Asymptotic Properties of Some Class of Solutions of Second Order Differential Equations with Rapidly and Regularly Varying Nonlinearities

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Let us consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y'). \tag{1}$$

In this equation $\alpha_0 \in \{-1;1\}$, functions $p : [a,\omega[\to]0,+\infty[$ $(-\infty < a < \omega \le +\infty)$, and $\varphi_i : \Delta_{Y_i} \to]0,+\infty[$ $(i \in \{0,1\})$ are continuous, $Y_i \in \{0,\pm\infty\}$, Δ_{Y_i} is either an interval $[y_i^0, Y_i[$ [1] or an interval $]Y_i, y_i^0]$.

We also suppose that the function φ_1 is a regularly varying function of index σ_1 as $y \to Y_1$ $(y \in \Delta_{Y_1})$ ([3, pp. 10–15]), the function φ_0 is twice continuously differentiable on Δ_{Y_0} and satisfies the conditions

$$\varphi_0'(y) \neq 0 \ \text{ as } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi_0(y) \in \{0,+\infty\}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi_0(y)\varphi_0''(y)}{(\varphi_0'(y))^2} = 1.$$

The solution y of the equation (1), that is defined on the interval $[t_0,\omega[\subset [a,\omega[$, is called $P_\omega(Y_0,Y_1,\lambda_0)$-solution $(-\infty \le \lambda_0 \le +\infty)$ if the following conditions take place

$$y^{(i)} : [t_0,\omega[\to \Delta_{Y_i}, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i=0,1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0.$$

The aim of the work is to find the necessary and sufficient conditions for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of the equation (1) if $\lambda_0 \in R \setminus \{0;1\}$, to find asymptotic representations of such solutions and its first order derivatives as $t \uparrow \omega$.

Definition 1. We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $\theta : \Delta_Y \to]0;+\infty[$ satisfies the condition S as $z \to Y$ if for any continuous differentiable normalized slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $L : \Delta_{Y_i} \to]0;+\infty[$ the following relation is valid

$$\theta(zL(z)) = \theta(z)(1+o(1)) \ \text{ as } \ z \to Y \ \ (z \in \Delta_Y).$$

Definition 2. We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $L : \Delta_Y \to]0;+\infty[$ satisfies the condition S_1 as $z \to Y$ if for any finite segment $[a;b] \subset]0;+\infty[$

$$\limsup_{\substack{z\to Y\\ z\in\Delta_Y}} \left| \ln|z| \cdot \Big(\frac{L(\lambda z)}{L(z)} - 1\Big)\right| < +\infty \ \text{ for all } \ \lambda \in [a;b].$$

[1] If $Y_i = +\infty$ $(Y_i = -\infty)$, we will take $y_i^0 > 0$ or $y_i^0 < 0$, respectively.

Conditions S and S_1 are satisfied, for example, for the functions: $\ln|y|$, $|\ln|y||^{\mu}$ $(\mu \in R)$, $\ln\ln|y|$.

Let us introduce the following notations.

$$\pi_\omega(t)=\begin{cases} t & \text{if } \omega=+\infty,\\ t-\omega & \text{if } \omega<+\infty, \end{cases} \quad \theta_1(y)=\varphi_1(y)|y|^{-\sigma_1},$$

$$\Phi_0(y)=\int\limits_{A_\omega}^{y}|\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\,dz,\quad A_\omega=\begin{cases} y_0^0 & \text{if } \int\limits_{y_0^0}^{Y_0}|\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\,dz=\pm\infty,\\ Y_0 & \text{if } \int\limits_{y_0^0}^{Y_0}|\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\,dz=const, \end{cases}$$

$$\Phi_1(y)=\int\limits_{A_\omega}^{y}\frac{\Phi_0(\tau)}{\tau}\,d\tau,\quad Z_1=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi_1(y),\quad F(t)=\frac{\Phi_1^{-1}(I_1(t))\Phi_1'(\Phi_1^{-1}(I_1(t)))}{\pi_\omega(t)I_1'(t)}\,.$$

If $y_1^0\cdot\lim\limits_{t\uparrow\omega}|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}=Y_1$, we put

$$I(t)=|\lambda_0-1|^{\frac{1}{1-\sigma_1}}\cdot y_1^0\cdot\int\limits_{B_\omega^0}^{t}\Big|\pi_\omega(\tau)p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}y_1^0\big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau,$$

$$B_\omega^0=\begin{cases} b & \text{if } \int\limits_{b}^{\omega}\Big|\pi_\omega(\tau)p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}y_1^0\big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau=+\infty,\\ \omega & \text{if } \int\limits_{b}^{\omega}\Big|\pi_\omega(\tau)p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}y_1^0\big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau<+\infty, \end{cases}$$

$$I_1(t)=\int\limits_{B_\omega^1}^{t}\frac{\lambda_0I(\tau)}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau,\quad B_\omega^1=\begin{cases} b & \text{if } \int\limits_{b}^{\omega}\frac{\lambda_0I(\tau)}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau=\pm\infty,\\ \omega & \text{if } \int\limits_{b}^{\omega}\frac{\lambda_0|I(\tau)|}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau=const, \end{cases}$$

where $b\in[a;\omega[$ is chosen in such a way that $y_1^0|\pi_\omega(t))|^{\frac{1}{\lambda_0-1}}\in\Delta_{Y_1}$ as $t\in[b;\omega]$.

The following conclusions take place for the equation (1).

Theorem 1. *Let $\sigma_1\neq1$, the function φ_1 satisfy the condition S, and the following limit relation be true*

$$\lim_{\substack{z\to Y_0\\ z\in\Delta_{Y_0}}}\frac{\big(\frac{\Phi_1'(z)}{\Phi_1(z)}\big)''\big(\frac{\Phi_1'(z)}{\Phi_1(z)}\big)}{\big(\big(\frac{\Phi_1'(z)}{\Phi_1(z)}\big)'\big)^2}=\gamma_0,\ \ \gamma_0\in R\setminus\{1,0\}. \tag{2}$$

The next conditions are necessary for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of the equation (1), if $\lambda_0\in\mathbb{R}\setminus\{0,1\}$:

$$\pi_\omega(t)y_1^0y_0^0\lambda_0(\lambda_0-1)>0,\quad \pi_\omega(t)y_1^0\alpha_0(\lambda_0-1)>0,\quad y_1^0\cdot\lim_{t\uparrow\omega}|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}=Y_1,$$

$$\lim_{t\uparrow\omega}\Phi_1^{-1}(I_1(t))=Y_0,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}=\infty,\quad \lim_{t\uparrow\omega}\frac{I_1'(t)\pi_\omega(t)}{\Phi_1'(\Phi_1^{-1}(I_1(t)))\Phi_1^{-1}(I_1(t))}=\frac{\lambda_0}{\lambda_0-1}\,.$$

These conditions are also sufficient for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the equation (1) *if*

$$I(t)I_1(t)\lambda_0(\sigma_1 - 1) > 0 \ \ as \ \ t \in [a;\omega[$$

and the function $\frac{|\pi_\omega(t)|^{1-\frac{(2-\gamma_0)\lambda_0}{(1-\gamma_0)(\lambda_0-1)}} I_1'(t)}{I_1(t)}$ *is a normalized slowly varying function as* $t \uparrow \omega$.

Moreover, for each such solution the following asymptotic representations take place as $t \uparrow \omega$

$$\Phi_1(y(t)) = I_1(t)[1 + o(1)], \quad \frac{y'(t)\Phi_1'(y(t))}{\Phi_1(y(t))} = \frac{I_1'(t)}{I_1(t)}\,[1 + o(1)].$$

Let us notice that the function $\Phi^{-1}(z) \cdot \frac{\Phi_1'(\Phi_1^{-1}(z))}{z}$ is a slowly varying function as $z \to Z_1$.

If the condition (2) is not true, the following theorem takes place.

Theorem 2. *Let for equation* (1) $\sigma_1 \neq 1$, *the function* $\theta_1(z)$ *satisfy the condition* S *as* $z \to Y_1$ ($z \in \Delta_{Y_1}$), *the function* $\Phi^{-1}(z) \cdot \frac{\Phi_1'(\Phi^{-1}(z))}{z}$ *satisfy the condition* S_1 *as* $z \to Z_1$. *Then for existence of the equation's* (1) $P_\omega(Y_0, Y_1, \lambda_0)$*-solutions, where* $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$, *it is necessary, and, if*

$$I(t)I_1(t)\lambda_0(\sigma_1 - 1) > 0 \ \ for \ \ t \in]b, \omega[\,,$$

and finite or infinite limits

$$\lim_{t\uparrow\omega} \pi_\omega(t)F'(t) \ \ and \ \ \lim_{t\uparrow\omega} \frac{\sqrt{|\frac{\pi_\omega(t)I_1'(t)}{}I_1(t)|}}{\ln|I_1(t)|}$$

exist, sufficient the fulfilment of the following conditions

$$\pi_\omega(t)y_1^0y_0^0\lambda_0(\lambda_0 - 1) > 0, \quad \pi_\omega(t)y_1^0\alpha_0(\lambda_0 - 1) > 0 \ \ as \ \ t \in [a;\omega[\,,$$

$$y_1^0 \cdot \lim_{t\uparrow\omega} |\pi_\omega(t)|^{\frac{1}{\lambda_0-1}} = Y_1, \quad \lim_{t\uparrow\omega} I_1(t) = Z_1, \quad \lim_{t\uparrow\omega} \frac{I_1''(t)I_1(t)}{(I_1'(t))^2} = 1, \quad \lim_{t\uparrow\omega} F(t) = \frac{\lambda_0 - 1}{\lambda_0}.$$

Moreover, for every such solution the following asymptotic representations as $t \uparrow \omega$ *take place*

$$\Phi_1(y(t)) = I_1(t)[1 + o(1)], \quad \frac{\pi_\omega(t)y'(t)}{y(t)} = \frac{\lambda_0}{\lambda_0 - 1}\,[1 + o(1)].$$

Note that if in the limit relation (2) $\gamma_0 = 1$, the function $\Phi^{-1}(z)\cdot\frac{\Phi_1'(\Phi^{-1}(z))}{z}$ satisfies the condition S_1 as $z \to Z_1$.

The next example illustrates the obtained results of Theorem 1.

Let's consider the following differential equation

$$y'' = \frac{1}{4}\,t^{-3}L(t)e^{|y|^4 - t^8}|y'|^3, \tag{3}$$

where $L : [2, +\infty[\, \to]0, +\infty[\,$, $\frac{2-\sigma_1}{\beta+a} > 0$, $\beta > 0$, $\beta \neq 1$ as $t \in [2, +\infty[\,$.

This is an equation of the form (1), where $a = 2$, $\alpha_0 = 1$, $p(t) = \frac{1}{4}\,t^{-3}L(t)e^{-t^8}$, $\varphi_0(y) = e^{|y|^4}$, $\varphi_1(y) = |y|^3$.

Theorem 1 implies that the equation (3) has a one-parameter family of $P_{+\infty}(+\infty, +\infty, 2)$-solutions, and every such solution and the derivative of such solution satisfy the following asymptotic representations

$$\frac{1}{y^7(t)}\,e^{\frac{1}{2}\,y^4(t)} = t^{-14}(L(t))^{-\frac{1}{2}}\,e^{\frac{1}{2}\,t^8}[1 + o(1)], \quad y'(t)y^3(t) = 2t^7[1 + o(1)] \ \ \text{as} \ \ t \uparrow \omega.$$

To illustrate the results of Theorem 2, we consider the differential equation

$$y'' = \psi(t)\exp\big(\exp(|y|^a) - \exp(t^d)\big)|y|^{\sigma_0}|y'|^{\sigma_1} \ \text{ as } \ t \in [2, +\infty[\,. \tag{4}$$

Here, $\sigma_0, \sigma_1 \in R$, $\sigma_1 > 1$, $a, d \in]0, +\infty[$, function $\psi : [2, +\infty[\to]0, +\infty[$ is continuous regularly varying on an infinity function of index γ, $\gamma \in R$.

This equation is an equation of the type (1), where

$$\alpha_0 = 1, \quad p(t) = \psi(t)\exp(-\exp(t^d)), \quad \varphi_0(y) = |y|^{\sigma_0}\exp(\exp(|y|^a)), \quad \varphi_1(y') = |y'|^{\sigma_1}.$$

We investigate the question of the existence and asymptotic behavior as $t \to +\infty$ of $P_{+\infty}(\infty, Y_1, \lambda_0)$-solutions of the equation (4) for which $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$.

In this case

$$\pi_\omega(t) = t, \quad \theta_1(y) = 1.$$

So, the function θ_1 satisfies the condition S.

Taking into account the choice of $B^0_{+\infty}$ as $t \to +\infty$, we have

$$I(t) = |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \cdot y_0^1 \cdot \frac{\sigma_1 - 1}{d} \cdot t^{1-d+\frac{1}{1-\sigma_1}} \cdot |\psi(t)|^{\frac{1}{1-\sigma_1}} \cdot \exp\Big(\frac{\exp(t^d)}{\sigma_1 - 1} - t^d\Big)[1 + o(1)].$$

At the same way as $t \to +\infty$ we have

$$I_1(t) = |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \cdot y_0^1 \cdot \Big(\frac{\sigma_1 - 1}{d}\Big)^2 \cdot t^{1-2d+\frac{1}{1-\sigma_1}} \cdot |\psi(t)|^{\frac{1}{1-\sigma_1}} \cdot \exp\Big(\frac{\exp(t^d)}{\sigma_1 - 1} - 2t^d\Big)[1 + o(1)].$$

In addition, in this case, since $Y_0 = \infty$, taking into account the choice of A^0_∞, we obtain

$$\Phi_0(y) = \frac{\sigma_1 - 1}{a} \cdot y^{\frac{\sigma_0}{\sigma_1 - 1}+1-a} \cdot \exp\Big(\frac{\exp(|y|^a)}{\sigma_1 - 1} - |y|^a\Big)[1 + o(1)] \ \text{ as } \ y \to \infty.$$

Similarly, we have

$$\Phi_1(y) = \Big(\frac{\sigma_1 - 1}{a}\Big)^2 \cdot y^{\frac{\sigma_0}{\sigma_1 - 1}+1-2a} \cdot \exp\Big(\frac{\exp(|y|^a)}{\sigma_1 - 1} - 2|y|^a\Big)[1 + o(1)] \ \text{ as } \ y \to \infty.$$

At the same time,

$$\Phi_1^{-1}(y) \cdot \frac{\Phi_1'(\Phi_1^{-1}(y))}{y} = \frac{(\sigma_1 - 1)^2}{a} \ln y \cdot \big(\ln\big((\sigma_1 - 1)\ln y\big)\big)^{\frac{\frac{\sigma_0}{\sigma_1 - 1} - 2a + 1}{a}}[1 + o(1)] \ \text{ as } \ y \to \infty.$$

It means that condition S_1 is satisfied.

Thus, all the conditions of Theorem 2 are satisfied. By virtue of this theorem, the equation (4) can have only $P_{+\infty}(+\infty, +\infty, \frac{d}{d-a})$-solutions of the class of $P_{+\infty}(\infty, Y_1, \lambda_0)$-solutions. From Theorem 2 it also follows that the equation (4) has one-parameter family of $P_{+\infty}(+\infty, +\infty, \frac{d}{d-a})$-solutions.

Also, taking into account the known asymptotic behavior of the function Φ_1^{-1}, it is easy to obtain that every $P_{+\infty}(+\infty, +\infty, \frac{d}{d-a})$-solution of the equation (4) and the derivative of such solution satisfy the following asymptotic representations

$$\begin{gathered}
(y(t))^{\frac{\sigma_0}{\sigma_1 - 1}+1-2a} \cdot \exp\Big(\frac{\exp(|y(t)|^a)}{\sigma_1 - 1} - 2|y(t)|^a\Big) \\
= \Big|\frac{a}{d-a}\Big|^{\frac{1}{1-\sigma_1}} \cdot \Big(\frac{a}{d}\Big)^2 \cdot t^{1-2d+\frac{1}{1-\sigma_1}} \cdot \psi^{\frac{1}{1-\sigma_1}}(t) \cdot \exp\Big(\frac{\exp(t^d)}{\sigma_1 - 1} - 2t^d\Big)[1 + o(1)] \ \text{ as } \ t \to +\infty, \\
y'(t) = \frac{y(t)}{t}\,[1 + o(1)] \ \text{ as } \ t \to +\infty.
\end{gathered}$$

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