Asymptotics of Solutions of Second-Order Differential Equations with Regularly and Rapidly Varying Nonlinearities

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Consider the differential equation

$$y'' = \sum_{i=1}^{m} \alpha_i p_i(t) \varphi_i(y), \tag{1}$$

where $\alpha_i \in \{-1, 1\}$ $(i = \overline{1,m}\,)$, $p_i : [a, \omega[\, \to]0, +\infty[$ $(i = \overline{1,m}\,)$ are continuous functions, $-\infty < a < \omega \le +\infty$, $\varphi_i : \Delta_{Y_0} \to]0, +\infty[$ $(i = \overline{1,m}\,)$, where Δ_{Y_0} is some one-sided neighborhood of the point Y_0, Y_0 is equal either to 0 or to $\pm\infty$, are continuous functions for $i = \overline{1,l}$ and twice continuously differentiable for $i = \overline{l+1,m}$, so that

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi_i(\lambda y)}{\varphi_i(y)} = \lambda^{\sigma_i} \;\; (i = \overline{1,l}\,) \;\text{ for any }\; \lambda > 0, \tag{2}$$

$$\varphi_i'(y) \neq 0 \;\text{ as }\; y \in \Delta_{Y_0}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \varphi_i(y) \in \{0, +\infty\}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi_i''(y)\varphi_i(y)}{\varphi_i'^2(y)} = 1 \;\; (i = \overline{l+1,m}\,). \tag{3}$$

It follows from the conditions (2) and (3) that φ_i $(i = \overline{1,l}\,)$ are regularly varying functions, as $y \to Y_0$, of orders σ_i and φ_i $(i = \overline{l+1,m}\,)$ are rapidly varying functions, as $y \to Y_0$ (see [5, Introduction, pp. 2, 4]).

Definition. A solution y of the differential equation (1) is called $P_\omega(Y_0, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on some interval $[t_0, \omega[\, \subset [a, \omega[$ and satisfies the following conditions

$$\lim_{t \uparrow \omega} y(t) = Y_0, \quad \lim_{t \uparrow \omega} y'(t) = \begin{cases} \text{either} & 0, \\ \text{or} & \pm\infty, \end{cases} \quad \lim_{t \uparrow \omega} \frac{y'^2(t)}{y''(t)y(t)} = \lambda_0.$$

There have been known the results of the asymptotic behavior of $P_\omega(Y_0, \lambda_0)$-solutions of differential equation (1) in case when there is only one item with a regularly or rapidly varying nonlinearity on the right-hand side of the equation (1) (see [1–3]). The case $l = m$ has been also investigated when all nonlinearities on the right-hand side of differential equation (1) are regularly varying functions (see [4]). The general case, when, in addition to items with regularly varying nonlinearities there are items with rapidly varying nonlinearities on the right-hand side of the equation (1), has not been studied yet.

In this paper, for $\lambda_0 \in \mathbb{R} \setminus \{0; 1\}$ the existence conditions of $P_\omega(Y_0, \lambda_0)$-solutions of the differential equation (1) and asymptotic representations, as $t \uparrow \omega$, of such solutions and their first-order derivatives, are established in case when on each such solution the right-hand side of equation is equivalent, as $t \uparrow \omega$, to the s-th item, that is when

$$\lim_{t \uparrow \omega} \frac{p_i(t)\varphi_i(y(t))}{p_s(t)\varphi_s(y(t))} = 0 \;\text{ for all }\; i \in \{1, \ldots, m\} \setminus \{s\}. \tag{4}$$

Let

$$\Delta_{Y_0}=\Delta_{Y_0}(b), \text{ where } \Delta_{Y_0}(b)=\begin{cases}[b,Y_0[& \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0,\\ ]Y_0,b] & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0,\end{cases}$$

and the number b satisfy the inequalities

$$|b|<1 \text{ as } Y_0=0 \quad \text{and} \quad b>1 \ (b<-1) \text{ as } Y_0=+\infty \ (Y_0=-\infty).$$

We set

$$\nu_0=\operatorname{sign} b, \quad \nu_1=\begin{cases}1 & \text{if } \Delta_{Y_0}(b)=[b,Y_0[\,,\\ -1 & \text{if } \Delta_{Y_0}(b)=]Y_0,b],\end{cases} \quad \mu_i=\operatorname{sign}\varphi_i'(y) \ \ (i=\overline{l+1,m}\,),$$

$$\pi_\omega(t)=\begin{cases}t & \text{if } \omega=+\infty,\\ t-\omega & \text{if } \omega<+\infty,\end{cases} \quad J_i(t)=\int\limits_{A_i}^{t}\pi_\omega(\tau)p_i(\tau)\,d\tau,$$

$$H_i(y)=\int\limits_{B_i}^{y}\frac{ds}{\varphi_i(s)}\,, \quad Z_i=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}(b)}} H_i(y) \ \ (i=\overline{1,m}\,),$$

where

$$A_i=\begin{cases}a & \text{if } \int\limits_a^\omega \pi_\omega(\tau)p_i(\tau)\,d\tau=\pm\infty,\\ \omega & \text{if } \int\limits_a^\omega \pi_\omega(\tau)p_i(\tau)\,d\tau=const,\end{cases} \qquad B_i=\begin{cases}b & \text{if } \int\limits_b^{Y_0}\frac{dy}{\varphi_i(y)}=\pm\infty,\\ Y_0 & \text{if } \int\limits_b^{Y_0}\frac{dy}{\varphi_i(y)}=const.\end{cases}$$

Theorem 1. *Let $\lambda_0\in\mathbb{R}\setminus\{0;1\}$ and $\sigma_s\neq1$ for some $s\in\{1,\dots,l\}$. For the existence of $P_\omega(Y_0,\lambda_0)$-solutions of the equation* (1)*, satisfied the limit relations* (4)*, it is necessary that the inequalities*

$$\alpha_s\nu_0\lambda_0>0, \quad \nu_0\nu_1\lambda_0(\lambda_0-1)\pi_\omega(t)>0 \ \text{ as } t\in]a,\omega[\tag{5}$$

and conditions

$$\alpha_s(\lambda_0-1)\lim_{t\uparrow\omega}J_s(t)=Z_s, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_s'(t)}{J_s(t)}=\frac{(1-\sigma_s)\lambda_0}{\lambda_0-1}\,, \tag{6}$$

$$\lim_{t\uparrow\omega}\frac{p_i(t)\varphi_i(H_s^{-1}(\alpha_s(\lambda_0-1)J_s(t)))}{p_s(t)\varphi_s(H_s^{-1}(\alpha_s(\lambda_0-1)J_s(t)))}=0 \ \text{ for all } i\in\{1,\dots,l\}\setminus\{s\}, \tag{7}$$

$$\lim_{t\uparrow\omega}\frac{p_i(t)\varphi_i(H_s^{-1}(\alpha_s(\lambda_0-1)J_s(t)(1+\delta_i)))}{p_s(t)\varphi_s(H_s^{-1}(\alpha_s(\lambda_0-1)J_s(t)))}=0 \ \text{ for all } i\in\{l+1,\dots,m\}$$

hold, where δ_i are arbitrary numbers of a one-sided neighborhood of zero. Moreover, for each of such solutions the following asymptotic representations hold

$$y(t)=H_s^{-1}\big(\alpha_s(\lambda_0-1)J_s(t)\big)[1+o(1)] \ \text{ at } t\uparrow\omega, \tag{8}$$

$$y'(t)=\frac{\lambda_0H_s^{-1}(\alpha_s(\lambda_0-1)J_s(t))}{(\lambda_0-1)\pi_\omega(t)}\,[1+o(1)] \ \text{ at } t\uparrow\omega. \tag{9}$$

Theorem 2. *Let $\lambda_0 \in \mathbb{R} \setminus \{0;1\}$ and $\sigma_s \neq 1$ for some $s \in \{1,\dots,l\}$, the conditions (5)–(7) hold and*
$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(H_s^{-1}(\alpha_s(\lambda_0-1)J_s(t)(1+u)))}{p_s(t)\varphi_s(H_s^{-1}(\alpha_s(\lambda_0-1)J_s(t)))} = 0 \ \ \text{for all} \ \ i \in \{l+1,\dots,m\}$$
uniformly with respect to $u \in [-\delta,\delta]$ for any $0<\delta<1$. Let also one of the following two conditions hold
$$\text{or} \ \ \lambda_0 \neq -1, \quad \text{or} \ \ \lambda_0 = -1 \ \ \text{and} \ \ \sigma_s < 1.$$
Then the differential equation (1) has $P_\omega(Y_0,\lambda_0)$-solutions that admit the asymptotic representations (8) and (9). Moreover, there is a one-parameter family of such solutions in case $\lambda_0(1-\sigma_s)<0$ and two-parameter one in case $\lambda_0(1-\sigma_s)>0$ and $\pi_\omega(t)(1-\lambda_0^2)<0$ as $t \in]a,\omega[$.

Besides the above-mentioned facts we also need the following auxiliary notations
$$J_{0i}(t) = \int_{A_i}^{t} \pi_\omega(\tau)p_{0i}(\tau)\,d\tau,$$
$$q_{0i}(t) = \frac{\alpha_i(\lambda_0-1)\pi_\omega^2(t)p_{0i}(t)\varphi_i(H_i^{-1}(\alpha_i(\lambda_0-1)J_{0i}(t)))}{H_i^{-1}(\alpha_i(\lambda_0-1)J_{0i}(t))},$$
$$G_{0i}(t) = \frac{y\varphi_i'(y)}{\varphi_i(y)}\bigg|_{y=H_i^{-1}(\alpha_i(\lambda_0-1)J_{0i}(t))}, \quad \psi_{0i}(t) = \int_{t_0}^{t} \frac{|G_{0i}(\tau)|^{\frac12}\,d\tau}{\pi_\omega(\tau)},$$
$$\Phi_{0i}(t) = \frac{y\big(\frac{\varphi_i'(y)}{\varphi_i(y)}\big)'}{\frac{\varphi_i'(y)}{\varphi_i(y)}}\Bigg|_{y=H_i^{-1}(\alpha_i(\lambda_0-1)J_{0i}(t))} \qquad (i=\overline{l+1,m}),$$
where $p_{0i} : [a,\omega[\to]0,+\infty[$ are continuous functions so that $p_{0i}(t) \sim p_i(t)$ as $t\uparrow\omega$, t_0 is some number of $[a,\omega[$.

Theorem 3. *Let $\lambda_0 \in \mathbb{R}\setminus\{0;1\}$ and for some $s \in \{l+1,\dots,m\}$ the conditions*
$$\frac{\varphi_s(y)\varphi_i'(y)}{\varphi_s'(y)\varphi_i(y)} = O(1) \ \ \text{as} \ \ y \to Y_0 \ \ (y \in \Delta_{Y_0}(b)) \ \ \text{for all} \ \ i \in \{l+1,\dots,m\} \tag{10}$$
hold. For the existence of $P_\omega(Y_0,\lambda_0)$-solutions of the equation (1) that admit the limit relations (4), it is necessary that for some continuous function $p_{0s} : [a,\omega[\to]0,+\infty[$ such that $p_{0s}(t) \sim p_i(t)$ as $t\uparrow\omega$ the conditions
$$\alpha_s\nu_0\lambda_0 > 0, \quad \alpha_s\mu_s(\lambda_0-1)J_{0s}(t) < 0 \ \ \text{at} \ \ t \in]a,\omega[, \tag{11}$$
$$\alpha_s(\lambda_0-1)\lim_{t\uparrow\omega} J_{0s}(t) = Z_s, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_{0s}'(t)}{J_{0s}(t)} = \pm\infty, \quad \lim_{t\uparrow\omega} q_{0s}(t) = \frac{\lambda_0}{\lambda_0-1}, \tag{12}$$
$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(H_s^{-1}(\alpha_s(\lambda_0-1)J_{0s}(t)))}{p_{0s}(t)\varphi_s(H_s^{-1}(\alpha_s(\lambda_0-1)J_{0s}(t)))} = 0 \ \ \text{for all} \ \ i \in \{1,\dots,m\}\setminus\{s\} \tag{13}$$
hold. Moreover, for each of such solutions the following asymptotic representations hold
$$y(t) = H_s^{-1}(\alpha_s(\lambda_0-1)J_{0s}(t))\Big[1+\frac{o(1)}{G_{0s}(t)}\Big] \ \ \text{at} \ \ t\uparrow\omega,$$
$$y'(t) = \frac{\lambda_0 H_s^{-1}(\alpha_s(\lambda_0-1)J_{0s}(t))}{(\lambda_0-1)\pi_\omega(t)}\,[1+o(1)] \ \ \text{at} \ \ t\uparrow\omega.$$

Theorem 4. *Let $\lambda_0 \in \mathbb{R} \setminus \{0;1\}$, for some $s \in \{l+1,\dots,m\}$ the function p_s might be represented in the form*

$$p_s(t) = p_{0s}(t)[1 + r_s(t)], \quad \textit{where} \quad \lim_{t\uparrow\omega} r_s(t) = 0,$$

$p_{0s} : [a,\omega[\to]0,+\infty[$ *is a continuously differentiable function,* $r_s : [a,\omega[\to]-1,+\infty[$ *is a continuous function, the conditions* (10)–(13) *hold and there exist finite or equal to infinity limits*

$$\gamma_s = \lim_{t\uparrow\omega} \Phi_{0s}(t), \quad \lim_{t\uparrow\omega} \pi_\omega(t) q'_{0s}(t), \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}(b)}} \frac{\big(\frac{\varphi'_s(y)}{\varphi_s(y)}\big)'}{\big(\frac{\varphi'_s(y)}{\varphi_s(y)}\big)^2}\sqrt{\Big|\frac{y\varphi'_s(y)}{\varphi_s(y)}\Big|}, \quad \lim_{t\uparrow\omega} \frac{\psi_{0s}(t)\psi''_{0s}(t)}{\psi'^2_{0s}(t)}\,.$$

Then

1) *if $\alpha_s\mu_s = 1$, the differential equation* (1) *has a one-parameter family of $P_\omega(Y_0,\lambda_0)$-solutions with asymptotic representations*

$$y(t) = H_s^{-1}(\alpha_s(\lambda_0 - 1)J_{0s}(t))\Big[1 + \frac{o(1)}{G_{0s}(t)}\Big] \quad \textit{at} \ \ t\uparrow\omega,$$
$$y'(t) = \frac{\lambda_0 H_s^{-1}(\alpha_s(\lambda_0 - 1)J_{0s}(t))}{(\lambda_0 - 1)\pi_\omega(t)}\Big[\frac{\lambda_0 - 1}{\lambda_0} q_{0s}(t) + |G_{0s}(t)|^{-\frac12} o(1)\Big] \quad \textit{at} \ \ t\uparrow\omega;$$

2) *if $\alpha_s\mu_s = -1$ and*

$$\gamma_s \neq \lim_{\lambda\to\lambda_0} \frac{(\lambda-1)(2-3\lambda)}{\lambda(5\lambda-4)}, \quad \lim_{t\uparrow\omega} \psi_{0s}(t)\Big[q_{0s}(t)[1 + r_s(t)] - \frac{\lambda_0}{\lambda_0 - 1}\Big] = 0,$$
$$\lim_{t\uparrow\omega} \psi^2_{0s}(t)\Big[\Big(\frac{\lambda_0}{\lambda_0 - 1} - q_{0s}(t)\Big) q_{0s}(t) + \frac{q_{0s}(t) r_s(t)}{\lambda_0 - 1} - \pi_\omega(t) q'_{0s}(t)\Big] = 0,$$
$$\lim_{t\uparrow\omega} \psi^2_{0s}(t) \sum_{\substack{i=1\\ i\neq s}}^{m} \frac{p_i(t)\varphi_i(H_s^{-1}(\alpha_s(\lambda_0 - 1)J_{0s}(t)))}{p_{0s}(t)\varphi_s(H_s^{-1}(\alpha_s(\lambda_0 - 1)J_{0s}(t)))} = 0,$$

the differential equation (1) *has a $P_\omega(Y_0,\lambda_0)$-solution with asymptotic at $t\uparrow\omega$ representations*

$$y(t) = H_s^{-1}(\alpha_s(\lambda_0 - 1)J_{0s}(t))\Big[1 + \frac{o(1)}{\psi_{0s}(t) G_{0s}(t)}\Big],$$
$$y'(t) = \frac{\lambda_0 H_s^{-1}(\alpha_s(\lambda_0 - 1)J_{0s}(t))}{(\lambda_0 - 1)\pi_\omega(t)}\Big[\frac{\lambda_0 - 1}{\lambda_0} q_{0s}(t) + \frac{o(1)}{\psi_{0s}(t)|G_{0s}(t)|^{\frac12}}\Big].$$

Moreover, there exists a two-parameter family of such solutions in case when

$$\beta\big(\lambda_0^2(5\gamma_s + 3) + \lambda_0(-4\gamma_s - 5) + 2\big) < 0 \ \ \textit{as} \ \ \gamma_s = const, \quad \frac{4}{5} < \lambda_0 < 1 \ \ \textit{as} \ \ \gamma_s = \pm\infty.$$

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