# Ill-Posed Multi Dimensional Boundary Value Problems for Linear Hyperbolic Equations of Higher Order

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Let $m_1,\dots,m_n$ be positive integers. In the $n$-dimensional box $\Omega=[0,\omega_1]\times\cdots\times[0,\omega_n]$ for the linear hyperbolic equation

$$u^{(\mathbf{m})}=\sum_{\boldsymbol{\alpha}<\mathbf{m}}p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}+q(\mathbf{x}) \tag{1}$$

consider the boundary conditions

$$h_{ik}\big(u(x_1,\dots,x_{i-1},\bullet,x_{i+1},\dots,x_n)\big)=\varphi_{ik}(\widehat{\mathbf{x}}_i)\ \text{ for }\ \widehat{\mathbf{x}}_i\in\Omega_i\ \ (k=1,\dots,m_i;\ \ i=1,\dots,n). \tag{2}$$

Here $\mathbf{x}=(x_1,\dots,x_n)$, $\widehat{\mathbf{x}}_i=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$, $\Omega_i=[0,\omega_1]\times\cdots\times[0,\omega_{i-1}]\times[0,\omega_{i+1}]\times\cdots\times[0,\omega_n]$, $\mathbf{m}=(m_1,\dots,m_n)$, $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)$, $\widehat{\mathbf{m}}_i=\mathbf{m}-\mathbf{m}_i$ and $\mathbf{m}_i=(0,\dots,m_i,\dots,0)$ are multi-indices,

$$u^{(\boldsymbol{\alpha})}(\mathbf{x})=\frac{\partial^{\alpha_1+\cdots+\alpha_n}u(\mathbf{x})}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\,,$$

$p_{\boldsymbol{\alpha}}\in C(\Omega)$ $(\boldsymbol{\alpha}<\mathbf{m})$, $q\in C(\Omega)$, $h_{ik}:C^{m_i-1}([0,\omega_i])\to\mathbb{R}$ $(k=1,\dots,m_i;\ i=1,\dots,n)$ are bounded linear functionals, and $\varphi_{ik}\in C^{\widehat{\mathbf{m}}_i}(\Omega_i)$ $(k=1,\dots,m_i;\ i=1,\dots,n)$. Furthermore, it is assumed that the functions $\varphi_{ik}$ satisfy the following consistency conditions:

$$h_{ik}(\varphi_{jl})(\widehat{\mathbf{x}}_{ij})\equiv h_{jl}(\varphi_{ik})(\widehat{\mathbf{x}}_{ij})\ \ (k=1,\dots,m_i;\ \ l=1,\dots,m_j;\ \ i,j=1,\dots,n),$$

where $\widehat{\mathbf{x}}_{ij}=\mathbf{x}-\widehat{\mathbf{x}}_i-\widehat{\mathbf{x}}_j$.

By a solution of problem (1), (2) we understand a classical solution, i.e., a function $u\in C^{\mathbf{m}}(\Omega)$ satisfying equation (1) and boundary conditions (2).

Along with problem (1), (2) consider its corresponding homogeneous problem

$$u^{(\mathbf{m})}=\sum_{\boldsymbol{\alpha}<\mathbf{m}}p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}, \tag{$1_0$}$$

$$h_{ik}\big(u(x_1,\dots,x_{i-1},\bullet,x_{i+1},\dots,x_n)\big)=0\ \text{ for }\ \widehat{\mathbf{x}}_i\in\Omega_i\ \ (k=1,\dots,m_i;\ \ i=1,\dots,n). \tag{$2_0$}$$

We make use of following notations and definitions.

- $\operatorname{supp}\boldsymbol{\alpha}=\{i\mid\ \alpha_i>0\}$, $\|\boldsymbol{\alpha}\|=|\alpha_1|+\cdots+|\alpha_n|$.
- $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)<\boldsymbol{\beta}=(\beta_1,\dots,\beta_n)\iff\alpha_i\le\beta_i$ $(i=1,\dots,n)$ and $\boldsymbol{\alpha}\neq\boldsymbol{\beta}$.
- $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)\le\boldsymbol{\beta}=(\beta_1,\dots,\beta_n)\iff\boldsymbol{\alpha}<\boldsymbol{\beta}$, or $\boldsymbol{\alpha}=\boldsymbol{\beta}$.
- $\mathbf{0}=(0,\dots,0)$, $\mathbf{1}=(1,\dots,1)$, $\mathbf{1}_i=(0,\dots,0,1,0,\dots,0)$.
- $\boldsymbol{\Xi}=\{\boldsymbol{\sigma}\mid\ \mathbf{0}<\boldsymbol{\sigma}<\mathbf{1}\}$.
- $\widehat{\boldsymbol{\alpha}}=\mathbf{m}-\boldsymbol{\alpha}$. If $\boldsymbol{\sigma}\in\boldsymbol{\Xi}$, then $\widehat{\boldsymbol{\sigma}}=\mathbf{1}-\boldsymbol{\sigma}$.

- $\mathbf{m}_{\boldsymbol{\sigma}} = (\sigma_1 m_1, \dots, \sigma_n m_n)$. It is clear that $\widehat{\mathbf{m}}_{\boldsymbol{\sigma}} = \mathbf{m} - \mathbf{m}_{\boldsymbol{\sigma}} = \mathbf{m}_{\widehat{\boldsymbol{\sigma}}}$.
- $\mathbf{x}_{\boldsymbol{\sigma}} = (\sigma_1 x_1, \dots, \sigma_n x_n)$. $\mathbf{x}_{\boldsymbol{\sigma}}$ will be identified with $(x_{i_1}, \dots, x_{i_l})$, as well as the set $\Omega_{\boldsymbol{\sigma}} = [0, \sigma_1\omega_1] \times \cdots \times [0, \sigma_n\omega_n]$ will be identified with the set $[0, \omega_{i_1}] \times \cdots \times [0, \omega_{i_l}]$, where $\{i_1, \dots, i_l\} = \operatorname{supp} \boldsymbol{\sigma}$.
- $C^{\mathbf{m}}(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{R}$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$, $\boldsymbol{\alpha} \leq \mathbf{m}$, with the norm
$$\|u\|_{C^{\mathbf{m}}(\Omega)} = \sum_{\boldsymbol{\alpha} \leq \mathbf{m}} \|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$
- $\widehat{C}^{\mathbf{m}}(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{R}$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$, $\boldsymbol{\alpha} < \mathbf{m}$, with the norm
$$\|u\|_{C^{\mathbf{m}}(\Omega)} = \sum_{\boldsymbol{\alpha} < \mathbf{m}} \|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

Let $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$. In the domain $\Omega_{\boldsymbol{\sigma}}$ consider the homogeneous boundary value problem depending on the parameter $\widehat{\mathbf{x}}_{\boldsymbol{\sigma}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$
$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha} + \widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{x}) v^{(\boldsymbol{\alpha})}, \tag{$3_{\boldsymbol{\sigma}}$}$$
$$h_{ik}\big(v(x_1, \dots, x_{i_1}, \bullet, x_{i+1}, \dots, x_n)\big) = 0 \ \ (k = 1, \dots, m_i; \ \ i \in \operatorname{supp} \boldsymbol{\sigma}). \tag{$4_{\boldsymbol{\sigma}}$}$$
Problem $(3_{\boldsymbol{\sigma}}), (4_{\boldsymbol{\sigma}})$ is called an associated problem of level $l = \|\boldsymbol{\sigma}\|$.

**Theorem 1.** *Let all of the coefficients of equation* (1) *be constants, and let for some* $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$ *associated problem* $(3_{\boldsymbol{\sigma}}), (4_{\boldsymbol{\sigma}})$ *be ill-posed. Furthermore, let*
$$p_{\boldsymbol{\alpha} + \boldsymbol{\beta}} + p_{a + \widehat{\mathbf{m}}_{\boldsymbol{\sigma}}} p_{\mathbf{m}_{\boldsymbol{\sigma}} + \boldsymbol{\beta}} = 0 \ \ \text{for} \ \ \mathbf{0} < \boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}}, \ \mathbf{0} < \boldsymbol{\beta} < \widehat{\mathbf{m}}_{\boldsymbol{\sigma}}.$$
*Then for solvability of problem* (1), (2) *it is necessary that for every* $l \in \operatorname{supp} \widehat{\boldsymbol{\sigma}}$ *and* $j \in \{1, \dots, m_l\}$ *the problem*
$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha} + \widehat{\mathbf{m}}_{\boldsymbol{\sigma}}} v^{(\boldsymbol{\alpha})} + Q_{lj}(\widehat{\mathbf{x}}_l), \tag{$5_{lj}$}$$
$$h_{ik}\big(v(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n)\big) = \Psi^{lj}_{ik}(\widehat{\mathbf{x}}_{il}) \quad (k = 1, \dots, m_i; \ \ i \in \operatorname{supp} \boldsymbol{\sigma}), \tag{$6_{lj}$}$$
*where*
$$Q_{lj}(\widehat{\mathbf{x}}_l) = (p_{\mathbf{0}} + p_{\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}} p_{\mathbf{m}_{\boldsymbol{\sigma}}}) \varphi_{lj}^{(\widehat{\mathbf{m}}_{\boldsymbol{\sigma}})}(\widehat{\mathbf{x}}_l) + h_{lj}(q)(\widehat{\mathbf{x}}_l)$$
*and*
$$\Psi^{lj}_{ik}(\widehat{\mathbf{x}}_{il}) = h_{lj}\big(\varphi_{ik}^{(\widehat{\mathbf{m}}_{\boldsymbol{\sigma}})}\big)(\widehat{\mathbf{x}}_{il}) - \sum_{\boldsymbol{\beta} < \widehat{\mathbf{m}}_{\boldsymbol{\sigma}}} p_{\mathbf{m}_{\boldsymbol{\sigma}} + \boldsymbol{\beta}} \varphi_{lj}^{(\boldsymbol{\beta})}(\widehat{\mathbf{x}}_l)$$
*is solvable.*

**Remark 1.** Solvability of ill-posed nonhomogenous associated problem $(5_{lj}), (6_{lj})$ is in fact additional consistency condition between the boundary values $\varphi_{ik}$, the coefficients $p_{\boldsymbol{\alpha}}$ and the free term $q$. These are necessary conditions of solvability and they do not guarantee solvability of problem (1), (2) even if the homogeneous problem $(1_0), (2_0)$ has only the trivial solution.

Indeed, consider the periodic problem

$$u^{(1,1,1)} = \cos^2 x_1 u - q(x_1), \tag{7}$$

$$u(\pi, x_2, x_3) = u(0, x_2, x_3), \quad u(x_1, \pi, x_3) = u(x_1, 0, x_3), \quad u(x_1, x_2, \pi) = u(x_1, x_2, 0), \tag{8}$$

where $q$ is a continuous function such that $q(\pi) = q(0)$. Problem $(7), (8)$ is ill-posed, and its corresponding homogeneous problem has only the trivial solution. Furthermore, for problem $(7), (8)$ all consistency conditions hold, Therefore, due to uniqueness, the only possible solution of problem $(7), (8)$ should be

$$u(x_1) = \frac{q(x_1)}{\cos^2 x_1}\,.$$

On the other hand, it is clear that problem $(7), (8)$ has a solution if and only if

$$q(x_1) = \cos^2 x_1 \widetilde{q}(x_1),$$

where $\widetilde{q} \in C^1([0, \pi])$. In particular, if $q(x_1) \equiv 1$, then problem $(7), (8)$ has no solution despite the fact that all coefficients of equation (7) and boundary data are analytic functions.

In the rectangle $\Omega = [0, \omega_1] \times [0, \omega_2]$ consider the problem

$$u^{(2m_1, 2m_2)} = p(x_1, x_2)\, u + q(x_1, x_2), \tag{9}$$

$$\begin{gathered} u^{(j-1,0)}(\omega_1, x_2) - u^{(j-1,0)}(0, x_2) = \varphi_j(x_2) \ \ (j = 1, \dots, 2m_1), \\ u^{(0,k-1)}(x_1, \omega_2) - u^{(0,k-1)}(x_1, 0) = \psi_k(x_1) \ \ (k = 1, \dots, 2m_2). \end{gathered} \tag{10}$$

**Theorem 2.** *Let* $p,\, q \in \widehat{C}^{2m_1, 2m_2}(\Omega)$, $\varphi_j \in C^{2m_2}([0, \omega_2])$ $(j = 1, \dots, 2m_1 - 1)$, $\varphi_{2m_1} \in C^{4m_2}([0, \omega_2])$, $\psi_k \in C^{2m_1}([0, \omega_1])$ $(k = 1, \dots, 2m_2 - 1)$, $\psi_{2m_2} \in C^{4m_1}([0, \omega_1])$,

$$\begin{gathered} p^{(j-1,0)}(\omega_1, x_2) - p^{(j-1,0)}(0, x_2) = 0 \ \ (j = 1, \dots, 2m_1), \\ p^{(0,k-1)}(x_1, \omega_2) - p^{(0,k-1)}(x_1, 0) = 0 \ \ (k = 1, \dots, 2m_2), \end{gathered} \tag{11}$$

*and let*

$$(-1)^{m_1+m_2} \int\limits_0^{\omega_1} p(s, x_2)\, ds < 0, \quad (-1)^{m_1+m_2} \int\limits_0^{\omega_2} p(x_1, t)\, dt < 0 \ \ \textit{for} \ \ (x_1, x_2) \in \Omega. \tag{12}$$

*Then problem* $(9), (10)$ *is solvable* ***if and only if***

$$\begin{aligned} \int\limits_0^{\omega_1} \bigg[ \sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!}\, p^{(0,k-i)}(s, 0) \psi_{i+1}(s) + q^{(0,k)}(s, \omega_2) - q^{(0,k)}(s, 0) \bigg]\, ds \\ = \varphi_{2m_1}^{(2m_2+k)}(\omega_2) - \varphi_{2m_1}^{(2m_2+k)}(0) \ \ (k = 0, \dots, 2m_2 - 1), \end{aligned} \tag{13}$$

$$\begin{aligned} \int\limits_0^{\omega_2} \bigg[ \sum_{i=0}^{j} \frac{j!}{i!\,(j-i)!}\, p^{(j-i,0)}(0, t) \varphi_{i+1}(t) + q^{(j,0)}(\omega_1, t) - q^{(j,0)}(0, t) \bigg]\, dt \\ = \psi_{2m_2}^{(2m_1+j)}(\omega_1) - \psi_{2m_2}^{(2m_1+j)}(0) \ \ \ (j = 0, \dots, 2m_1 - 1). \end{aligned} \tag{14}$$

*Moreover, if the equalities* (13) *and* (14) *hold, problem* (9), (10) *has a unique solution $u$ admitting the estimate*

$$\|u\|_{C^{2m_1,2m_2}(\Omega)} \leq M\bigg(\sum_{j=1}^{2m_1-1}\|\varphi_j\|_{C^{2m_2}([0,\omega_2])} + \|\varphi_{2m_2}\|_{C^{4m_2}([0,\omega_2])} + \sum_{k=1}^{2m_2-1}\|\psi_k\|_{C^{2m_1}([0,\omega_1])} + \|\psi_{2m_2}\|_{C^{4m_1}([0,\omega_1])} + \|q\|_{\widehat{C}^{2m_1,2m_2}(\Omega)}\bigg), \quad (15)$$

*where $M$ is a positive constant independent of $q$.*

**Remark 2.** Estimate (15) for a solution of problem (9), (10) is sharp, and regularity requirements on functions $\varphi_k$, $\psi_j$, $p$ and $q$ cannot be relaxed. For the sake of comparison, for equation (9) consider the Dirichlet boundary conditions

$$\begin{aligned} &u^{(j-1,0)}(\omega_1,x_2)=0, \quad u^{(j-1,0)}(0,x_2)=0 \ \ (j=1,\dots,m_1),\\ &u^{(0,k-1)}(x_1,\omega_2)=0, \quad u^{(0,k-1)}(x_1,0)=0 \ \ (k=1,\dots,m_2). \end{aligned} \quad (16)$$

Unlike to problem (9), (10), by Theorem 2 from [1], problem (9), (16) is well-posed and its solution $u$ admits the estimate

$$\|u\|_{C^{2m_1,2m_2}(\Omega)} \leq M\|q\|_{C(\Omega)},$$

where $M$ is a positive constant independent of $q$.

This clearly demonstrates that ill-posedness of associated problems $(3_{\boldsymbol\sigma}), (4_{\boldsymbol\sigma})$ not only creates additional consistency conditions, but also increases regularity requirements on coefficients of the equation and the boundary data.

In order to better illustrate the affect of ill-posedness of associated problems on the regularity of solutions to problem (1), (2), consider the following examples.

**Example 1.** Let $\mathcal{E}=\{\mathbf{m}_{\boldsymbol\sigma} \mid \ \boldsymbol\sigma\in\boldsymbol\Xi\}$, $p_{\boldsymbol\alpha}$ be constants ($\boldsymbol\alpha\in\mathcal{E}$), and let $p_0$ and $q$ be continuous functions such that

$$p_0(x_1,\dots,x_i+\omega_i,\dots,x_n)\equiv p_0(x_1,\dots,x_i,\dots,x_n) \ \ (i=1,\dots,n), \quad (17)$$

$$q(x_1,\dots,x_i+\omega_i,\dots,x_n)\equiv q(x_1,\dots,x_i,\dots,x_n) \ \ (i=1,\dots,n). \quad (18)$$

For the equation

$$u^{(2\mathbf{m})}=\sum_{\boldsymbol\alpha\in\mathcal{E}} p_{\boldsymbol\alpha}\, u^{(2\boldsymbol\alpha)}+p_0(\mathbf{x})u+q(\mathbf{x}) \quad (19)$$

consider the periodic problem

$$u^{(k\mathbf{1}_i)}(x_1,\dots,0,\dots,x_n)=u^{(k\mathbf{1}_i)}(x_1,\dots,\omega_i,\dots,x_n) \ \ (k=0,\dots,2m_i-1; \ \ i=1,\dots,n). \quad (20)$$

Assume that $(-1)^{\|\mathbf{m}\|+\|\boldsymbol\alpha\|}p_{\boldsymbol\alpha}<0$ for $\boldsymbol\alpha\neq\mathbf{m}_n$, $p_{\mathbf{m}_n}=0$, and

$$(-1)^{\|\mathbf{m}\|}p_0(\mathbf{x})<0, \ \ \mathbf{x}\in\Omega. \quad (21)$$

Then for $\boldsymbol\sigma=\mathbf{1}_n$ the associated problem $(3_{\boldsymbol\sigma}), (4_{\boldsymbol\sigma})$ (problem of level $n-1$) has a nontrivial solution. As a result, problem (19), (20) is ill-posed. It is solvable if $p_0,q\in C^{\mathbf{m}_n}(\Omega)$ and its unique solution $u$ admits the estimates

$$\|u\|_{C^{\widehat{2\mathbf{m}_n}}(\Omega)} \leq M\|q\|_{C(\Omega)}$$

and

$$\|u\|_{C^{2\mathbf{m}}(\Omega)} \leq M\|q\|_{C^{2\mathbf{m}_n}}.$$

**Example 2.** Let $m_1 = m_2 = \cdots = m_n = m$, $p_0$ and $q$ satisfy (17), (18) and (21). The problem

$$u^{(2\mathbf{m})} = (-1)^{mn+m-1} \sum_{i=1}^{n} u^{(2\mathbf{m}_i)} + p_0(\mathbf{x})u + q(\mathbf{x}), \tag{22}$$

$$u^{(k\mathbf{1}_i)}(x_1, \dots, 0, \dots, x_n) = u^{(k\mathbf{1}_i)}(x_1, \dots, \omega_i, \dots, x_n) \;\; (k = 0, \dots, 2m-1; \;\; i = 1, \dots, n). \tag{23}$$

is ill-posed because all of its associated problems are ill-posed. As a result, problem (22), (23) is solvable if $p_0, q \in C^{2\mathbf{m}}(\Omega)$ and its unique solution $u$ admits the estimates

$$\|u\|_{C^{2\mathbf{m}}(\Omega)} \leq M \|q\|_{C^{2\mathbf{m}}},$$

and

$$\|u\|_{C^{m,\gamma}(\Omega)} \leq M \|q\|_{C^{\gamma}(\Omega)},$$

where $\gamma \in (0,1)$, and $C^{k,\gamma}(\Omega)$ is the space of $k$ times continuously differentiable functions whose $k$th derivative is Hölder continuous with the exponent $\gamma$.

## References


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