On the Relationships Between Stieltjes Type Integrals

Umi Mahnuna Hanung

Universitas Gadjah Mada, Indonesia
E-mail: hanungum@ugm.ac.id

Milan Tvrdý

Institute of Mathematics of the Czech Academy of Sciences, Praha, Czech Republic
E-mail: tvrdy@math.cas.cz

Integral equations of the form

$$x(t) = x(t_0) + \int_{t_0}^{t} \mathrm{d}[A]\,x = f(t) - f(t_0)$$

are natural generalizations of systems of linear differential equations. Their main goal is that they admit solutions which need not be absolutely continuous. Up to now such equations have been considered by several authors starting with J. Kurzweil [6] and T. H. Hildebrandt [3]. For further contributions see e.g. [1,5,8,9,11–14] and the references therein. These papers worked with several different concepts of the Stieltjes type integral like Young's (Hildebrandt), Kurzweil's (Kurzweil, Schwabik and Tvrdý), Dushnik's (Hönig) or Lebesgue's (Ashordia, Meng and Zhang). Thus an interesting question arises: what are the relationships between all these concepts?

It is known that (cf. [6, Theorem 1.2.1]) the Kurzweil–Stieltjes integral is in finite dimensional setting equivalent with the Perron–Stieltjes, while the relationship between the Perron–Stieltjes and the Lebesgue–Stieltjes integrals has been described already in [10, Theorem VI.8.1]. Furthermore, the relationship between the Young–Stieltjes and the Dushnik–Stieltjes integrals (DS) follows from [7, Theorem B]. Finally, the relationship between the Young–Stieltjes (YS) integral and the Kurzweil–Stieltjes (KS) one has been described in [11] and [12].

In this paper the symbols like $\mathbb{R}$, $\mathbb{N}$, $[a,b]$, (a,b), $\operatorname{var}_a^b f$ and $\|f\|_\infty$ have their usual and traditional meaning. For more details we refer to the preliminary version of the monograph [9]. In addition, recall that a finite ordered set $\boldsymbol{\alpha} = \{\alpha_0,\dots,\alpha_{\nu(P)}\}$ of points from $[a,b]$ is a division of $[a,b]$ if $a=\alpha_0 < \dots < \alpha_{\nu(P)} = b$. The couple of ordered sets $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$ is a partition of $[a,b]$ if $\boldsymbol{\alpha}$ is a division of $[a,b]$ and $\boldsymbol{\xi} = \{\xi_1,\dots,\xi_{\nu(P)}\}$ is such that $\xi_j \in [\alpha_{j-1},\alpha_j]$ for all j. If $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$ is a division of $[a,b]$, the elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\xi}$ are always denoted respectively as α_j and ξ_j. At the same time the number of elements of $\boldsymbol{\xi}$ is always denoted by $\nu(P)$. For functions $f,g\colon [a,b]\to\mathbb{R}$ and a partition $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$ of $[a,b]$ we set

$$S(P) = \sum_{j=1}^{\nu(P)} f(\xi_j)\big[g(\alpha_j)-g(\alpha_{j-1})\big]$$

and, if g is regulated,

$$S_Y(P) = \sum_{j=1}^{\nu(P)} \Big(f(\alpha_{j-1})\Delta^+ g(\alpha_{j-1}) + f(\xi_j)\big[g(\alpha_j-)-g(\alpha_{j-1}+)\big] + f(\alpha_j)\Delta^- g(\alpha_j)\Big).$$

- *YS integral* (Y)$\int\limits_a^b f\ \mathrm{d}g$ $\big($*DS integral* (D)$\int\limits_a^b f\ \mathrm{d}g\big)$ exists and equals $I \in \mathbb{R}$ if

$$\begin{gathered}\text{for every } \varepsilon > 0 \text{ there is a division } \boldsymbol{\alpha}_\varepsilon \text{ of } [a,b] \text{ such that}\\ |S_Y(P) - I| < \varepsilon \quad \big(\text{or } \ |S(P) - I| < \varepsilon\big)\\ \text{holds for all partitions } P = (\boldsymbol{\alpha}, \boldsymbol{\xi}) \text{ of } [a,b] \text{ such that } \boldsymbol{\alpha} \supset \boldsymbol{\alpha}_\varepsilon \text{ and}\\ \alpha_{j-1} < \xi_j < \alpha_j \ \text{ for all } \ j \in \{1, \dots, \nu(\boldsymbol{\alpha})\}.\end{gathered}$$

- *KS integral* (K)$\int\limits_a^b f\ \mathrm{d}g$ exists and has a value $I \in \mathbb{R}$ if

$$\begin{gathered}\text{for every } \varepsilon > 0 \text{ there exists a function } \delta_\varepsilon : [a,b] \to (0,1) \text{ such that}\\ |I - S(P)| < \varepsilon\\ \text{holds for all partitions } P = (\boldsymbol{\alpha}, \boldsymbol{\xi}) \text{ of } [a,b] \text{ such that}\\ [\alpha_{j-1}, \alpha_j] \subset [\xi_j - \delta_\varepsilon(\xi_j), \xi_j + \delta_\varepsilon(\xi_j)].\end{gathered}$$

It is not difficult to see that for all the three integrals under consideration the estimates

$$\bigg|\int_a^b f\ \mathrm{d}g\bigg| \le \|f\|_\infty \operatorname{var}_a^b g \ \text{ and } \ \bigg|\int_a^b f\ \mathrm{d}g\bigg| \le \big(|g(a)| + |g(b)| + \operatorname{var}_a^b g\big)\|f\|_\infty$$

are true whenever the corresponding integral exists. Indeed, it is enough to show that analogous inequalities are satisfied by the sums $|S(P)|$ and $|S_Y(P)|$ for arbitrary compatible partitions. To see how to prove the latter inequality for the YS integral, it helps to observe that the relation

$$\begin{aligned}f(\alpha)\big[g(\alpha+)-g(\alpha)\big] + f(\xi)\big[g(\beta-)-g(\alpha+)\big] + f(\beta)\big[g(\beta)-g(\beta-)\big]&\\ = \big[f(\alpha)-f(\xi)\big]g(\alpha+) + \big[f(\xi)-f(\beta)\big]g(\beta-) + f(\beta)g(\beta) - f(\alpha)g(\alpha)&\end{aligned}$$

is true if $f : [a,b] \to \mathbb{R}$, g is regulated on $[a,b]$, and $a \le \alpha \le \xi \le \beta \le b$.

Next convergence results are also true for all the three integrals under consideration.

Proposition. *Let $f : [a,b] \to \mathbb{R}$, $g \in \mathrm{BV}([a,b])$ and let the sequence $\{f_n\}$ tend uniformly to f on $[a,b]$. Then:*

- *If all the integrals $\int\limits_a^b f_n\ \mathrm{d}g$, $n \in \mathbb{N}$, exist, then the integral $\int\limits_a^b f\ \mathrm{d}g$ exists, too, and*

$$\lim_{n\to\infty} \int_a^b f_n\ \mathrm{d}g = \int_a^b f\ \mathrm{d}g.$$

- *If all the integrals $\int\limits_a^b f\ \mathrm{d}g_n$, $n \in \mathbb{N}$, exist, then the integral $\int\limits_a^b f\ \mathrm{d}g$ exists, too, and*

$$\lim_{n\to\infty} \int_a^b f\ \mathrm{d}g_n = \int_a^b f\ \mathrm{d}g.$$

For KS integrals the proofs are available in Section 6.3 *of* [9]*. Their ideas are pretty transparent and applicable also to YS and DS integrals: First, we notice that in both situations the sequences of integrals depending on* n *are Cauchy sequences in* $\mathbb{R}$ *and therefore they have a limit* $I \in \mathbb{R}$*. Further, uniform convergence and the above estimates implies that the limit integrals exist and equals* I*.*

Now we can formulate and justify our main result.

Theorem. *Suppose* f *and* g *are regulated on* $[a,b]$ *and at least one of them has a bounded variation on* $[a,b]$*. Then all the integrals* $(\mathrm{K})\int\limits_a^b f\,\mathrm{d}g$*,* $(\mathrm{Y})\int\limits_a^b f\,\mathrm{d}g$ *and* $(\mathrm{D})\int\limits_a^b f\,\mathrm{d}g$ *exist and*

$$(\mathrm{K})\int\limits_a^b f\,\mathrm{d}g = (\mathrm{Y})\int\limits_a^b f\,\mathrm{d}g = f(b)g(b) - f(a)g(a) - (\mathrm{D})\int\limits_a^b g\,\mathrm{d}f. \tag{1}$$

Sketch of the proof:

- It is not difficult to verify that the equalities (1) hold for every $f:[a,b]\to\mathbb{R}$ whenever g is a finite step function and, similarly, they are also true whenever g is regulated and f is a finite step function. (For analogous arguments see Examples 6.3.1 in [9].)
- Approximate uniformly regulated functions by sequences of finite step functions.
- Applying convergence results stated in Proposition, it is easy to complete the proof.

References

[1] M. Ashordia, On the Opial type criterion for the well-posedness of the Cauchy problem for linear systems of generalized ordinary differential equations. *Math. Bohem.* **141** (2016), no. 2, 183–215.

[2] B. Dushnik, On the Stieltjes integral. *Thesis (Ph.D.) – University of Michigan. ProQuest LLC, Ann Arbor, MI*, 1931.

[3] T. H. Hildebrandt, On systems of linear differentio-Stieltjes-integral equations. *Illinois J. Math.* **3** (1959), 352–373.

[4] T. H. Hildebrandt, *Introduction to the Theory of Integration.* Pure and Applied Mathematics, Vol. XIII Academic Press, New York–London, 1963.

[5] Ch. S. Hönig, Volterra–Stieltjes integral equations. *Functional differential equations and bifurcation (Proc. Conf., Inst. Ciênc. Mat. São Carlos, Univ. São Paulo, São Carlos, 1979)*, pp. 173–216, Lecture Notes in Math., 799, Springer, Berlin, 1980.

[6] J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter. (Russian) *Czechoslovak Math. J.* **7 (82)** (1957), 418–449.

[7] J. S. MacNerney, An integration-by-parts formula. *Bull. Amer. Math. Soc.* **69** (1963), 803–805.

[8] G. Meng and M. Zhang, Measure differential equations I. Continuity of solutions in measures with weak topology.
`https://www.researchgate.net/profile/Meirong_Zhang/publication/228839316_Measure_differential_equations_I_Continuity_of_solutions_in_measures_with_weak_topology/links/00b7d52abb88809580000000/Measure-differential-equations-I-Continuity-of-solutions-in-measures-with-weak-topology.pdf`

[9] G. Antunes Monteiro, A. Slavík and M. Tvrdý, *Kurzweil–Stieltjes Integral and its Applications.* `http://users.math.cas.cz/ tvrdy/ks-dekk.pdf`.

[10] S. Saks, *Théorie de l'intégrale.* (French) Monografie Matematyczne 2. Seminarium Matematyczne Uniwersytetu Warszawskiego, Instytut Matematyczny PAN, Warszawa, 1933; English translation: *Theory of the integral.* G. E. Stechert & Co., New York, 1937.

[11] Š. Schwabik, On the relation between Young's and Kurzweil's concept of Stieltjes integral. *Časopis Pěst. Mat.* **98** (1973), 237–251.

[12] Š. Schwabik, On a modified sum integral of Stieltjes type. *Časopis Pěst. Mat.* **98** (1973), 274–277.

[13] Š. Schwabik, *Generalized Ordinary Differential Equations.* Series in Real Analysis, 5. World Scientific Publishing Co., Inc., River Edge, NJ, 1992.

[14] Š. Schwabik, M. Tvrdý and O. Vejvoda, *Differential and Integral Equations. Boundary Value Problems and Adjoints.* D. Reidel Publishing Co., Dordrecht–Boston, Mass.–London, 1979.