On the Numerical Solvability of the Cauchy Problem for Systems of Linear Ordinary Differential Equations

Malkhaz Ashordia

A. Razmadze Mathematical Institute of I. Javakhishvili Tbiisi State University, Tbilisi, Georgia;
Sokhumi State University, Tbilisi, Georgia
E-mail: `ashord@rmi.ge`

There is investigated the numerical solvability question of the Cauchy problem for the system of ordinary differential equations

$$\frac{dx}{dt} = P(t)x + q(t), \tag{1}$$

$$x(t_0) = c_0, \tag{2}$$

where P and q are, respectively, real matrix- and vector-functions with the Lebesque integrable components defined on a closed interval $[a,b]$, $t_0 \in [a,b]$, $c_0 \in \mathbb{R}^n$ is a real vector.

Let the absolutely continuous vector function $x^0 : [a,b] \to \mathbb{R}^n$ be the unique solution of the problem $(1),(2)$.

Along with the problem $(1),(2)$ we consider the difference scheme

$$\Delta y(k-1) = \frac{1}{m}\big(G_{1m}(k)\,y(k) + G_{1m}(k)\,y(k-1) + g_m(k)\big) \ \ (k = 1,\dots,m), \tag{1_m}$$

$$y(k_m) = \gamma_m \tag{2_m}$$

$(m = 2,3,\dots)$, where G_{jm} $(j = 1,2)$ and q_m are, respectively, discrete real matrix- and vector-functions acting from the set $\{1,\dots,m\}$ into $\mathbb{R}^{n\times n}$, $k_m \in \{0,\dots,m\}$ and $\gamma_m \in \mathbb{R}^n$ for every $m \in \{2,3,\dots\}$.

In the work, the necessary and sufficient and effective sufficient conditions are given for the convergence of the difference scheme $(1_m),(2_m)$ $(m = 2,3,\dots)$ to the solution x^0 of the Cauchy problem $(1),(2)$.

The following notations and definitions will be used.

$\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ are, respectively, the sets of all natural, integer and real numbers. $\mathbb{R}_+ = [0,+\infty[$. $[a,b]$ is a closed interval.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$.

$O_{n\times m}$ is the zero $n\times m$-matrix. I_n is an identity $n\times n$ matrix.

O_n is the zero n-vector.

$\det(X)$ is the determinant of the $n\times n$-matrix X.

$\bigvee\limits_a^b(X)$ is the sum of total variations of the components x_{ij} $(i = 1,\dots,m;\ j = 1,\dots,m)$ of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$.

$\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$ is the set of all bounded variation matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$, i.e. such that $\bigvee\limits_a^b(X)<+\infty$.

$L([a,b];\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ whose components are Lebesgue integrable.

If $m\in\mathbb{N}$, then $\mathbb{N}_m=\{1,\dots,m\}$ and $\widetilde{\mathbb{N}}_m=\{0,1,\dots,m\}$.

If $J\subset\mathbb{Z}$, then $\mathrm{E}(J;\mathbb{R}^{n\times m})$ is the space of all matrix-functions $Y:J\to\mathbb{R}^{n\times m}$ with the norm

$$\|Y\|_J=\max\big\{\|Y(k)\|:\ k\in J\big\}.$$

Δ is the first order difference operator, i.e.

$$\Delta Y(i-1)=Y(i)-Y(i-1)\ \text{ for }\ Y\in\mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times m}),\ \ i\in\mathbb{N}_m.$$

Set

$$I_{10m}=\Big[a,a+\frac{\tau_m}{2}\Big[,\ \ I_{1km}=\Big[\tau_{km}-\frac{\tau_m}{2},\tau_{km}+\frac{\tau_m}{2}\Big[,\ \ I_{1mm}=\Big[b-\frac{\tau_m}{2},b\Big[,\ \ (k=1,\dots,m-1),$$
$$I_{20m}=\Big[a,a+\frac{\tau_m}{2}\Big],\ \ I_{1km}=\Big]\tau_{km}-\frac{\tau_m}{2},\tau_{km}+\frac{\tau_m}{2}\Big],\ \ I_{1mm}=\Big]b-\frac{\tau_m}{2},b\Big],\ \ (k=1,\dots,m-1),$$

where

$$\tau_{km}=a+k\tau_m\ \ (k=0,\dots,m),\quad \tau_m=\frac{b-a}{m}\,.$$

We introduce the operators $p_m:\mathrm{BV}([a,b];\mathbb{R}^n)\to\mathrm{E}(\widetilde{N}_m,\mathbb{R}^n)$ and $q_{jm}:\mathrm{E}(\widetilde{N}_m,\mathbb{R}^n)\to\mathrm{BV}([a,b];\mathbb{R}^n)$ $(j=1,2)$ defined by

$$p_m(x)(k)=x(\tau_{km})\ \text{ for }\ k\in\widetilde{N}_m$$

and

$$q_{jm}(y)(t)=y(k)\ \text{ for }\ t\in I_{jkm},\ \ k\in\widetilde{N}_m\ \ (j=1,2)$$

for every $m\in\{2,3,\dots\}$.

Definition. We say that a sequence $(G_{1m},G_{2m},g_m;k_m)$ $(m=2,3,\dots)$ belongs to the set $\mathcal{CS}(P,q,t_0)$ if for every $c_0\in\mathbb{R}^n$ and the sequence $\gamma_m\in\mathbb{R}^n$ $(m=2,3,\dots)$, satisfying the condition

$$\lim_{m\to+\infty}\gamma_m=c_0,$$

the difference problem $(1_m),(2_m)$ has a unique solution $y_m\in\mathrm{E}(\widetilde{N}_m;\mathbb{R}^n)$ for any sufficiently large m and the condition

$$\lim_{m\to+\infty}\big\|y_m-p_m(x^0)\big\|_{\widetilde{N}_m}=0$$

holds.

We assume that $P\in L([a,b];\mathbb{R}^{n\times n})$, $q\in L([a,b];\mathbb{R}^n)$, $G_{jm}\in\mathrm{E}(N_m;\mathbb{R}^{n\times n})$ $(j=1,2)$ and $g_m\in\mathrm{E}(N_m;\mathbb{R}^n)$ $(m=2,3,\dots)$. In addition, we define G_{jm} $(j=1,2)$ and g_m at the point zero by

$$G_{jm}(0)=O_{n\times n},\quad g_m(0)=0_n\ \ (j=1,2;\ \ m=2,3,\dots).$$

Theorem 1. *Let*

$$\lim_{m\to+\infty}\tau_{k_m m}=t_0. \tag{3}$$

Then

$$\big((G_{1m},G_{2m},g_m;k_m)\big)_{m=2}^{+\infty}\in\mathcal{CS}(P,q,t_0) \tag{4}$$

if and only if there exist matrix-functions $P_j \in L([a,b];\mathbb{R}^{n\times n})$ $(j=1,2)$ *and a sequence of matrix-functions* $H_m \in \mathrm{E}(\mathbb{N}_m;\mathbb{R}^{n\times n})$ $(m=2,3,\dots)$ *such that*

$$P_1(t)+P_2(t)=P(t) \ \text{ for } \ t\in[a,b], \tag{5}$$

$$\limsup_{m\to+\infty}\Big(\frac{1}{m}\sum_{k=1}^{m}\|H_m(k)G_{jm}(k)\|\Big)<+\infty \ \ (j=1,2), \tag{6}$$

$$\lim_{k\to+\infty}\max\big\{\|H_m(k)-I_n\|:\ k\in\mathbb{N}_m\big\}=0, \tag{7}$$

$$\lim_{k\to+\infty}\max\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}H_m(l)G_{jm}(l)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}P_j(\tau)\,d\tau\bigg\|:\ \sigma<k;\ \sigma,k\in\widetilde{N}_m\bigg\}=0 \ \ (j=1,2) \tag{8}$$

and

$$\lim_{k\to+\infty}\max\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}H_m(l)g_m(l)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}q(\tau)\,d\tau\bigg\|:\ \sigma<k;\ \sigma,k\in\widetilde{N}_m\bigg\}=0. \tag{9}$$

Corollary 1. *Let the conditions* (3), (5)–(7) *hold and let*

$$\lim_{k\to+\infty}\max\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}H_m(l+i)G_{jm}(l)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}P_j(\tau)\,d\tau\bigg\|:\ \sigma<k;\ \sigma,\ k\in\widetilde{N}_m\bigg\}=0$$
$$(i=-1,1;\ \ j=1,2)$$

and

$$\lim_{k\to+\infty}\max\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}H_m(l+i)g_m(l)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}q(\tau)\,d\tau\bigg\|:\ \sigma<k;\ \sigma,k\in\widetilde{N}_m\bigg\}=0 \ \ (i=-1,1),$$

where $P_j\in L([a,b];\mathbb{R}^{n\times n})$ $(j=1,2)$, $H_m\in\mathrm{E}(\mathbb{N}_m;\mathbb{R}^{n\times n})$ $(m=2,3,\dots)$. *Let, moreover, either*

$$\limsup_{k\to+\infty}\bigg(\frac{1}{m}\sum_{k=1}^{m}\big(\|G_{1m}(k)\|+\|G_{2m}(k)\|+\|g_m(k)\|\big)\bigg)<+\infty$$

or

$$\limsup_{k\to+\infty}\bigg(\sum_{k=1}^{m}\big\|\Delta H_m(k-1)\big\|\bigg)<+\infty.$$

Then the inclusion (4) *holds.*

Theorem 2. *Let the conditions* (3), (5)–(7) *hold and let*

$$\lim_{k\to+\infty}\max\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}G_{jm}(l)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}P_j(\tau)\,d\tau\bigg\|:\ \sigma<k;\ \sigma,k\in\widetilde{N}_m\bigg\}=0 \ \ (j=1,2), \tag{10}$$

$$\lim_{k\to+\infty}\max\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}g_m(l)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}q(\tau)\,d\tau\bigg\|:\ \sigma<k;\ \sigma,k\in\widetilde{N}_m\bigg\}=0, \tag{11}$$

$$\limsup_{k\to+\infty}\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}\sum_{i=1}^{l}\Delta H(i)G_{jm}(i)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}P_{*j}(\tau)\,d\tau\bigg\|:\ \sigma<k;\ \sigma,k\in\widetilde{N}_m\bigg\}=0$$

and

$$\lim_{k\to+\infty}\sup\left\{\left\|\frac{1}{m}\sum_{l=\sigma+1}^{k}\sum_{i=1}^{l}\Delta H(i)g_m(i)-\int_{\tau_{\sigma m}}^{\tau_{km}}q_*(\tau)\,d\tau\right\|:\ \sigma<k;\ \sigma,k\in\widetilde{N}_m\right\}=0,$$

where $P_j,P_{*j}\in L([a,b];\mathbb{R}^{n\times n})$ $(j=1,2)$, $q_*\in L([a,b];\mathbb{R}^n)$, $H_m\in \mathrm{E}(\mathbb{N}_m;\mathbb{R}^{n\times n})$ $(m=2,3,\dots)$. *Then*

$$\big((G_{1m},G_{2m},g_m;k_m)\big)_{m=2}^{+\infty}\in\mathcal{CS}(P-P_*,q-q_*,t_0),$$

where $P_*(t)\equiv P_{*1}(t)+P_{*2}(t)$.

Corollary 2. *Let the conditions* (3), (5) *hold and let there exist a natural* μ *and matrix-functions* $B_i\in\mathrm{E}(\mathbb{N}_m;\mathbb{R}^{n\times n})$ $(i=0,\dots,\mu-1)$ *such that*

$$\lim_{k\to+\infty}\sup\left(\frac{1}{m}\sum_{k=1}^{m}(\|G_{1m\mu}(k)\|+\|G_{2m\mu}(k)\|)\right)<+\infty,$$

$$\lim_{k\to+\infty}\max\{\|H_{m\,\mu-1}(k)-I_n\|:k\in\mathbb{N}_m\}=0,$$

$$\lim_{k\to+\infty}\max\left\{\left\|\frac{1}{m}\sum_{l=\sigma+1}^{k}G_{jm\mu}(l)-\int_{\tau_{\sigma m}}^{\tau_{km}}P_j(\tau)d\tau\right\|:\sigma<k;\ \sigma,k\in\widetilde{N}_m\right\}=0\ \ (j=1,2)$$

and

$$\lim_{k\to+\infty}\max\left\{\left\|\frac{1}{m}\sum_{l=\sigma+1}^{k}g_{m\mu}(l)-\int_{\tau_{\sigma m}}^{\tau_{km}}q_j(\tau)d\tau\right\|:\sigma<k;\ \sigma,k\in\widetilde{N}_m\right\}=0,$$

where $P_j\in L([a,b];\mathbb{R}^{n\times n})$ $(j=1,2)$,

$$H_{m0}(k)\equiv I_n,$$

$$H_{m\,i+1}(k)\equiv\left(I_n-H_{mi}(k)-\frac{1}{m}\sum_{l=1}^{k}H_{mi}(l)G_{\sigma m}(l)-B_{i+1}(k)\right)H_{mi}(k),$$

$$G_{jm\,i+1}(k)\equiv H_{mi}(k)G_{jm}(k),\quad g_{m\,i+1}(k)\equiv H_{mi}(k)g_m(k)$$

$$(\sigma=1,2;\ \ i=1,\dots,\mu-1;\ \ m=2,3,\dots).$$

Then the inclusion (4) *holds.*

If $\mu=1$, then Corollary 2 has the following form.

Corollary 3. *Let the conditions* (3), (5), (10), (11) *and*

$$\lim_{k\to+\infty}\sup\left(\frac{1}{m}\sum_{k=1}^{m}\big(\|G_{1m}(k)\|+\|G_{2m}(k)\|\big)\right)<+\infty$$

hold, where $P_j\in L([a,b];\mathbb{R}^{n\times n})$ $(j=1,2)$. *Then the inclusion* (4) *holds.*

Corollary 4. *Let the conditions* (3), (5), (7)–(9) *hold and let there exist sequences of matrix-functions* $Q_{jm}\in\mathrm{E}(\widetilde{N}_m,\mathbb{R}^{n\times n})$ $(j=1,2;\ m=2,3,\dots)$ *such that*

$$\det\big(I_n+(-1)^jQ_{jm}(k)\big)\neq0\ \ \textit{for}\ \ k\in\mathbb{N}_m\ \ (j=1,2;\ m=2,3,\dots)$$

and

$$\lim_{m\to+\infty} \sup \left(\frac{1}{m} \sum_{\sigma=1}^{2} \sum_{k=1}^{m} \| G_{\sigma m}(k) - Q_{\sigma m}(k) \| \right) < +\infty,$$

where

$$H_m(k) \equiv \prod_{l=1}^{k} (I_n + Q_{2m}(l))^{-1} (I_n - Q_{1m}(l)).$$

Then the inclusion (4) *holds.*

The question considered in the work is classical. There are a lot of papers where the problem has been investigated (see, for example, [5, 6] and the references therein). But we could not find the necessary and sufficient conditions for the convergence of the difference schemes. Note that there are obtained the necessary and sufficient conditions for stability of the difference schemes circumscribed above, as well.

The proofs of the results are based on the following concept. We rewrite the problems (1), (2) and $(1_m), (2_m)$ $(m = 2, 3, \dots)$ as the Cauchy problem for the systems of so called generalized ordinary differential equations in the sense of Kurzweil ([5, 7]). Therefore, the convergence of the differential scheme is equivalent to the well-posedness question for the Cauchy problem for the last systems. So, using the results of the papers [1–3] we established the present results.

The analogous results are valid for general linear boundary value problems, as well.

References

[1] M. Ashordia, On the correctness of linear boundary value problems for systems of generalized ordinary differential equations. *Georgian Math. J.* **1** (1994), no. 4, 343–351.

[2] M. Ashordia, Criteria of correctness of linear boundary value problems for systems of generalized ordinary differential equations. *Czechoslovak Math. J.* **46(121)** (1996), no. 3, 385–404.

[3] M. T. Ashordia and N. A. Kekelia, On the well-posedness of the Cauchy problem for linear systems of generalized ordinary differential equations on an infinite interval. (Russian) *Differ. Uravn.* **40** (2004), no. 4, 443–454, 574; translation in *Differ. Equ.* **40** (2004), no. 4, 477–490.

[4] S. K. Godunov and V. S. Ryaben'kií, *Difference Schemes.* (Russian) Nauka, Moskow, 1973.

[5] J. Kurzweil and Z. Vorel, Continuous dependence of solutions of differential equations on a parameter. (Russian) *Czechoslovak Math. J.* **7 (82)** (1957), 568–583.

[6] A. A. Samarskiĭ, *Theory of Difference Schemes.* (Russian) Izdat. "Nauka", Moscow, 1977.

[7] Š. Schwabik, M. Tvrdý and O. Vejvoda, *Differential and Integral Equations. Boundary Value Problems and Adjoints.* D. Reidel Publishing Co., Dordrecht-Boston, Mass.-London, 1979.