mu e^{[1+o(1)]\ln|y|}\big)=L(y)[1+o(1)] \ \text{ as } \ y\to Y_0 \ \ (y\in\Delta Y_0),$$
where $\mu=\operatorname{sign} y$.

The condition S_0 is necessarily satisfied for functions L that have a nonzero finite limit, as $y\to Y_0$, for functions of the form
$$L(y)=|\ln|y||^{\gamma_1}, \quad L(y)=|\ln|y||^{\gamma_1}\big|\ln|\ln|y||\big|^{\gamma_2},$$
where $\gamma_1,\gamma_2\neq 0$, and for many other functions.

Consider the case $\lambda_0=\pm\infty$. The following statement holds for equation (1).

Theorem 1. *For $k\in\{3,\dots,n\}$ equation* (1) *doesn't have $\mathcal{P}^k_{+\infty}(\pm\infty)$-solutions.*

To investigate the case $\lambda_0=1$, besides the above-mentioned facts about the functions, regularly and slowly varying as $y^{(j)}\to Y_j$ $(j=\overline{0,n-1})$, we need the following auxiliary notations:
$$\gamma_k=1-\sum_{j=n-k+1}^{n-1}\sigma_j, \quad \nu_k=\sum_{j=n-k+1}^{n-2}\sigma_j(n-j-1), \quad M_k(c)=\prod_{j=1}^{n-k}\Big|\frac{c}{(n-j-k+1)!}\Big|^{\sigma_{j-1}},$$
$$I_k(t)=\varphi_{n-k}(c)M_k(c)\int\limits_{A_{0k}}^{t}p(\tau)\prod_{j=0}^{n-k-1}\varphi_j(\mu_j\tau^{n-k-j})\,d\tau, \quad I_{1k}(t)=\int\limits_{A_{1k}}^{t}I_k(\tau)\,d\tau,$$
where A_{0k} (A_{1k}) is chosen equal either to $a_{0k}\geq a$ $(a_{1k}\geq a_{0k})$ or to $+\infty$ so as to ensure that the integral tends either to zero or to $+\infty$ as $t\to+\infty$.

Theorem 2. *Let $k\in\{3,\dots,n\}$ and $\gamma_k\neq 0$. Then, for existence of $\mathcal{P}^k_{+\infty}(1)$-solutions of equation* (1), *it is necessary that $c\in\Delta Y_{n-k}$, along with* (4)–(6), (8) *the following conditions*
$$\frac{I_k'(t)}{I_k(t)}\sim\frac{I_k(t)}{I_{1k}(t)} \ \ \text{as} \ \ t\to+\infty, \quad \lim_{t\to+\infty}|I_k(t)|^{\frac{1}{\gamma_k}}=0 \ \ (j=\overline{n-k+1,n-1}) \tag{9}$$
and the inequalities, as $t\in]a,+\infty[$,
$$\gamma_kI_k(t)<0, \quad I_{1k}(t)>0, \quad (-1)^{n-j-1}\mu_j\mu_{n-1}>0 \ \ (j=\overline{n-k+1,n-3}) \tag{10}$$
hold. Moreover, each solution of that kind admits along with (2) *and* (7) *the asymptotic, as $t\to+\infty$, representations*
$$y^{(j)}(t)=\Big(\frac{\gamma_kI_{1k}(t)}{I_k(t)}\Big)^{n-j-1}y^{(n-1)}(t)[1+o(1)] \ \ (j=\overline{n-k+1,n-2}),$$
$$\frac{|y^{(n-1)}(t)|^{\gamma_k}}{\prod\limits_{j=n-k+1}^{n-1}L_j\Big(\big(\frac{\gamma_kI_{1k}(t)}{I_k(t)}\big)^{n-j-1}y^{(n-1)}(t)\Big)}=\alpha\mu_{n-1}\gamma_kI_k(t)\Big|\frac{\gamma_kI_{1k}(t)}{I_k(t)}\Big|^{\nu_k}[1+o(1)].$$

Theorem 3. *Let $k\in\{3,\dots,n\}$, $\gamma_k\neq 0$ and functions L_j $(j=\overline{n-k+1,n-1})$, slowly varying as $y^{(j)}\to Y_j$, satisfy the condition S_0. Then, in case of existence of $\mathcal{P}^k_{+\infty}(1)$-solutions of equation* (1), *the following condition*
$$\int\limits_{a_{2k}}^{+\infty}\Big(\frac{I_{1k}(\tau)}{I_k(\tau)}\Big)^{k-2}\Big|\gamma_kI_k(\tau)\Big|\frac{\gamma_kI_{1k}(\tau)}{I_k(\tau)}\Big|^{\nu_k}\prod_{j=n-k+1}^{n-1}L_j\big(\mu_j|I_k(\tau)|^{\frac{1}{\gamma_k}}\big)\Big|^{\frac{1}{\gamma_k}}\,d\tau<+\infty \tag{11}$$

holds, where $a_{2k} \geq a_{1k}$ *such that* $\mu_{j-1}|I_k(t)|^{\frac{1}{\gamma_k}} \in \Delta Y_{j-1}$ $(j = \overline{n-k+2, n}\,)$ *for* $t \geq a_{2k}$, *and each solution of that kind admits along with* (7) *the following asymptotic, as* $t \to +\infty$, *representations*

$$y^{(n-k)}(t) = c + \mu_{n-1}\gamma_k^{k-2} W_k(t)[1+o(1)], \tag{12}$$

$$y^{(l-1)}(t) = \mu_{n-1}\gamma_k^{n-l}\Big(\frac{I_{1k}(t)}{I_k(t)}\Big)^{n-l-k+2} W_k'(t)[1+o(1)] \ \ (l = \overline{n-k+2, n}\,), \tag{13}$$

where

$$W_k(t) = \int\limits_{+\infty}^{t} \Big(\frac{I_{1k}(\tau)}{I_k(\tau)}\Big)^{k-2} \bigg|\gamma_k I_k(\tau)\bigg|\frac{\gamma_k I_{1k}(\tau)}{I_k(\tau)}\bigg|^{\nu_k} \prod_{j=n-k+1}^{n-1} L_j\big(\mu_j |I_k(\tau)|^{\frac{1}{\gamma_k}}\big)\bigg|^{\frac{1}{\gamma_k}} d\tau.$$

In the next theorem the sufficient existence conditions of $\mathcal{P}_{+\infty}^{k}(1)$-solutions of equation (1) with mentioned in Theorem 3 asymptotic representations are presented.

Theorem 4. *Let* $k \in \{3, \dots, n\}$, $\gamma_k \neq 0$, $c \in \Delta Y_{n-k}$, *the conditions* (4)–(6), (8)–(10), (11) *hold and the functions* L_j $(j = \overline{n-k+1, n-1}\,)$, *slowly varying as* $y^{(j)} \to Y_j$, *satisfy the condition* S_0. *In addition, let the inequality* $\sigma_{n-1} \neq 1$ *holds and the algebraic relative to* ρ *equation*

$$\sum_{l=2}^{k-1} \sigma_{n-l}(\rho+1)^{k-l-1} - (1-\sigma_{n-1}+\rho)(\rho+1)^{k-2} = 0 \tag{14}$$

has no roots with zero real part. Then equation (1) *has a* $(n-k+m)$*-parameter family of* $\mathcal{P}_{+\infty}^{k}(1)$*-solutions that admit the asymptotic, as* $t \to +\infty$, *representations* (7), (12), (13), *where* m *is a number of roots* (*taking into account divisible*) *with positive real part of the algebraic equation* (14).

Remark. In fact, the algebraic equation (14) has no roots with zero real part if

$$\sum_{l=2}^{k-1} |\sigma_{n-l}| < |1-\sigma_{n-1}|.$$

References

[1] V. M. Evtukhov, Asymptotic representations of solutions of non-autonomous differential equations. (Russian) *Doctoral (Phys. Math.) Dissertation*, 01.01.02, Kiev, 1998.

[2] V. M. Evtukhov and K. S. Korepanova, Asymptotic representation of solutions of differential equations with rightly varying nonlinearities. (Russian) *Ukr. Mat. Zh.* **69** (2017), no. 9, 1198–1216.

[3] V. M. Evtukhov and K. S. Korepanova, Asymptotic behaviour of solutions of one class of n-th order differential equations. *Mem. Differ. Equ. Math. Phys.* **71** (2017), 111–124.

[4] I. T. Kiguradze and T. A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations.* (Russian) Nauka, Moscow, 1990.

[5] K. S. Korepanova, Conditions of existence of power-mode solutions of differential equation with regularly varying nonlinearities. (Ukrainian) *Bukovyn. Mat. Zh.* **4** (2016), no. 3-4, 75–79.