III-Posed Multi Dimensional Boundary Value Problems for Linear Hyperbolic Equations of Higher Order

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Let \( m_1, \ldots, m_n \) be positive integers. In the \( n \)-dimensional box \( \Omega = [0, \omega_1] \times \cdots \times [0, \omega_n] \) for the linear hyperbolic equation

\[
    u^{(m)} = \sum_{\alpha < m} p_\alpha(x)u^{(\alpha)} + q(x)
\]

consider the boundary conditions

\[
    h_{ik}(u(x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n)) = \varphi_{ik}(\hat{x}_i) \quad \text{for} \quad \hat{x}_i \in \Omega_i \quad (k = 1, \ldots, m_i; \ i = 1, \ldots, n). \tag{2}
\]

Here \( x = (x_1, \ldots, x_n) \), \( \hat{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \), \( \Omega_i = [0, \omega_1] \times \cdots \times [0, \omega_{i-1}] \times [0, \omega_{i+1}] \times \cdots \times [0, \omega_n] \), \( m = (m_1, \ldots, m_n) \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \hat{m}_i = m - m_i \) and \( m_i = (0, \ldots, m_i, \ldots, 0) \) are multi-indices,

\[
    u^{(\alpha)}(x) = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},
\]

\( p_\alpha \in C(\Omega) \) \((\alpha < m)\), \( q \in C(\Omega) \), \( h_{ik} : C^{m_i-1}([0, \omega_i]) \to \mathbb{R} \) \((k = 1, \ldots, m_i; \ i = 1, \ldots, n)\) are bounded linear functionals, and \( \varphi_{ik} \in C^m(\Omega_i) \) \((k = 1, \ldots, m_i; \ i = 1, \ldots, n)\). Furthermore, it is assumed that the functions \( \varphi_{ik} \) satisfy the following consistency conditions:

\[
    h_{ik}(\varphi_{jl})(\hat{x}_{ij}) = h_{jl}(\varphi_{ik})(\hat{x}_{ij}) \quad (k = 1, \ldots, m_i; \ l = 1, \ldots, m_j; \ i, j = 1, \ldots, n),
\]

where \( \hat{x}_{ij} = x - \hat{x}_i - \hat{x}_j \).

By a solution of problem (1), (2) we understand a classical solution, i.e., a function \( u \in C^m(\Omega) \) satisfying equation (1) and boundary conditions (2).

Along with problem (1), (2) consider its corresponding homogeneous problem

\[
    u^{(m)} = \sum_{\alpha < m} p_\alpha(x)u^{(\alpha)}, \tag{10}
\]

\[
    h_{ik}(u(x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n)) = 0 \quad \text{for} \quad \hat{x}_i \in \Omega_i \quad (k = 1, \ldots, m_i; \ i = 1, \ldots, n). \tag{20}
\]

We make use of following notations and definitions.

- \( \text{supp} \alpha = \{ i : \alpha_i > 0 \} \), \( \| \alpha \| = |\alpha_1| + \cdots + |\alpha_n| \).
- \( \alpha = (\alpha_1, \ldots, \alpha_n) < \beta = (\beta_1, \ldots, \beta_n) \iff \alpha_i \leq \beta_i \ (i = 1, \ldots, n) \) and \( \alpha \neq \beta \).
- \( \alpha = (\alpha_1, \ldots, \alpha_n) \leq \beta = (\beta_1, \ldots, \beta_n) \iff \alpha < \beta, \) or \( \alpha = \beta \).
- \( 0 = (0, \ldots, 0), \ 1 = (1, \ldots, 1), \ 1_i = (0, \ldots, 0, 1, 0, \ldots, 0). \)
- \( \Xi = \{ \sigma : 0 < \sigma < 1 \} \).
- \( \hat{\alpha} = m - \alpha. \) If \( \sigma \in \Xi, \) then \( \hat{\sigma} = 1 - \sigma. \)
- $\mathbf{m}_\sigma = (\sigma_1 m_1, \ldots, \sigma_n m_n)$. It is clear that $\hat{\mathbf{m}}_\sigma = \mathbf{m} - \mathbf{m}_\sigma = \mathbf{m}_{\tilde{\sigma}}$.

- $\mathbf{x}_\sigma = (\sigma_1 x_1, \ldots, \sigma_n x_n)$. $\mathbf{x}_\sigma$ will be identified with $(x_{i_1}, \ldots, x_{i_l})$, as well as the set $\Omega_\sigma = [0, \sigma_1 \omega_1] \times \cdots \times [0, \sigma_n \omega_n]$ will be identified with the set $[0, \omega_{i_1}] \times \cdots \times [0, \omega_{i_l}]$, where $\{i_1, \ldots, i_l\} = \text{supp } \sigma$.

- $C^m(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{R}$, having continuous partial derivatives $u^{(\alpha)}$, $\alpha \leq \mathbf{m}$, with the norm

$$\|u\|_{C^m(\Omega)} = \sum_{\alpha \leq \mathbf{m}} \|u^{(\alpha)}\|_{C(\Omega)}.$$ 

- $\tilde{C}^m(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{R}$, having continuous partial derivatives $u^{(\alpha)}$, $\alpha < \mathbf{m}$, with the norm

$$\|u\|_{\tilde{C}^m(\Omega)} = \sum_{\alpha < \mathbf{m}} \|u^{(\alpha)}\|_{C(\Omega)}.$$ 

Let $\sigma \in \Xi$. In the domain $\Omega_\sigma$ consider the homogeneous boundary value problem depending on the parameter $\tilde{x}_\sigma \in \Omega_{\tilde{\sigma}}$

$$v^{(\mathbf{m}_\sigma)} = \sum_{\alpha < \mathbf{m}_\sigma} p_{\alpha + \hat{\mathbf{m}}_\sigma} (x) v^{(\alpha)}, \quad (3_{\sigma})$$

$$h_{ik}(v(x_1, \ldots, x_{i_l}; \bullet, x_{i_1+i_l}, \ldots, x_n)) = 0 \quad (k = 1, \ldots, m_i; \ i \in \text{supp } \sigma). \quad (4_{\sigma})$$

Problem $(3_{\sigma}), (4_{\sigma})$ is called an associated problem of level $l = \|\sigma\|$. 

**Theorem 1.** Let all of the coefficients of equation $(1)$ be constants, and let for some $\sigma \in \Xi$ associated problem $(3_{\sigma}), (4_{\sigma})$ be ill-posed. Furthermore, let

$$p_{\alpha + \beta} + p_{\alpha + \hat{\mathbf{m}}_\sigma} p_{\mathbf{m}_\sigma + \beta} = 0 \quad \text{for } 0 < \alpha < \mathbf{m}_\sigma, \ 0 < \beta < \hat{\mathbf{m}}_\sigma.$$

Then for solvability of problem $(1), (2)$ it is necessary that for every $l \in \text{supp } \tilde{\sigma}$ and $j \in \{1, \ldots, m_l\}$ the problem

$$v^{(\mathbf{m}_\sigma)} = \sum_{\alpha < \mathbf{m}_\sigma} p_{\alpha + \hat{\mathbf{m}}_\sigma} v^{(\alpha)} + Q_{lj}(\tilde{x}_l), \quad (5_{lj})$$

$$h_{ik}(v(x_1, \ldots, x_{i_l}; \bullet, x_{i_1+i_l}, \ldots, x_n)) = \Psi_{ik}^{lj}(\tilde{x}_{il}) \quad (k = 1, \ldots, m_i; \ i \in \text{supp } \sigma), \quad (6_{lj})$$

where

$$Q_{lj}(\tilde{x}_l) = (p_0 + p_{\hat{\mathbf{m}}_\sigma} p_{\mathbf{m}_\sigma}) \varphi^{(\hat{\mathbf{m}}_\sigma)}(\tilde{x}_l) + h_{lj}(q)(\tilde{x}_l)$$

and

$$\Psi_{ik}^{lj}(\tilde{x}_{il}) = h_{ij}(\varphi^{(\hat{\mathbf{m}}_\sigma)}(\tilde{x}_{il})) - \sum_{\beta < \hat{\mathbf{m}}_\sigma} p_{\mathbf{m}_\sigma + \beta} \varphi^{(\beta)}(\tilde{x}_l)$$

is solvable.

**Remark 1.** Solvability of ill-posed nonhomogenous associated problem $(5_{lj}), (6_{lj})$ is in fact additional consistency condition between the boundary values $\varphi_{ik}$, the coefficients $p_{\alpha}$ and the free term $q$. These are necessary conditions of solvability and they do not guarantee solvability of problem $(1), (2)$ even if the homogeneous problem $(1_0), (2_0)$ has only the trivial solution.
Indeed, consider the periodic problem
\[
\begin{align*}
  u^{(1,1,1)} &= \cos^2 x_1 u - q(x_1), & (7) \\
  u(\pi, x_2, x_3) &= u(0, x_2, x_3), & \quad u(x_1, \pi, x_3) = u(x_1, 0, x_3), & \quad u(x_1, x_2, \pi) = u(x_1, x_2, 0), & (8)
\end{align*}
\]

where \( q \) is a continuous function such that \( q(\pi) = q(0) \). Problem (7), (8) is ill-posed, and its corresponding homogeneous problem has only the trivial solution. Furthermore, for problem (7), (8) all consistency conditions hold, Therefore, due to uniqueness, the only possible solution of problem (7), (8) should be
\[
u(x_1) = \frac{q(x_1)}{\cos^2 x_1}.
\]

On the other hand, it is clear that problem (7), (8) has a solution if and only if
\[
q(x_1) = \cos^2 x_1 \tilde{q}(x_1),
\]

where \( \tilde{q} \in C^1([0, \pi]) \). In particular, if \( q(x_1) \equiv 1 \), then problem (7), (8) has no solution despite the fact that all coefficients of equation (7) and boundary data are analytic functions.

In the rectangle \( \Omega = [0, \omega_1] \times [0, \omega_2] \) consider the problem
\[
\begin{align*}
  u^{(2m_1,2m_2)} &= p(x_1, x_2) u + q(x_1, x_2), & (9) \\
  u^{(j-1,0)}(x_1, x_2) - u^{(j-1,0)}(0, x_2) &= \varphi_j(x_2) \quad (j = 1, \ldots, 2m_1), & (10) \\
  u^{(0,k-1)}(x_1, x_2) - u^{(0,k-1)}(x_1, 0) &= \psi_k(x_1) \quad (k = 1, \ldots, 2m_2).
\end{align*}
\]

**Theorem 2.** Let \( p, q \in \tilde{C}^{2m_1,2m_2}(\Omega), \varphi_j \in C^{2m_2}([0, \omega_2]) \quad (j = 1, \ldots, 2m_1-1), \varphi_{2m_1} \in C^{4m_2}([0, \omega_2]), \psi_k \in C^{2m_1}([0, \omega_1]) \quad (k = 1, \ldots, 2m_2-1), \psi_{2m_2} \in C^{4m_1}([0, \omega_1]), \)
\[
p^{(j-1,0)}(x_1, x_2) - p^{(j-1,0)}(0, x_2) = 0 \quad (j = 1, \ldots, 2m_1), \]
\[
p^{(0,k-1)}(x_1, x_2) - p^{(0,k-1)}(x_1, 0) = 0 \quad (k = 1, \ldots, 2m_2),
\]

and let
\[
(-1)^{m_1+m_2} \int_0^\omega_1 p(s, x_2) \, ds < 0, \quad (-1)^{m_1+m_2} \int_0^\omega_2 p(x_1, t) \, dt < 0 \quad \text{for} \quad (x_1, x_2) \in \Omega.
\]

Then problem (9), (10) is solvable if and only if
\[
\begin{align*}
  \int_0^\omega_1 \left[ \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \left( p^{(0,k-i)}(s, 0) \psi_{i+1}(s) + q^{(0,k)}(s, \omega_2) - q^{(0,k)}(s, 0) \right) \right] \, ds \\
  &= \varphi^{(2m_2+k)}_{2m_1}(\omega_2) - \varphi^{(2m_2+k)}_{2m_1}(0) \quad (k = 0, \ldots, 2m_2 - 1), \quad (13) \\
  \int_0^\omega_2 \left[ \sum_{i=0}^{j} \frac{j!}{i!(j-i)!} \left( p^{(j-i,0)}(0, t) \varphi_{i+1}(t) + q^{(j,0)}(\omega_1, t) - q^{(j,0)}(0, t) \right) \right] \, dt \\
  &= \psi^{(2m_1+j)}_{2m_2}(\omega_1) - \psi^{(2m_1+j)}_{2m_2}(0) \quad (j = 0, \ldots, 2m_1 - 1).
\end{align*}
\]
Moreover, if the equalities (13) and (14) hold, problem (9), (10) has a unique solution \( u \) admitting the estimate
\[
\|u\|_{C^{2m_1,2m_2}(\Omega)} \leq M \left( \sum_{j=1}^{2m_1-1} \|\varphi_j\|_{C^{2m_2}([0,\omega_2])} + \|\varphi_{2m_2}\|_{C^{4m_2}([0,\omega_2])} + \sum_{k=1}^{2m_2-1} \|\psi_k\|_{C^{2m_1}([0,\omega_1])} + \|\psi_{2m_2}\|_{C^{4m_1}([0,\omega_1])} + \|q\|_{C^{2m_1,2m_2}(\Omega)} \right),
\]
where \( M \) is a positive constant independent of \( q \).

**Remark 2.** Estimate (15) for a solution of problem (9), (10) is sharp, and regularity requirements on functions \( \varphi_k, \psi_j, p \) and \( q \) cannot be relaxed. For the sake of comparison, for equation (9) consider the Dirichlet boundary conditions
\[
\begin{align*}
u^{(j-1,0)}(\omega_1, x_2) &= 0, & u^{(j-1,0)}(0, x_2) &= 0 \quad (j = 1, \ldots, m_1), \\
u^{(0,k-1)}(x_1, \omega_2) &= 0, & u^{(0,k-1)}(x_1, 0) &= 0 \quad (k = 1, \ldots, m_2).
\end{align*}
\]

Unlike to problem (9), (10), by Theorem 2 from [1], problem (9), (16) is well-posed and its solution \( u \) admits the estimate
\[
\|u\|_{C^{2m_1,2m_2}(\Omega)} \leq M\|q\|_{C(\Omega)},
\]
where \( M \) is a positive constant independent of \( q \).

This clearly demonstrates that ill-posedness of associated problems (3\( \sigma \)), (4\( \sigma \)) not only creates additional consistency conditions, but also increases regularity requirements on coefficients of the equation and the boundary data.

In order to better illustrate the affect of ill-posedness of associated problems on the regularity of solutions to problem (1), (2), consider the following examples.

**Example 1.** Let \( \mathcal{E} = \{m_\sigma \mid \sigma \in \Xi \} \), \( p_\alpha \) be constants \( (\alpha \in \mathcal{E}) \), and let \( p_0 \) and \( q \) be continuous functions such that
\[
\begin{align*}
p_0(x_1, \ldots, x_i + \omega_i, \ldots, x_n) &= p_0(x_1, \ldots, x_i, \ldots, x_n) \quad (i = 1, \ldots, n), \\
q(x_1, \ldots, x_i + \omega_i, \ldots, x_n) &= q(x_1, \ldots, x_i, \ldots, x_n) \quad (i = 1, \ldots, n).
\end{align*}
\]

For the equation
\[
u^{(2m)} = \sum_{\alpha \in \mathcal{E}} p_\alpha u^{(2\alpha)} + p_0(x)u + q(x)
\]
consider the periodic problem
\[
u^{(k_1)}(x_1, \ldots, 0, \ldots, x_n) = u^{(k_1)}(x_1, \ldots, \omega_i, \ldots, x_n) \quad (k = 0, \ldots, 2m_i - 1; \ i = 1, \ldots, n).
\]

Assume that \((-1)^{\|\mathbf{m}\| + \|\alpha\|} p_\alpha < 0 \) for \( \alpha \neq m_n, p_{m_n} = 0 \), and
\[
(-1)^{\|\mathbf{m}\|} p_0(x) < 0, \quad x \in \Omega.
\]

Then for \( \sigma = 1_n \) the associated problem (3\( \sigma \)), (4\( \sigma \)) (problem of level \( n - 1 \)) has a nontrivial solution. As a result, problem (19), (20) is ill-posed. It is solvable if \( p_0, q \in C^{m_n}(\Omega) \) and its unique solution \( u \) admits the estimates
\[
\|u\|_{C^{2m_n}(\Omega)} \leq M\|q\|_{C(\Omega)}
\]
and
\[
\|u\|_{C^{2m}(\Omega)} \leq M\|q\|_{C^{2m_n}}.
\]
Example 2. Let $m_1 = m_2 = \cdots = m_n = m$, $p_0$ and $q$ satisfy (17), (18) and (21). The problem

$$u^{(2m)} = (-1)^{mn+m-1} \sum_{i=1}^{n} u^{(2m)} + p_0(x) u + q(x),$$

$$u^{(k_1)}(x_1, \ldots, 0, \ldots, x_n) = u^{(k_1)}(x_1, \ldots, \omega_i, \ldots, x_n) \quad (k = 0, \ldots, 2m - 1; \ i = 1, \ldots, n).$$

is ill-posed because all of its associated problems are ill-posed. As a result, problem (22), (23) is solvable if $p_0, q \in C^{2m}(\Omega)$ and its unique solution $u$ admits the estimates

$$\|u\|_{C^{2m}(\Omega)} \leq M \|q\|_{C^{2m}},$$

and

$$\|u\|_{C^{m,\gamma}(\Omega)} \leq M \|q\|_{C^{\gamma}(\Omega)},$$

where $\gamma \in (0, 1)$, and $C^{k,\gamma}(\Omega)$ is the space of $k$ times continuously differentiable functions whose $k$th derivative is Hölder continuous with the exponent $\gamma$.

References

