Asymptotic Properties of Some Class of Solutions of Second Order Differential Equations with Rapidly and Regularly Varying Nonlinearities

O. O. Chepok

Odessa I. I. Mechnikov National University, Odessa, Ukraine
E-mail: olachepek@ukr.net

Let us consider the differential equation

\[ y'' = \alpha_0 p(t) \varphi_0(y) \varphi_1(y'). \]  

(1)

In this equation \( \alpha_0 \in \{-1; 1\} \), functions \( p : [a, \omega[ \rightarrow ]0, +\infty[ \) \((-\infty < a < \omega \leq +\infty)\), and \( \varphi_i : \Delta_{Y_i} \rightarrow ]0, +\infty[ \) \((i \in \{0, 1\})\) are continuous, \( Y_i \in \{0, \pm\infty\} \), \( \Delta_{Y_i} \) is either an interval \( [y_i^0, y_i^1] \) or an interval \( ]Y_i, y_i^0[ \).

We also suppose that the function \( \varphi_1 \) is a regularly varying function of index \( \sigma_1 \) as \( y \rightarrow Y_1 \) \((y \in \Delta_{Y_1}) \) \([3, \text{pp. 10–15}]\), the function \( \varphi_0 \) is twice continuously differentiable on \( \Delta_{Y_0} \) and satisfies the conditions

\[ \varphi_0'(y) \neq 0 \text{ as } y \in \Delta_{Y_0}, \quad \lim_{y \rightarrow Y_0} \varphi_0(y) \in \{0, +\infty\}, \quad \lim_{y \rightarrow Y_0} \frac{\varphi_0(y) \varphi_0''(y)}{(\varphi_0'(y))^2} = 1. \]

The solution \( y \) of the equation (1), that is defined on the interval \( [t_0, \omega[ \subset [a, \omega[, \) is called \( P_\omega(Y_0, Y_1, \lambda_0)-\)solution \((-\infty \leq \lambda_0 \leq +\infty)\) if the following conditions take place

\[ y^{(i)} : [t_0, \omega[ \rightarrow \Delta_{Y_i}, \quad \lim_{t \rightarrow Y_0} y^{(i)}(t) = Y_i \quad (i = 0, 1), \quad \lim_{t \rightarrow \omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0. \]

The aim of the work is to find the necessary and sufficient conditions for the existence of \( P_\omega(Y_0, Y_1, \lambda_0)\)-solutions of the equation (1) if \( \lambda_0 \in R \setminus \{0; 1\} \), to find asymptotic representations of such solutions and its first order derivatives as \( t \uparrow \omega \).

**Definition 1.** We say that a slowly varying as \( z \rightarrow Y \) \((z \in \Delta_Y)\) function \( \theta : \Delta_Y \rightarrow ]0; +\infty[ \) satisfies the condition \( S \) as \( z \rightarrow Y \) if for any continuous differentiable normalized slowly varying as \( z \rightarrow Y \) \((z \in \Delta_Y)\) function \( L : \Delta_{Y_i} \rightarrow ]0; +\infty[ \) the following relation is valid

\[ \theta(zL(z)) = \theta(z)(1 + o(1)) \text{ as } z \rightarrow Y \quad (z \in \Delta_Y). \]

**Definition 2.** We say that a slowly varying as \( z \rightarrow Y \) \((z \in \Delta_Y)\) function \( L : \Delta_Y \rightarrow ]0; +\infty[ \) satisfies the condition \( S_1 \) as \( z \rightarrow Y \) if for any finite segment \([a; b] \subset ]0; +\infty[ \)

\[ \limsup_{z \rightarrow Y} \left| \ln |z| \cdot \left( \frac{L(\lambda z)}{L(z)} - 1 \right) \right| < +\infty \text{ for all } \lambda \in [a; b]. \]

\[ ^1 \text{If } Y_i = +\infty \text{ (} Y_i = -\infty \text{), we will take } y_i^0 > 0 \text{ or } y_i^0 < 0, \text{ respectively.} \]
Theorem 1. Let \( \sigma_1 \neq 1 \), the function \( \varphi_1 \) satisfy the condition \( S \), and the following limit relation be true

\[
\lim_{z \to Y_0} \left( \frac{\Phi_1'(z)}{\Phi_1(z)} \right)' = \gamma_0, \quad \gamma_0 \in R \setminus \{1, 0\}.
\]

The next conditions are necessary for the existence of \( P_{\omega}(Y_0, Y_1, \lambda_0) \)-solutions of the equation (1), if \( \lambda_0 \in R \setminus \{0, 1\} \):

\[
\pi_\omega(t) y_0^1 \lambda_0 (\lambda_0 - 1) > 0, \quad \pi_\omega(t) y_0^1 \alpha_0 (\lambda_0 - 1) > 0, \quad y_1^0 \cdot \lim_{t \to \omega} \pi_\omega(t) |y_1^0|^{1 - \sigma_1} = Y_1,
\]

\[
\lim_{t \to \omega} \Phi^{-1}_1(I_1(t)) = Y_0, \quad \lim_{t \to \omega} \frac{\pi_\omega(t) I_1'(t)}{I_1(t)} = \infty, \quad \lim_{t \to \omega} \frac{I_1'(t) \pi_\omega(t)}{\Phi_1'(\Phi^{-1}_1(I_1(t))) \Phi^{-1}_1(I_1(t))} = \frac{\lambda_0}{\lambda_0 - 1}.
\]
These conditions are also sufficient for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the equation (1) if
\[ I(t)I_1(t)\lambda_0(\sigma_1 - 1) > 0 \text{ as } t \in [a; \omega] \]
and the function $\frac{\pi_\omega(t)^{\frac{(2-\gamma_0)\lambda_0}{(1-\gamma_0)(\lambda_0-1)}} I_1(t)}{I(t)}$ is a normalized slowly varying function as $t \uparrow \omega$.

Moreover, for each such solution the following asymptotic representations take place as $t \uparrow \omega$
\[ \Phi_1(y(t)) = I_1(t)[1 + o(1)], \quad \frac{y'(t)\Phi_1'(y(t))}{\Phi_1(y(t))} = \frac{I_1'(t)}{I_1(t)}[1 + o(1)]. \]

Let us notice that the function $\Phi^{-1}(z) \cdot \frac{\Phi_1'(\Phi^{-1}(z))}{\Phi_1'(z)}$ is a slowly varying function as $z \to Z_1$.

If the condition (2) is not true, the following theorem takes place.

**Theorem 2.** Let for equation (1) $\sigma_1 \neq 1$, the function $\theta_1(z)$ satisfy the condition $S$ as $z \to Y_1$ ($z \in \Delta Y_1$), the function $\Phi^{-1}(z) \cdot \frac{\Phi_1'(\Phi^{-1}(z))}{\Phi_1'(z)}$ satisfy the condition $S_1$ as $z \to Z_1$. Then for existence of the equation’s (1) $P_\omega(Y_0, Y_1, \lambda_0)$-solutions, where $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$, it is necessary, and, if
\[ I(t)I_1(t)\lambda_0(\sigma_1 - 1) > 0 \text{ for } t \in ]b, \omega[, \]
and finite or infinite limits
\[ \lim_{t \uparrow \omega} \pi_\omega(t)F'(t) \text{ and } \lim_{t \uparrow \omega} \frac{\sqrt{[\pi_\omega(t)I_1'(t)]^2}}{\ln |I_1(t)|} \]
exist, sufficient the fulfilment of the following conditions
\[ \pi_\omega(t)y_1^0y_0\lambda_0(\lambda_0 - 1) > 0, \quad \pi_\omega(t)y_1^0\alpha_0(\lambda_0 - 1) > 0 \text{ as } t \in [a; \omega[, \]
\[ y_1^0 \cdot \lim_{t \uparrow \omega} |\pi_\omega(t)|^{\frac{1}{\lambda_0 - 1}} = Y_1, \quad \lim_{t \uparrow \omega} I_1(t) = Z_1, \quad \lim_{t \uparrow \omega} \frac{I_1''(t)I_1(t)}{(I_1'(t))^2} = 1, \quad \lim_{t \uparrow \omega} F(t) = \frac{\lambda_0 - 1}{\lambda_0}. \]

Moreover, for every such solution the following asymptotic representations as $t \uparrow \omega$ take place
\[ \Phi_1(y(t)) = I_1(t)[1 + o(1)], \quad \frac{\pi_\omega(t)y'(t)}{y(t)} = \frac{\lambda_0}{\lambda_0 - 1}[1 + o(1)]. \]

Note that if in the limit relation (2) $\gamma_0 = 1$, the function $\Phi^{-1}(z) \cdot \frac{\Phi_1'(\Phi^{-1}(z))}{\Phi_1'(z)}$ satisfies the condition $S_1$ as $z \to Z_1$.

The next example illustrates the obtained results of Theorem 1.

Let’s consider the following differential equation
\[ y'' = \frac{1}{4} t^{-3}L(t)e^{\frac{\gamma}{t^4} - t^8}|y|^3, \quad (3) \]
where $L : [2, +\infty[ \to [0, +\infty[, \quad \frac{\gamma}{\beta + \alpha} > 0, \quad \beta > 0, \quad \beta \neq 1 \text{ as } t \in [2, +\infty[.$

This is an equation of the form (1), where $a = 2, \alpha_0 = 1, p(t) = \frac{1}{4} t^{-3}L(t)e^{-t^8}, \varphi_0(y) = e^{\frac{\gamma}{t^4}}, \varphi_1(y) = |y|^3$.

Theorem 1 implies that the equation (3) has a one-parameter family of $P_{+\infty}(+\infty, +\infty, 2)$-solutions, and every such solution and the derivative of such solution satisfy the following asymptotic representations
\[ \frac{1}{y'(t)}e^{\frac{\gamma}{t^4}}y^4(t) = t^{-14}(L(t))^{-\frac{1}{2}}e^{\frac{\gamma}{t^8}}[1 + o(1)], \quad y'(t)y^3(t) = 2t^7[1 + o(1)] \text{ as } t \uparrow \omega. \]
To illustrate the results of Theorem 2, we consider the differential equation
\[
y'' = \psi(t) \exp\left(\exp(|y|^a) - \exp(t^d)\right)|y|^\sigma_0 |y|^{\sigma_1} \quad \text{as} \quad t \in [2, +\infty[. \tag{4}
\]

Here, \(\sigma_0, \sigma_1 \in R, \sigma_1 > 1, a, d \in [0, +\infty[, \) function \(\psi : [2, +\infty[ \to [0, +\infty[\) is continuous regularly varying on an infinity function of index \(\gamma, \gamma \in R.\)

This equation is an equation of the type (1), where
\[
\alpha_0 = 1, \quad p(t) = \psi(t) \exp(-\exp(t^d)), \quad \varphi_0(y) = |y|^\sigma_0 \exp(\exp(|y|^a)), \quad \varphi_1(y) = |y|^\sigma_1.
\]

We investigate the question of the existence and asymptotic behavior as \(t \to +\infty\) of \(P_{+\infty}(\infty, Y_1, \lambda_0)\)-solutions of the equation (4) for which \(\lambda_0 \in \mathbb{R} \setminus \{0, 1\}.\)

In this case
\[
\pi_\omega(t) = t, \quad \theta_1(y) = 1.
\]

So, the function \(\theta_1\) satisfies the condition \(S.\)

Taking into account the choice of \(B^0_{+\infty}\) as \(t \to +\infty,\) we have
\[
I(t) = |\lambda_0 - 1|^{-1/\sigma_1} \cdot y_1 \cdot \frac{\sigma_1 - 1}{d} \cdot t^{1-d+1/\sigma_1} \cdot |\psi(t)|^{1/\sigma_1} \cdot \exp\left(\frac{\exp(t^d)}{\sigma_1 - 1} - t^d\right) \left[1 + o(1)\right].
\]

At the same way as \(t \to +\infty\) we have
\[
I_1(t) = |\lambda_0 - 1|^{-1/\sigma_1} \cdot y_1^2 \cdot \left(\frac{\sigma_1 - 1}{d}\right)^2 \cdot t^{1-2d+1/\sigma_1} \cdot |\psi(t)|^{1/\sigma_1} \cdot \exp\left(\frac{\exp(t^d)}{\sigma_1 - 1} - 2t^d\right) \left[1 + o(1)\right].
\]

In addition, in this case, since \(Y_0 = \infty,\) taking into account the choice of \(A^0_{\infty},\) we obtain
\[
\Phi_0(y) = \frac{\sigma_1 - 1}{a} \cdot y^{\sigma_0/\sigma_1 + 1-a} \cdot \exp\left(\frac{\exp(|y|^a)}{\sigma_1 - 1} - |y|^a\right) \left[1 + o(1)\right] \quad \text{as} \quad y \to \infty.
\]

Similarly, we have
\[
\Phi_1(y) = \left(\frac{\sigma_1 - 1}{a}\right)^2 \cdot y^{\sigma_0/\sigma_1 + 1-2a} \cdot \exp\left(\frac{\exp(|y|^a)}{\sigma_1 - 1} - 2|y|^a\right) \left[1 + o(1)\right] \quad \text{as} \quad y \to \infty.
\]

At the same time,
\[
\Phi_0^{-1}(y) \cdot \Phi_0'(\Phi_0^{-1}(y)) \cdot \frac{\left(\sigma_1 - 1\right)^2}{a} \cdot \ln y \cdot \left(\ln \left(\left(\sigma_1 - 1\right) \ln y\right)\right)^{-\frac{\sigma_0}{\sigma_1 - 1}} \cdot \left[1 + o(1)\right] \quad \text{as} \quad y \to \infty.
\]

It means that condition \(S_1\) is satisfied.

Thus, all the conditions of Theorem 2 are satisfied. By virtue of this theorem, the equation (4) can have only \(P_{+\infty}(+\infty, +\infty, \frac{d}{d-a})\)-solutions of the class of \(P_{+\infty}(\infty, Y_1, \lambda_0)\)-solutions. From Theorem 2 it also follows that the equation (4) has one-parameter family of \(P_{+\infty}(+\infty, +\infty, \frac{d}{d-a})\)-solutions.

Also, taking into account the known asymptotic behavior of the function \(\Phi_0^{-1},\) it is easy to obtain that every \(P_{+\infty}(+\infty, +\infty, \frac{d}{d-a})\)-solution of the equation (4) and the derivative of such solution satisfy the following asymptotic representations
\[
y(t) = \left|\frac{\sigma_0}{\sigma_1 - 1 + 1 - 2a} \cdot \exp\left(\frac{\exp(|y|^a)}{\sigma_1 - 1} - 2|y|^a\right)\right|
\]
\[
\frac{\left(\sigma_1 - 1\right)^2}{a} \cdot \ln y \cdot \left(\ln \left(\left(\sigma_1 - 1\right) \ln y\right)\right)^{-\frac{\sigma_0}{\sigma_1 - 1}} \cdot \left[1 + o(1)\right] \quad \text{as} \quad t \to +\infty, \quad y(t) = \frac{y(t)}{t} \left[1 + o(1)\right] \quad \text{as} \quad t \to +\infty.
\]
References

