On Exponential Equivalence of Solutions to Nonlinear Differential Equations

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1 Introduction

The equations

$$y^{(n)} + \frac{a}{x^2}\,y + p(x)y|y|^{k-1} = f(x), \tag{1.1}$$

$$z^{(n)} + \frac{a}{x^2}\,z + p(x)z|z|^{k-1} = 0 \tag{1.2}$$

with $k > 1$, $a \in \mathbb{R} \setminus \{0\}$ are considered. Functions $p(x)$ and $f(x)$ are assumed to be continuous as $x > x_0 > 0$, $p(x) \not\equiv 0$. Exponential equivalence of solutions to equations (1.1), (1.2) is proved under some assumptions on the function $f(x)$. If $a = 0$, equation (1.2) is well-known Emden–Fowler equation:

$$z^{(n)} + p(x)z|z|^{k-1} = 0.$$

A lot of results on the asymptotic behaviour of solutions to this equation and its generalizations were obtained in [1,2,4–6]. Note that equation (1.2) with $a \neq 0$ can't be reduced to Emden–Fowler differential equation by any substitution of dependent or independent variables.

2 Exponential equivalence of solutions to nonlinear differential equations

Consider the differential equations

$$y^{(n)} + \frac{a}{x^2}\,y + p(x)y|y|^{k-1} = e^{-\alpha x}f(x), \tag{2.1}$$

$$z^{(n)} + \frac{a}{x^2}\,z + p(x)z|z|^{k-1} = e^{-\alpha x}g(x). \tag{2.2}$$

with $n \geq 2$, $k > 1$, $a \in \mathbb{R} \setminus \{0\}$, $\alpha > 0$.

Lemma 2.1 ([3])**.** *If function $y(x)$ and its n-th derivative $y^{(n)}(x)$ tend to zero as $x \to +\infty$, then the same holds for $y^{(j)}(x)$, $0 < j < n$.*

Lemma 2.2. *Let $y(x)$ be a solution to equation* (2.1) *such that $y(x)$ tends to zero as $x \to +\infty$. Then it holds*

$$y(x) = \mathbf{J^n}\Big[e^{-\alpha x}f(x) - \frac{a}{x^2}\,y(x) - p(x)[y(x)]^k_{\pm}\Big]$$

with $[y(x)]^k_{\pm} = |y|^{k-1}y$. $\mathbf{J}$ is the operator that maps tending to zero as $x \to +\infty$ function $\varphi(x)$ to its antiderivative:

$$\mathbf{J}[\varphi](x) = -\int\limits_x^{+\infty} \varphi(t)\,dt.$$

Theorem 2.1. *Let $p(x)$, $f(x)$, $g(x)$ be continuous bounded functions defined as $x > x_0 > 0$, $p(x) \not\equiv 0$. Then for any solution $y(x)$ to equation* (2.1) *that tends to zero as $x \to +\infty$ there exists a unique solution $z(x)$ to equation* (2.2) *such that*

$$|z(x) - y(x)| = O(e^{-\alpha x}), \ \ x \to +\infty.$$

Remark 2.1. Obviously, equations (2.1) and (2.2) in Theorem 2.1 can be swapped.

Back to equations (1.1), (1.2):

$$y^{(n)} + \frac{a}{x^2}\,y + p(x)y|y|^{k-1} = f(x),$$
$$z^{(n)} + \frac{a}{x^2}\,z + p(x)z|z|^{k-1} = 0$$

with $k > 1$, $a \in \mathbb{R} \setminus \{0\}$.

Corollary 2.1.1. *Suppose continuous function $f(x)$ satisfies the following condition*

$$f(x) = O(e^{-\alpha x}), \ \ \alpha > 0.$$

Let function $p(x)$ be a continuous bounded function, $p(x) \not\equiv 0$. Then for any solution $y(x)$ to equation (1.1) *that tends to zero as $x \to +\infty$ there exists a unique solution $z(x)$ to equation* (1.2) *such that*

$$|y(x) - z(x)| = O(e^{-\alpha x}), \ \ x \to +\infty.$$

References

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