Three-Layer Fully Linearized Difference Scheme for Symmetric Regularized Long Wave Equations

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The symmetric regularized-long-wave (SRLW) equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\,\frac{\partial^2 (u)^2}{\partial x\partial t}-\frac{\partial^4 u}{\partial x^2\partial t^2}=0 \tag{1}$$

was first derived in [7]. Such equation arises in different physical applications, including ion sound waves in plasma. The solvability and uniqueness of the solution of SRLW equation were studied in works [4–6].

Equation (1) can be rewritten in the form of equivalent first order system:

$$\begin{gathered}\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial x^2\partial t}+\frac{\partial \rho}{\partial x}+u\,\frac{\partial u}{\partial x}=0,\\ \frac{\partial \rho}{\partial t}+\frac{\partial u}{\partial x}=0.\end{gathered} \tag{2}$$

In the domain $x\in[a,b]$, $t\in[0,T]$, let us define boundary and initial conditions for system (2) as follows:

$$u(a,t)=u(b,t)=0,\quad \rho(a,t)=\rho(b,t)=0,\ \ t\in[0,T], \tag{3}$$

$$u(x,0)=u_0(x),\quad \rho(x,0)=\rho_0(x),\ \ x\in[a,b]. \tag{4}$$

The domain $\overline{Q}$ is divided into rectangular grid by the points $(x_i,t_j)=(a+ih,j\tau)$, $i=0,1,2,\dots,n$, $j=0,1,\dots,J$, where $h=(b-a)/n$ and $\tau=T/J$ denote the spatial and temporal mesh sizes, respectively.

The value of mesh function U at the node (x_i,t_j) is denoted by U_i^j, that is $U_i^j=U(x_i,t_j)$.

We define the difference quotients (forward, backward, and central, respectively) in x and t directions as follows:

$$\begin{gathered}(U_i^j)_x:=\frac{U_{i+1}^j-U_i^j}{h}\,,\quad (U_i^j)_{\overline{x}}:=\frac{U_i^j-U_{i-1}^j}{h}\,,\quad (U_i^j)_{\overset{\circ}{x}}:=\frac{1}{2}\,\big((U_i^j)_x+(U_i^j)_{\overline{x}}\big),\\ (U_i^j)_t:=\frac{U_i^{j+1}-U_i^j}{\tau}\,,\quad (U_i^j)_{\overline{t}}:=\frac{U_i^j-U_i^{j-1}}{\tau}\,,\quad (U_i^j)_{\overset{\circ}{t}}:=\frac{1}{2}\,\big((U_i^j)_t+(U_i^j)_{\overline{t}}\big).\end{gathered}$$

We approximate the problem (2)-(4) by the difference scheme

$$(U_i^j)_{\overset{\circ}{t}}-(U_i^j)_{\overline{x}x\overset{\circ}{t}}+\frac{1}{2}\,(\Phi_i^{j+1}+\Phi_i^{j-1})_{\overset{\circ}{x}}+\frac{1}{6}\,(\Lambda U)_i^j=0, \tag{5}$$

$$(\Phi_i^j)_{\overset{\circ}{t}}+\frac{1}{2}\,(U_i^{j+1}+U_i^{j-1})_{\overset{\circ}{x}}=0 \tag{6}$$

for $i = 1, 2, \dots, n-1$, $j = 1, 2, \dots J-1$, and

$$(U_i^0)_t - (U_i^0)_{\overline{x}xt} + \frac{1}{2}\,(\Phi_i^1 + \Phi_i^0)_{\overset{\circ}{x}} + \frac{1}{6}\,(\Lambda U)_i^0 = 0, \tag{7}$$

$$(\Phi_i^0)_t + \frac{1}{2}\,(U_i^1 + U_i^0)_{\overset{\circ}{x}} = 0 \tag{8}$$

for $i = 1, 2, \dots, n-1$, $j = 0$, with

$$U_i^0 = u_0(x_i), \quad \Phi_i^0 = \rho_0(x_i), \quad U_0^j = U_n^j = \Phi_0^j = \Phi_n^j = 0. \tag{9}$$

Here

$$(\Lambda U)_i^j := U_i^j (U_i^{j+1} + U_i^{j-1})_{\overset{\circ}{x}} + \big(U_i^j (U_i^{j+1} + U_i^{j-1})\big)_{\overset{\circ}{x}}, \ \ j = 1, 2, \dots, J-1,$$
$$(\Lambda U)_i^0 := (U_i^0)(U_i^1 + U_i^0)_{\overset{\circ}{x}} + \big(U_i^0 (U_i^1 + U_i^0)\big)_{\overset{\circ}{x}}.$$

Let H_0 be the set of functions defined on the mesh $\overline{\omega} = \{x_0, x_1, \dots, x_n\}$ and equal to zero at x_0, x_n. On H_0 we define the following inner products and norms

$$(U^j, V^j) := \sum_{i=1}^{n-1} h U_i^j V_i^j, \quad (U^j, V^j] := \sum_{i=1}^{n} h U_i^j V_i^j,$$
$$\|U^j\|^2 := (U^j, U^j), \quad \|U^j]|^2 := (U^j, U^j], \quad \|U^j\|_\infty := \max_{0 \le i \le n} |U_i^j|.$$

Theorem. *The finite difference scheme* (5)–(9) *is uniquely solvable and possesses the following invariant*

$$E^j := \|U^j\|^2 + \|U_{\overline{x}}^j]|^2 + \|\Phi^j\|^2 = \|u_0\|^2 + \|u_{0,\overline{x}}]|^2 + \|\rho_0\|^2 := E^0, \ \ j = 1, 2, \dots . \tag{10}$$

Proof. Multiplying (5) by $\tau(U_i^{j+1} + U_i^{j-1})$ and summing over i, we obtain

$$A - B + \frac{1}{2}\,C + \frac{\tau}{6}\,D = 0, \tag{11}$$

where

$$A := \tau(U_{\overset{\circ}{t}}^j, U^{j+1} + U^{j-1}) = \frac{1}{2}\,\big(\|U^{j+1}\|^2 - \|U^{j-1}\|^2\big),$$
$$B := \tau(U_{\overline{x}x\overset{\circ}{t}}^j, U^{j+1} + U^{j-1}) = -\frac{1}{2}\,\big(\|U_{\overline{x}}^{j+1}]|^2 - \|U_{\overline{x}}^{j-1}]|^2\big),$$
$$C := \tau\big((\Phi^{j+1} + \Phi^{j-1})_{\overset{\circ}{x}}, U^{j+1} + U^{j-1}\big)$$
$$= -\tau\big(\Phi^{j+1} + \Phi^{j-1}, (U^{j+1} + U^{j-1})_{\overset{\circ}{x}}\big) = 2\tau\big(\Phi^{j+1} + \Phi^{j-1}, \Phi_{\overset{\circ}{t}}^j\big) = \|\Phi^{j+1}\|^2 - \|\Phi^{j-1}\|^2,$$
$$D := (\Lambda U^j, U^{j+1} + U^{j-1}) = 0.$$

Thus, from (11) we have

$$\|U^{j+1}\|^2 + \|U_{\overline{x}}^{j+1}]|^2 + \|\Phi^{j+1}\|^2 = \|U^{j-1}\|^2 + \|U_{\overline{x}}^{j-1}]|^2 + \|\Phi^{j-1}\|^2, \ \ j = 1, 2, \dots . \tag{12}$$

Multiplying (7) by $\tau(U_i^1 + U_i^0)$ and summing over i, we obtain

$$A_0 - B_0 + \frac{1}{2}\,C_0 + \frac{\tau}{6}\,D_0 = 0, \tag{13}$$

where

$$\begin{aligned}
A_0 &:= \tau(U_t^0, U^1+U^0) = \big(\|U^1\|^2 - \|U^0\|^2\big),\\
B_0 &:= \tau(U^0_{\overline{x}xt}, U^1+U^0) = \big((U^1-U^0)_{\overline{x}x}, U^1+U^0\big) = -\|U^1_{\overline{x}}]|^2 + \|U^0_{\overline{x}}]|^2,\\
C_0 &:= \tau\big((\Phi^1+\Phi^0)_{\overset{\circ}{x}}, U^1+U^0\big) = -\tau\big(\Phi^1+\Phi^0, (U^1+U^0)_{\overset{\circ}{x}}\big)\\
&\qquad = 2\tau(\Phi^1+\Phi^0, \Phi_t^0) = 2\big(\|\Phi^1\|^2 - \|\Phi^0\|^2\big),\\
D_0 &:= (\Lambda U^0, U^1+U^0) = 0.
\end{aligned}$$

Thus, from (13) we have

$$\|U^1\|^2 + \|U^1_{\overline{x}}]|^2 + \|\Phi^1\|^2 = \|U^0\|^2 + \|U^0_{\overline{x}}]|^2 + \|\Phi^0\|^2,$$

which together with (12) confirms the validity of (10).

Because the difference scheme is linear on each new level with respect to the unknown values, its unique solvability follows from (10). □

Theorem. *Difference scheme* (5)–(9) *is absolutely stable with respect to initial data.*

Theorem. *If the solution of problem* (2)–(4) *belongs to W_2^3 Sobolev space, then the order of convergence of the difference scheme equals $O(\tau^2+h^2)$.*

Remark 1. Note that scheme (5), (6) is studied by Wang, Zhang, Chen in [8]. But there, for obtaining additional initial conditions on the first layer, they offer nonlinear two-layer scheme, requiring additional iterations, and which essentially worsens the result. Our approach uses an idea developed in [1–3], by which we obtain approximations (7), (8).

Remark 2. Note that equation (10) represents an perfect analogy of the well-known conservation law for SRLW equation

$$E(t) = \int_a^b \Big(|u|^2 + \Big|\frac{\partial u}{\partial x}\Big|^2 + |\rho|^2\Big)\,dx = \|u_0\|^2_{L_2} + \Big\|\frac{\partial u_0}{\partial x}\Big\|^2_{L_2} + \|\rho_0\|^2_{L_2} = E(0).$$

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