Local Solvability of Multi Dimensional Initial-Boundary Value Problems for Higher Order Nonlinear Hyperbolic Equations

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Let $m_1, \dots, m_n$ be positive integers. In the n-dimensional box $\Omega = [0, \omega_1] \times \dots \times [0, \omega_n]$ for the nonlinear hyperbolic equation

$$u^{(\mathbf{m})} = f(\mathbf{x}, \widehat{\mathcal{D}}^{\mathbf{m}}[u]) \tag{1}$$

consider the initial-boundary conditions

$$\begin{gathered} h_{ik}(u(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n)) = \varphi_{ik}(\widehat{\mathbf{x}}_i) \;\; (k = 1, \dots, m_i, \;\; i = 1, \dots, n-1), \\ u^{(0,\dots,0,k-1)}(x_1, \dots, x_{n-1}, 0) = \varphi_{nk}(\widehat{\mathbf{x}}_n) \;\; (k = 1, \dots, m_n). \end{gathered} \tag{2}$$

Here $\mathbf{x} = (x_1, \dots, x_n)$, $\widehat{\mathbf{x}}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\widehat{\mathbf{m}}_i = \mathbf{m} - \mathbf{m}_i$ and $\mathbf{m}_i = (0, \dots, m_i, \dots, 0)$ are multi-indices,

$$u^{(\alpha)}(\mathbf{x}) = \frac{\partial^{\alpha_1 + \dots + \alpha_n} u(\mathbf{x})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

$\mathcal{D}^{\mathbf{m}}[u] = (u^{(\alpha)})_{\alpha \le \mathbf{m}}$, $\widehat{\mathcal{D}}^{\mathbf{m}}[u] = (u^{(\alpha)})_{\alpha < \mathbf{m}}$, $\Omega_i = [0, \omega_1] \times \dots \times [0, \omega_{i-1}] \times [0, \omega_{i+1}] \times \dots \times [0, \omega_n]$, $f \in C(\Omega \times \mathbb{R}^{m_1 \times \dots \times m_n})$, $h_{ik} : C^{m_i - 1}([0, \omega_i]) \to \mathbb{R}$ ($k = 1, \dots, m_i$; $i = 1, \dots, n-1$) are bounded linear functionals, and $\varphi_{ik} \in C^{\widehat{\mathbf{m}}_i}(\Omega_i)$ ($k = 1, \dots, m_i$; $i = 1, \dots, n$). Furthermore, it is assumed that the functions φ_{ik} satisfy the following consistency conditions:

$$h_{ik}(\varphi_{jl})(\widehat{\mathbf{x}}_{ij}) \equiv h_{jl}(\varphi_{ik})(\widehat{\mathbf{x}}_{ij}) \;\; (k = 1, \dots, m_i; \;\; l = 1, \dots, m_j; \;\; i, j = 1, \dots, n),$$

where $\widehat{\mathbf{x}}_{ij} = \mathbf{x} - \widehat{\mathbf{x}}_i - \widehat{\mathbf{x}}_j$.

Set:

$$\mathbf{Z} = (z_\alpha)_{\alpha < \mathbf{m}}; \quad f_\alpha(\mathbf{x}, \mathbf{Z}) = \frac{\partial f(\mathbf{x}, \mathbf{Z})}{\partial z_\alpha}.$$

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \Upsilon_{\mathbf{m}} \iff \alpha_i = m_i \text{ for some } (i = 1, \dots, n).$$

The variables z_α ($\alpha \in \Upsilon_{\mathbf{m}}$) are called ***principal phase variables*** of the function $f(\mathbf{x}, \mathbf{Z})$.

By a solution of problem (1), (2) we understand a classical solution, i.e., a function $u \in C^{\mathbf{m}}(\Omega)$ satisfying equation (1) and boundary conditions (2).

Two-dimensional initial-boundary value problems were studied in [4, 5].

Definition. Let $\mathbf{n} = (n_1, \dots, n_r)$, $\Omega = [0, \omega_1] \times [0, \dots, \omega_r]$, $\mathbf{y} = (y_1, \dots, y_r)$, and let the function $g : C(\Omega \times \mathbb{R}^{n_1 \times \dots \times n_r})$ be continuously differentiable with respect to the phase variables. A solution $v_0 \in C^n(\Omega)$ of the problem

$$v^{(\mathbf{n})} = g(\mathbf{y}, \widehat{\mathcal{D}}^{\mathbf{n}}[v]), \tag{3}$$

$$h_{ij}\big(v(y_1, \dots, y_{i-1}, \bullet, y_{i+1}, \dots, y_r)\big) = \psi_{ij}(\widehat{\mathbf{y}}_i) \;\; (j = 1, \dots, n_i; \;\; i = 1, \dots, r) \tag{4}$$

is called ***strongly isolated*** if the linearized problem

$$v^{(\mathbf{n})}=\sum_{\alpha<\mathbf{n}}p_\alpha(\mathbf{y})v^{(\alpha)},$$
$$h_{ij}\big(v(y_1,\dots,y_{i-1},\bullet,y_{i+1},\dots,y_r)\big)=0\ \ (j=1,\dots,n_i;\ \ i=1,\dots,r),$$

where $p_\alpha(\mathbf{y})=g_\alpha(\mathbf{y},\widehat{\mathcal{D}}^{\mathbf{k}}[v_0(\mathbf{y})])$, is well-posed.

Well-posed multi-dimensional boundary value problems for higher order linear hyperbolic equations were studied in [2].

The concept of a strongly isolated solution is closely related to the concept of strongly well-posedness. Strong well-posedness of two-dimensional boundary value problems for higher order nonlinear hyperbolic equations were introduced in [3].

Theorem 1. *Let the function f be continuously differentiable with respect to the phase variables, and let v_0 be a strongly isolated solution of the problem*

$$v^{(\widehat{\mathbf{m}}_n)}=p\big(\widehat{\mathbf{x}}_{\mathbf{n}},\widehat{\mathcal{D}}^{\widehat{\mathbf{m}}_n}[v]\big), \tag{5}$$
$$h_{ik}\big(u(x_1,\dots,x_{i-1},\bullet,x_{i+1},\dots,x_{n-1})\big)=\varphi_{ik}^{(m_n)}(\widehat{\mathbf{x}}_{ni})\ \ (k=1,\dots,m_i;\ \ i=1,\dots,n-1), \tag{6}$$

where

$$p\big(\widehat{\mathbf{x}}_{\mathbf{n}},\widehat{\mathcal{D}}^{\widehat{\mathbf{m}}_n}[v]\big)=f\Big(x_1,\dots,x_{n-1},0,\mathcal{D}^{\mathbf{m}-\mathbf{1}_n}[u_0](x_1,\dots,x_{n-1},0),\widehat{\mathcal{D}}^{\widehat{\mathbf{m}}_n}[v]\Big),$$

$\widehat{\mathbf{x}}_{ni}=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n-1})$ *and* $\mathbf{1}_n=(0,\dots,0,1)$. *Then there exists $\delta\in(0,\omega_n]$ such that in the set $\Omega_\delta=[0,\omega_1]\times\dots\times[0,\omega_{n-1}]\times[0,\delta]$ problem* (1), (2) *has a unique solution u satisfying the condition*

$$u^{(\mathbf{m}_n)}(x_1,\dots,x_{n-1},0)=v_0(x_1,\dots,x_{n-1}). \tag{7}$$

Consider the "perturbed" equation

$$u^{(\mathbf{m})}=f\big(\mathbf{x},\widehat{\mathcal{D}}^{\mathbf{m}}[u]\big)+x_nq\big(\mathbf{x},\widehat{\mathcal{D}}^{\mathbf{m}}[u]\big). \tag{8}$$

Theorem 2. *Let the conditions of Theorem 1 hold, and let the function $q(\mathbf{x},\mathbf{Z})$ be continuously differentiable with respect to the principal phase variables z_α ($\alpha\in\Upsilon_{\mathbf{m}}$). Then there exists $\delta\in(0,\omega_n]$ such that in the set $\Omega_\delta=[0,\omega_1]\times\dots\times[0,\omega_{n-1}]\times[0,\delta]$ problem* (8), (2) *has a at least one solution u satisfying condition* (7). *Moreover, if the function q is locally Lipschitz continuous with respect to the rest of the phase variables, then such solution is unique.*

The following is a particular case of conditions (2):

$$h_{1k}\big(u(x_1,\dots,x_{i-1},\bullet,x_{i+1},\dots,x_n)\big)=\varphi_{1k}(\widehat{\mathbf{x}}_i)\ \ (k=1,\dots,m_i),$$
$$u^{(0,\dots,k_i-1,\dots,0)}(x_1,\dots,0,\dots,x_n)=\varphi_{ik_i}(\widehat{\mathbf{x}}_i)\ \ (k_i=1,\dots,m_i;\ \ i=2,\dots,n). \tag{9}$$

Corollary. *Let the function f be continuously differentiable with respect to the phase variables, and let v_0 be a strongly isolated solution of the problem*

$$v^{(m_1)}=p(x_1,v,v',\dots,v^{m_1-1}),$$
$$h_{1k}(v)=\varphi_{1k}^{(\widehat{\mathbf{m}}_1)}(\mathbf{0})\ \ (k=1,\dots,m_1),$$

where

$$p(x_1,v,v',\dots,v^{m_1-1})=f\Big(x_1,0,\dots,0,\widehat{\mathcal{D}}^{\widehat{\mathbf{m}}_1}\big[\mathcal{D}^{\mathbf{m}_1}[u_0]\big](x_1,0,\dots,0),v,v',\dots,v^{m_1-1}\Big).$$

Then there exist $\delta_i\in(0,\omega_i]$ ($i=2,\dots,n$) such that in the set $\Omega_{\delta_2\cdots\delta_n}=[0,\omega_1]\times[0,\delta_2]\times\dots\times[0,\delta_n]$ problem (1), (7) *has a unique solution u satisfying the condition*

$$u^{(\widehat{\mathbf{m}}_1)}(x_1,0,\dots,0)=v_0(x_1).$$

Remark. In Theorem 1 the requirement of strong isolation of the solution v_0 cannot be replaced by well-posedness of problem $(5),(6)$. In order to illustrate this, consider the problem

$$u^{(1,1)} = (u^{(0,1)})^3 - y^2 u^{(0,1)}, \tag{10}$$

$$u(\omega_1, y) - u(0, y) = \int\limits_0^y t \sin \frac{1}{t}\, dt, \quad u(x, 0) = 0. \tag{11}$$

For this case problem $(5),(6)$ is the following one:

$$v' = v^3, \quad v(\omega_1) - v(0) = 0. \tag{12}$$

By Corollary 4.2 and Theorem 4.4 from [1], problem (12) has a unique solution $v_0(y) \equiv 0$ and is well-posed. On the other hand, it is clear, that $v_0(y) \equiv 0$ is not strongly isolated.

Our goal is to show that problem $(10),(11)$ has no solution in the rectangle $\Omega_\delta = [0, \omega_1] \times [0, \delta]$ no matter how small $\delta > 0$ is.

Assume the contrary that problem $(10),(11)$ has a solution u in Ω_δ for some $\delta > 0$. Then for an arbitrarily fixed $y \in (0, \delta]$, the function $v(\,\cdot\,) = u^{(0,1)}(\,\cdot\,, y)$ is a solution of the problem

$$v' = v^3 - y^2 v, \tag{13}$$

$$v(\omega_1) - v(0) = y \sin \frac{1}{y}\,. \tag{14}$$

containing the parameter $y \in [0, \omega_2]$. Moreover, if problem $(10),(11)$ has a solution, then v is a solution $(13),(14)$ depending continuously on the parameter y.

For every fixed $y \in (0, \delta]$ equation (13) has three constant solutions: $v_0(x) = 0$, $v_1(x) = y$ and $v_2(x) = -y$. Due to the existence and uniqueness theorem, a nonconstant solution v of equation (13) intersects v_0, v_1 or v_2, and thus $v'(x) \neq 0$ for $x \in [0, \omega_1]$. Let

$$k > \frac{1}{2\pi\delta} \ \text{ and } \ x \in \Big(\frac{1}{\pi + 2\pi k}, \frac{1}{2\pi k}\Big).$$

Then $v(\omega_1) > v(0)$ and $v'(x) > 0$ for $x \in [0, \omega_1]$. Therefore, either

$$v(x) > y \ \text{ for } \ x \in [0, \omega_1],$$

or

$$v(x) \in (-y, 0) \ \text{ for } \ x \in [0, \omega_1].$$

If $y = \frac{1}{\frac{\pi}{2} + 2\pi k}$, then $v(\omega_1) - v(0) = y$, and consequently,

$$v(x) \notin (-y, 0) \ \text{ for } \ x \in [0, \omega_1].$$

From the aforesaid, in view of continuity of $u^{(0,1)}$ in Ω_δ, it follows that

$$u^{(0,1)}(x, y) > y \ \text{ for } \ y \in \Big(\frac{1}{\pi + 2\pi k}, \frac{1}{2\pi k}\Big).$$

Similarly, one can show that

$$u^{(0,1)}(x, y) < -y \ \text{ for } \ y \in \Big(\frac{1}{2\pi(k+1)}, \frac{1}{\pi + 2\pi k}\Big).$$

However, the latter two inequalities imply that $u^{(0,1)}(x, y)$ is discontinuous along the lines $y = \frac{1}{\pi k}$ $(k = 1, 2, \dots)$. Thus we have proved that problem $(10),(11)$ has no solution in Ω_δ for any $\delta > 0$.

In conclusion, as examples, consider the following initial-boundary value problems.

Example 1.

$$u^{(2,2,1)} = u^2u^{(2,0,1)} + (u^{(1,1,0)})^4u^{(0,2,1)} - (u^{(0,0,1)})^6 + q\big(x_1, x_2, x_3, u, u^{(1,0,0)}, u^{(0,1,0)}, u^{(1,1,0)}\big), \quad (15)$$

$$u(0, x_2, x_3) = 0, \quad u(\omega_1, x_2, x_3) = 0; \quad u(x_1, 0, x_3) = 0, \quad u(x_1, \omega_2, x_3) = 0;$$
$$u(x_1, x_2, 0) = \psi(x_1, x_2). \quad (16)$$

Let the function q be continuous. Then, by Corollary 4 from [2] and Theorem 2, there exists $\delta \in (0, \omega_3]$ such that in the set $\Omega_\delta = [0, \omega_1] \times [0, \omega_2] \times [0, \delta]$ problem $(15), (16)$ has a at least one solution. Moreover, if the function q is locally Lipschitz continuous with respect to the phase variables, then problem $(15), (16)$ is uniquely solvable.

Example 2.

$$u^{(2,2,1)} = u^{(2,0,1)} + (u^{(2,0,1)})^5 + u^{(0,2,1)} - u^{(0,0,1)} + q\big(x_1, x_2, x_3, u, u^{(1,0,0)}, u^{(0,1,0)}, u^{(1,1,0)}\big), \quad (17)$$

$$u^{(i,0,0)}(0, x_2, x_3) = u^{(i,0,0)}(\omega_1, x_2, x_3); \quad u^{(0,i,0)}(x_1, 0, x_3) = u^{(0,i,0)}(x_1, \omega_2, x_3) \ \ (i = 0, 1);$$
$$u(x_1, x_2, 0) = \psi(x_1, x_2). \quad (18)$$

Let the function q be continuous. Then, by Corollary 5 from [2] and Theorem 2, there exists $\delta \in (0, \omega_3]$ such that in the set $\Omega_\delta = [0, \omega_1] \times [0, \omega_2] \times [0, \delta]$ problem $(17), (18)$ has a at least one solution. Moreover, if the function q is locally Lipschitz continuous with respect to the phase variables, then problem $(17), (18)$ is uniquely solvable.

Example 3. Let p_α be smooth functions, q be a continuous function, $\mathbf{m} = (m_1, \dots, m_n, 0)$ and $\mathbf{1}_{n+1}) = (0, \dots, 0)$. For the equation

$$u^{(2\mathbf{m}+\mathbf{1}_{n+1})} = \sum_{\alpha<\mathbf{m}} \big(p_\alpha(x, u)u^{(\alpha+\mathbf{1}_{n+1})}\big)^{(\alpha)} + q\big(\mathbf{x}, \mathcal{D}^{\mathbf{2m}\text{ - }\mathbf{1}}[u]\big). \quad (19)$$

consider the initial-boundary value problems with the Dirichlet and periodic boundary conditions

$$u^{(k\mathbf{1}_i)}(x_1, \dots, 0, \dots, x_{n+1}) = 0, \quad u^{(k\mathbf{1}_i)}(x_1, \dots, \omega_i, \dots, x_{n+1}) = 0$$
$$(k = 0, \dots, m_i - 1;\ i = 1, \dots, n); \quad u(x_1, \dots, x_n, 0) = \varphi(x), \quad (20)$$

and

$$u^{(k\mathbf{1}_i)}(x_1, \dots, 0, \dots, x_{n+1}) = u^{(k\mathbf{1}_i)}(x_1, \dots, \omega_i, \dots, x_{n+1})$$
$$(k = 0, \dots, 2m_i - 1;\ \ i = 1, \dots, n); \quad u(x_1, \dots, x_n, 0) = \varphi(x). \quad (21)$$

Let $(-1)^{\|\mathbf{m}\|+\|\alpha\|}p_\alpha \leq 0$ $((-1)^{\|\mathbf{m}\|+\|\alpha\|}p_\alpha < 0)$ for $\alpha < \mathbf{m}$. Then, by Theorem 2 from [2] and Theorem 2, there exists $\delta \in (0, \omega_{n+1}]$ such that in the set $\Omega_\delta = [0, \omega_1] \times \cdots \times [0, \omega_n] \times [0, \delta]$ problem $(19), (20)$ (problem $(19), (21)$) has a at least one solution. Moreover, if the function q is locally Lipschitz continuous with respect to the phase variables, then problem $(19), (20)$ (problem $(19), (21)$) is uniquely solvable.

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