The Boundary Value Problem for One Class of Semilinear Partial Differential Equations

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In the Euclidean space $\mathbb{R}^{n+1}$ of the variables $x=(x_1,x_2,\dots,x_n)$ and t we consider the semilinear equation of the type

$$\frac{\partial^{4k}u}{\partial t^{4k}}-\Delta u+f(u)=F, \tag{1}$$

where $f:\mathbb{R}\to\mathbb{R}$ is a given continuous function, $F=F(x,t)$ is a given, and $u=u(x,t)$ is an unknown real functions, k is a natural number, $\Delta:=\sum\limits_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$, $n\geq 2$.

For the equation (1) we consider the boundary value problem: find in the cylindrical domain $D_T=\Omega\times(0,T)$, where Ω is an open Lipschitz domain in $\mathbb{R}^n$, a solution $u(x,t)$ of that equation according to the boundary conditions

$$u\big|_{\Gamma}=0, \tag{2}$$

$$\frac{\partial^i u}{\partial t^i}\Big|_{\Omega_0\cup\Omega_T}=0,\ \ i=1,\dots,2k-1, \tag{3}$$

where $\Gamma:=\partial\Omega\times(0,T)$ is the lateral face of the cylinder D_T, $\Omega_0:\ x\in\Omega,\ t=0$ and $\Omega_T:\ x\in\Omega,\ t=T$ are the lower and upper bases of this cylinder, respectively.

Denote by $C^{2,4k}(\overline{D_T},\partial D_T)$ the space of functions u continuous in $\overline{D_T}$ and having continuous partial derivatives $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_i\partial x_j}$, $\frac{\partial^l u}{\partial t^l}$ in $\overline{D_T}$, $i,j=1,\dots,n$; $l=1,\dots,4k$. Let

$$C_0^{2,4k}(\overline{D_T},\partial D_T):=\Big\{u\in C^{2,4k}(\overline{D_T},\partial D_T):\ u\big|_{\Gamma}=0,\ \frac{\partial^i u}{\partial t^i}\Big|_{\Omega_0\cup\Omega_T}=0\ \ i=1,\dots,2k-1\Big\}.$$

Let $u\in C_0^{2,4k}(\overline{D_T},\partial D_T)$ be a classical solution of the problem (1), (2), (3). Multiplying both parts of the equation (1) by an arbitrary function $\varphi\in C_0^{2,4k}(\overline{D_T},\partial D_T)$ and integrating the obtained equation by parts over the domain D_T, we obtain

$$\int\limits_{D_T}\bigg[\frac{\partial^{2k}u}{\partial t^{2k}}\,\frac{\partial^{2k}\varphi}{\partial t^{2k}}+\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\,\frac{\partial \varphi}{\partial x_i}\bigg]\,dx\,dt+\int\limits_{D_T}f(u)\varphi\,dx\,dt=\int\limits_{D_T}F\varphi\,dx\,dt. \tag{4}$$

Introduce the Hilbert space $W_0^{1,2k}(D_T)$ as a completion with respect to the norm

$$\|u\|^2_{W_0^{1,2k}(D_T)}=\int\limits_{D_T}\bigg[u^2+\sum_{i=1}^{2k}\Big(\frac{\partial^i u}{\partial t^i}\Big)^2+\sum_{i=1}^{n}\Big(\frac{\partial u}{\partial x_i}\Big)^2\bigg]\,dx\,dt$$

of the classical space $C_0^{2,4k}(\overline{D_T},\partial D_T)$.

We take the equality (4) as a basis for our definition of the weak generalized solution u of the problem (1), (2), (3): the function $u\in W_0^{1,2k}(D_T)$ is said to be a weak generalized solution of the problem (1), (2), (3) if for any function $\varphi\in W_0^{1,2k}(D_T)$ the integral equality (4) is valid.

It is not difficult to verify that if the weak generalized solution of the problem (1), (2), (3) belongs to the class $C_0^{2,4k}(\overline{D_T}, \partial D_T)$, then it will also be a classical solution of this problem.

Below, on the function $f = f(u)$ we impose the following requirements

$$f \in C(\mathbb{R}), \ \ |f(u)| \leq M_1 + M_2|u|^\alpha, \ \ u \in \mathbb{R}, \tag{5}$$

where

$$0 \leq \alpha = const < \frac{n+1}{n-1}\,, \tag{6}$$

and

$$uf(u) \geq 0 \ \ \forall\, u \in \mathbb{R}. \tag{7}$$

Theorem. *Let the conditions* (5)–(7) *be fulfilled. Then for any* $F \in L_2(D)$ *the problem* (1), (2), (3) *has at least one weak generalized solution* $u \in W_0^{1,2k}(D_T)$.

Remark. Let us note that if along with the conditions (5)–(7) imposed on function f to demand that it is monotonous, then the solution $u \in W_0^{1,2k}(D_T)$ of the problem (1), (2), (3), the existence of which is stated in the theorem, is unique. As show the examples, when the conditions imposed on nonlinear function f are violated, then the problem (1), (2), (3) may not have a solution.

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