On Power-Law Asymptotic Behavior of Solutions to Weakly Superlinear Equations with Potential of General Form

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1 Introduction

Consider the equation

$$y^{(n)} = p(x, y, y', \dots, y^{(n-1)})|y|^k \operatorname{sign} y, \quad n > 4, \quad k > 1. \tag{1.1}$$

New results are proved on asymptotic behavior of blow-up and Kneser (see [7, Definition 13.1]) solutions to this equation. The same results concerning equation (1.1) with the constant potential $p = p_0 > 0$ are proved in [6]. In this paper one can also find the history of these problems. To prove the results, the equation is reduced to a dynamical system on an $(n-1)$-dimensional compact sphere (see [6]). We study the behavior of the trajectories of this system corresponding to constant-sign parts of solutions to (1.1). It is a modification of the method applied for the first time in [1] for the description of the asymptotic behavior of blow-up solutions to equation (1.1) with $n = 3, 4$. See also [2]. Later an asymptotic classification of solutions to (1.1) with $n = 3, 4$ was obtained by that method (see [3, 5] and the references here).

In particular, it was proved that for $n = 3, 4$ all blow-up and Kneser solutions to equation (1.1) have the power-law asymptotic behavior (see [2, 3]), namely, for blow-up at some point x^* solutions $y(x)$ it was obtained that

$$y(x) = C(x^* - x)^{-\alpha}(1 + o(1)) \tag{1.2}$$

with

$$\alpha = \frac{n}{k-1}\,, \quad C^{k-1} = \frac{1}{p_0} \prod_{j=0}^{n-1} (j + \alpha). \tag{1.3}$$

It was also proved for equation (1.1) with $(-1)^n p \equiv p_0 > 0$ for sufficiently large n (see [8]) and for $n = 12, 13, 14$ (see [4]) that there exists $k > 1$ such that equation (1.1) has a solution with non-power-law behavior, namely,

$$y(x) = (x^* - x)^{-\alpha}\ h(\log(x^* - x)),$$

where h is a positive periodic non-constant function on $\mathbf{R}$. We will discuss this problem for $n \geq 15$.

2 Main Results

Theorem 2.1. *Suppose $p \in C(\mathbf{R}^{n+1}) \cap \mathrm{Lip}_{y_0,\dots,y_{n-1}}(\mathbf{R}^n)$ and $p \to p_0 > 0$ as $x \to x^*$, $y_0 \to \infty, \dots, y_{n-1} \to \infty$. Then for any integer $n > 4$ there exists $K > 1$ such that for any real $k \in (1, K)$, any solution to equation* (1.1) *tending to $+\infty$ as $x \to x^* - 0$ has power-law asymptotic behavior* (1.2), (1.3).

Theorem 2.2. *Suppose $p \in C(\mathbf{R}^{n+1}) \cap \mathrm{Lip}_{y_0,\dots,y_{n-1}}(\mathbf{R}^n)$ and $(-1)^n p \to p_0 > 0$ as $x \to \infty$, $y_0 \to 0, \dots, y_{n-1} \to 0$. Then for any integer $n > 4$ there exists $K > 1$ such that all Kneser solutions to equation (1.1) with any real $k \in (1, K)$ tend to zero with power-law asymptotic behavior, namely,*

$$y(x) = C|x|^{-\alpha}(1 + o(1)), \quad x \to \infty,$$

with α and C given by (1.3).

3 Sketch of the Proof

Proof. To prove Theorem 2.1, as in the proof of Theorem 3.1 (see [6]), we put

$$\alpha = \frac{n}{k-1}\,, \quad \gamma = \frac{1}{\alpha}\,, \quad m = n - 1. \tag{3.1}$$

Consider equation (1.1) with $p = p_0 > 0$. Without loss of generality we can assume that $p_0 = 1$. To prove the theorem, an auxiliary dynamical system is investigated on the m-dimensional sphere. To define it note that if a function $y(x)$ is a solution to equation (1.1) with $p = p_0 > 0$, the same is true for the function

$$z(x) = Ay(A^\gamma x + B) \tag{3.2}$$

with any constants $A > 0$ and B.

Any non-trivial solution $y(x)$ of equation (1.1) with $p = p_0 > 0$ generates in $\mathbb{R}^n \setminus \{0\}$ the curve given parametrically by

$$\big(y(x), y'(x), y''(x), \dots, y^{(m)}(x)\big).$$

We can define an equivalence relation on $\mathbb{R}^n \setminus \{0\}$ such that all solutions obtained from $y(x)$ by (3.2) with $A > 0$ generate equivalent curves, i.e., curves passing through equivalent points (maybe for different x). We assume the points $(y_0, y_1, y_2, \dots, y_m)$ and $(z_0, z_1, z_2, \dots, z_m)$ in $\mathbb{R}^n \setminus \{0\}$ to be equivalent if and only if there exists a constant $\lambda > 0$ such that

$$z_j = \lambda^{n+j(k-1)} y_j, \quad j \in \{0, 1, \dots, m\}.$$

The obtained quotient space is homeomorphic to the m-dimensional sphere

$$S^m = \big\{y \in \mathbb{R}^n : \ y_0^2 + y_1^2 + y_2^2 + \dots + y_m^2 = 1\big\},$$

having exactly one representative of each equivalence class since the equation

$$\lambda^{2n} y_0^2 + \lambda^{2(n+2(k-1))} y_1^2 + \dots + \lambda^{2(n+m(k-1))} y_m^2 = 1$$

has exactly one positive root λ for any $(y_0, y_1, y_2, \dots, y_m) \in \mathbb{R}^n \setminus \{0\}$.

Equivalent curves in $\mathbb{R}^n \setminus \{0\}$ generate the same curves in the quotient space. The last ones are trajectories of an appropriate dynamical system, which can be described, in different charts covering the quotient space, by different formulae using different independent variables. A unique common independent variable can be obtained from those ones by using a partition of unity.

Within the chart that covers the points corresponding to positive values of solutions and has the coordinate functions

$$u_j = y^{(j)} y^{-1-\gamma j}, \quad j \in \{1, \dots, m\},$$

the dynamical system can be written as

$$\begin{cases} \dfrac{du_1}{dt} = u_2 - (1+\gamma)u_1^2, \\ \dfrac{du_j}{dt} = u_{j+1} - (1+\gamma j)u_1 u_j, \quad j \in \{2,\dots,m-1\}, \\ \dfrac{du_m}{dt} = 1 - (1+\gamma m)u_1 u_m \end{cases} \tag{3.3}$$

with the independent variable

$$t = \int_{x_0}^{x} y(\xi)^\gamma \, d\xi.$$

The described dynamical system has some equilibrium points corresponding to the solutions to equation (1.1) with $p = p_0 > 0$ having the exact power-law behavior. One of them, which corresponds to the n-positive solutions with exact power-law behavior, can be found in terms of its u_j coordinates noted by $(a_1,\dots,a_m)$:

$$\begin{cases} a_{j+1} = (1+\gamma j)a_1 a_j = a_1^{j+1} \prod\limits_{l=1}^{j}(1+\gamma l), \quad j \in \{1,\dots,m-1\}, \\ \quad a_1 = \Big(\prod\limits_{l=1}^{m}(1+\gamma l)\Big)^{-1/n}. \end{cases} \tag{3.4}$$

Instead of system (3.3) it is more convenient for our current purposes to use another one obtained by the substitution $\tau = a_1 t$, $u_j = a_j v_j$, $j \in \{1,\dots,m\}$:

$$\begin{cases} \dfrac{dv_1}{d\tau} = (1+\gamma)(v_2 - v_1^2), \\ \dfrac{dv_j}{d\tau} = (1+\gamma j)(v_{j+1} - v_1 v_j), \quad j \in \{2,\dots,m-1\}, \\ \dfrac{dv_m}{d\tau} = (1+\gamma m)(1 - v_1 v_m). \end{cases}$$

The above equilibrium point has in the new chart all coordinates equal to 1.

Up to the moment, we actually considered, for each $\gamma > 0$, its own dynamical system defined on its own quotient space homeomorphic to the m-dimensional sphere. In what follows, we need one sphere with a γ-parameterized dynamical system having an equilibrium point common for all γ in consideration. Thus, the points $(y_0, y_1, \dots, y_m) \in \mathbb{R} \setminus \{0\}$ obtained while treating solutions to (1.1) with $p = p_0 > 0$ and different k will generate the same point on S^m if their corresponding coordinates have the same sign and the tuples

$$\bigg(|y| \ : \ \Big|\frac{y'}{a_1}\Big|^{\frac{1}{1+\gamma}} \ : \ \cdots \ : \ \Big|\frac{y^{(j)}}{a_j}\Big|^{\frac{1}{1+\gamma j}} \ : \cdots : \ \Big|\frac{y^{(m)}}{a_m}\Big|^{\frac{1}{1+\gamma m}}\bigg),$$

if considered as sets of projective coordinates, define the same point in the projective space $\mathbb{R}P^m$. In particular, for points corresponding to n-positive solutions this means that they have the same v_j coordinates in the related charts. Hereafter, the domain consisting of all points with positive v_j coordinates is denoted by S^m_+. The only equilibrium point in S^m_+, which has all v_j coordinates equal to 1, is denoted by v^*.

For further proof we need the following

Lemma 3.1 (see [6]). *There exist $\gamma_2 > 0$ and an open neighborhood U of the point v^* such that for any positive $\gamma < \gamma_2$, any trajectory of the global dynamical system passing through the closure $\overline{U}$ tends to v^*. If such a trajectory does not coincide with v^*, then it passes transversally, at some time, through the boundary ∂U.*

Now let us consider a solution $y(x)$ to equation (1.1), in suggestion that $P \to 1$ as $x \to x^*$, $y_0 \to \infty, \dots, y_{n-1} \to \infty$. This solution generates in S^m a curve described in the same chart by the system

$$\begin{cases} \dfrac{dv_1}{d\tau} = (1+\gamma)(v_2 - v_1^2), \\ \dfrac{dv_j}{d\tau} = (1+\gamma j)(v_{j+1} - v_1 v_j), \quad j \in \{2, \dots, m-1\}, \\ \dfrac{dv_m}{d\tau} = (1+\gamma m)(q(\tau) - v_1 v_m), \end{cases} \tag{3.5}$$

with the function $q(\tau)$ obtained by the correspondent substitution in P, and it tends to 1 as $\tau \to \infty$.

Lemma 3.2. *The set of all ω-limit points of the trajectory described by* (3.5) *with $q(\tau)$ tending to* 1 *as $\tau \to \infty$ is the union of some whole trajectories of system* (3.5).

The proof of this lemma is almost the same as the proof of Lemma 5.6 in [3].

Since S^m is a compact set, any trajectory $s(\tau)$ on it has at least one ω-limit point. If this ω-limit point is unique, then it is the limit of the trajectory. So, if the trajectory does not tend to v^*, then it has at least one ω-limit point $w \neq v^*$. If the trajectory $s(\tau)$ is generated by a solution to equation (1.1) tending to $+\infty$ as $x \to x^* - 0$, then we can assume that $w \in S^m_+$. According to Lemma 3.1, the trajectory $s_1(\tau)$ of (3.5), passing through the point w, then it passes transversally, at some time, through the boundary ∂U for some $\gamma \in (0, \gamma_2)$. When the function $q(\tau)$ is sufficiently close to 1, then the trajectory $s(\tau)$ also passes transversally through ∂U. In this case it can enter U but cannot leave it. So, the points $s_1(\tau)$, outside of U, cannot be ω-limit points of $s(\tau)$. This contradiction to Lemma 3.2 shows that $s(\tau) \to v^*$ as $\tau \to \infty$. In particular,

$$v_1 = \Big(\frac{z_1}{z_0}\Big)^{1+\gamma} \longrightarrow 1 \ \text{ as } \ \tau \to \infty.$$

It means that the corresponding solution $y(x)$ to equation (1.1) satisfies the condition

$$\frac{y'}{a_1 y^{1+\gamma}} \longrightarrow 1 \ \text{ as } \ x \to x^* - 0.$$

So,

$$y' \sim a_1 y^{1+\gamma} \ \text{ as } \ x \to x^* - 0,$$
$$y \sim (a_1\gamma)^{-\frac{1}{\gamma}} (x^* - x)^{-\frac{1}{\gamma}},$$

and from (3.1) and (3.4) we obtain

$$y \sim \big(\alpha(\alpha+1)\cdots(\alpha+n-1)\big)^{\frac{1}{k-1}} (x^* - x)^{-\alpha}, \ \ x \to x^* - 0. \tag{3.6}$$

It means that Theorem 2.1 for $p_0 = 1$ is proved.

If $y(x)$ is a solution to equation (1.1) with P tending to an arbitrary $p_0 > 0$, then $y{p_0}^{\frac{1}{k-1}}$ is a solution to equation (1.1) with a similar function P tending to 1. So, $y{p_0}^{\frac{1}{k-1}}$ satisfies (3.6), and,

$$y = \Big(\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{p_0}\Big)^{\frac{1}{k-1}} (x^* - x)^{-\alpha}(1 + o(1)) \ \text{ as } \ x \to x^* - 0.$$

Theorem 2.1 is proved. □

By similar considerations we can prove Theorem 2.2.

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