Effects of Several Delays Perturbations in the Variation Formulas of Solution for a Functional Differential Equation with the Discontinuous Initial Condition

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Let $\theta_{i2} > \theta_{i1} > 0$, $i = \overline{1,s}$, be given numbers and $O \subset \mathbb{R}^n$ be an open set. Let E_f be the set of functions $f : I \times O^{1+s} \to \mathbb{R}^n$, $I = [a,b]$, satisfying the following conditions: for almost all fixed $t \in I$ the function $f(t,\,\cdot\,) : O^{1+s} \to \mathbb{R}^n$ is continuously differentiable; for each fixed $(x, x_1, \dots, x_s) \in O^{1+s}$ the functions $f(t, x, x_1, \dots, x_s)$, $f_x(t,\,\cdot\,)$ and $f_{x_i}(t,\,\cdot\,)$, $i = \overline{1,s}$, are measurable on I; for any $f \in E_f$ and compact set $K \subset O$ there exists a function $m_{f,K}(t) \in L_1(I, \mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$, such that

$$\big|f(t, x, x_1, \dots, x_s)\big| + |f_x(t,\,\cdot\,)| + \sum_{i=1}^{s} \big|f_{x_i}(t,\,\cdot\,)\big| \le m_{f,K}(t)$$

for all $(x, x_1, \dots, x_s) \in K^{1+s}$ and for almost all $t \in I$.

Let Φ be the set of continuous initial functions $\varphi : I_1 = [\widehat{\tau}, b] \to O$, where $\widehat{\tau} = a - \max\{\theta_{12}, \dots, \theta_{s2}\}$. To each element $\mu = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, f) \in \Lambda = [a,b) \times [\theta_{11}, \theta_{12}] \times \cdots \times [\theta_{s1}, \theta_{s2}] \times O \times \Phi \times E_f$ we set in correspondence the delay functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s)\big) \tag{1}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0), \ \ x(t_0) = x_0. \tag{2}$$

The condition (2) is said to be the discontinuous initial condition since, in general, $x(t_0) \ne \varphi(t_0)$.

Definition. Let $\mu = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, f) \in \Lambda$. A function $x(t) = x(t;\mu) \in O, t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to the element μ and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Let us introduce the set of variation:

$$V = \bigg\{\delta\mu = (\delta t_0, \delta\tau_1, \dots, \delta\tau_s, \delta x_0, \delta\varphi, \delta f) : \ |\delta t_0| \le \alpha, \ |\delta\tau_i| \le \alpha, \ i = \overline{1,s},$$
$$|\delta x_0| \le \alpha, \ \delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i, \ \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, \ |\lambda_i| \le \alpha, \ i = \overline{1,k}\bigg\},$$

where $\delta\varphi_i \in \Phi - \varphi_0$, $\delta f_i \in E_f - f_0$, $i = \overline{1,k}$, $\varphi_0 \in \Phi$, $f_0 \in E_f$ are fixed functions; $\alpha > 0$ is a fixed number.

Let

$$\mu_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, x_0, \varphi_0, f_0) \in \Lambda \tag{3}$$

be a fixed element, where $t_{00}, t_{10} \in (a,b)$, $t_{00} < t_{10}$ and $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = \overline{1,s}$. Let $x_0(t)$ be the solution corresponding to μ_0. There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times V$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$, and the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to it (see [4, Theorem 1.2]). By the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ to the interval $[\widehat{\tau}, t_{10} + \delta_1]$. Therefore, we can assume that the solution $x_0(t)$ is defined on the whole interval $[\widehat{\tau}, t_{10} + \delta_1]$. Now we introduce the increment of the solution $x_0(t) = x(t; \mu_0)$:

$$\Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), \ \ (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times V.$$

Theorem 1. *Let the following conditions hold:*

1) $\tau_{10} < \dots < \tau_{s0}$ *(see (3)) and* $t_{00} + \tau_{s0} < t_{10}$;
2) *the function* $\varphi_0(t)$ *is absolutely continuous and* $\dot{\varphi}_0(t)$, $t \in I_1$, *is bounded;*
3) *the function* $f_0(w)$, $w = (t, x, x_1, \dots, x_s) \in I \times O^{1+s}$, *is bounded;*
4) *there exists the finite limit*

$$\lim_{w \to w_0} f_0(w) = f_0^-, \ \ w \in (a, t_{00}] \times O^{1+s},$$

where $w_0 = (t_{00}, x_{00}, \varphi_0(t_{00} - \tau_{10}), \dots, \varphi_0(t_{00} - \tau_{s0}))$;

5) *there exist the finite limits*

$$\lim_{(w_{1i}, w_{2i}) \to (w_{1i}^0, w_{2i}^0)} \big[f_0(w_{1i}) - f_0(w_{2i})\big] = f_{0i}, \ \ w_{1i}, w_{2i} \in (a,b) \times O^{1+s}, \ \ i = \overline{1,s},$$

where

$$\begin{aligned} w_{1i}^0 = \Big(&t_{00} + \tau_{i0}, x_0(t_{00} + \tau_{i0}), x_0(t_{00} + \tau_{i0} - \tau_{10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{i-10}), \\ &x_{00}, x_0(t_{00} + \tau_{i0} - \tau_{i+10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{s0})\Big), \\ w_{2i}^0 = \Big(&t_{00} + \tau_{i0}, x_0(t_{00} + \tau_{i0}), x_0(t_{00} + \tau_{i0} - \tau_{10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{i-10}), \\ &\varphi_0(t_{00}), x_0(t_{00} + \tau_{i0} - \tau_{i+10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{s0})\Big). \end{aligned}$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1), t_{00} + \tau_{s0} < t_{10} - \delta_2$, *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V^-$, *where* $V^- = \{\delta\mu \in V : \delta t_0 \le 0\}$, *we have*

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu). \tag{4}$$

Here

$$\begin{aligned} &\delta x(t; \delta\mu) = -Y(t_{00}; t) f_0^- \delta t_0 + \beta(t; \delta\mu), \qquad (5) \\ &\beta(t; \delta\mu) = Y(t_{00}; t)\delta x_0 - \Big[\sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t) f_{0i}\Big]\delta t_0 \end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{s}\bigg[Y(t_{00}+\tau_{i0};t)f_{0i}+\int\limits_{t_{00}}^{t_{00}+\tau_{i0}}Y(\xi;t)f_{0x_i}[\xi]\dot{\varphi}_0(\xi-\tau_{i0})\,d\xi \\
&\qquad+\int\limits_{t_{00}+\tau_{i0}}^{t}Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i \\
&+\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi+\int\limits_{t_{00}}^{t}Y(\xi;t)\delta f[\xi]\,d\xi,
\end{aligned}
$$

where $Y(\xi;t)$ is the $n\times n$-matrix function satisfying the equation

$$
Y_\xi(\xi;t)=-Y(\xi;t)f_{0x}[\xi]-\sum_{i=1}^{s}Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}],\ \ \xi\in[t_{00},t]
$$

and the condition:

$$
Y(\xi;t)=H\ \text{ for }\ \xi=t,\quad Y(\xi;t)=\Theta\ \text{ for }\ \xi>t;
$$

H is the identity matrix and Θ is the zero matrix;

$$
\begin{aligned}
f_{0x_i}[\xi]&=f_{0x_i}\big(\xi,x_0(\xi),x_0(\xi-\tau_{10}),\dots,x_0(\xi-\tau_{s0})\big),\\
\delta f[\xi]&=\delta f\big(\xi,x_0(\xi),x_0(\xi-\tau_{10}),\dots,x_0(\xi-\tau_{s0})\big).
\end{aligned}
$$

The expression (5) is called the variation formula of solution. The addend

$$
-\sum_{i=1}^{s}\bigg[Y(t_{00}+\tau_{i0};t)f_{0i}+\int\limits_{t_{00}}^{t}Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i
$$

in the formula (5) is the effects of perturbations of the delays τ_{i0}, $i=\overline{1,s}$. For the ordinary differential equation the variation formula of solution has been proved in the monograph R. V. Gamkrelidze [1]. In [3] variation formulas of solution were proved for the equation $\dot{x}(t)=f(t,x(t),x(t-\tau))$ with the condition (2) in the case when the initial moment and delay variations have the same signs. In the present paper, the equation with several delays is considered and variation formulas of solution are obtained with respect to wide classes of variations (see V^- and V^+). Variation formulas of solution for various classes of delay functional differential equations, without perturbations of delays, are proved in [2].

Theorem 2. *Let the conditions* 1)–3) *and* 5) *of the Theorem* 1 *hold. Moreover, there exists the finite limit*

$$
\lim_{w\to w_0}f_0(w)=f_0^+,\ \ w\in[t_{00},b)\times O^{1+s}. \tag{6}
$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^+$, where $V^+=\{\delta\mu\in V:\delta t_0\le 0\}$, the formula (4) *holds. Here*

$$
\delta x(t;\delta\mu)=-Y(t_{00};t)f_0^+\delta t_0+\beta(t;\delta\mu).
$$

Theorem 3. *Let the conditions* 1)–4) *of the Theorem* 1 *hold. Moreover, there exists the finite limits:*

$$
\lim_{(w_{1i},w_{2i})\to(w_{1i}^0,w_{2i}^0)}\big[f_0(w_{1i})-f_0(w_{2i})\big]=f_{0i}^-,\ \ w_{1i},w_{2i}\in(a,t_{00}+\tau_{i0}]\times O^{1+s},\ \ i=\overline{1,s},
$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V_1^-$, where $V_1^- = \{\delta\mu \in V : \ \delta t_0 \le 0, \delta\tau_i \le 0, \ i = \overline{1, s}\}$ the formula (4) *holds. Here*

$$\delta x(t; \delta\mu) = -\Big[Y(t_{00}; t) f_0^- + \sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t) f_{0i}^-\Big]\delta t_0 - \sum_{i=1}^{s} \big[Y(t_{00} + \tau_{i0}; t) f_{0i}^-\big]\delta\tau_i + \beta_1(t; \delta\mu),$$

where

$$\begin{aligned}\beta_1(t; \delta\mu) &= Y(t_{00}; t)\delta x_0 \\ &+ \sum_{i=1}^{s} \Bigg[\int\limits_{t_{00}}^{t_{00}+\tau_{i0}} Y(\xi; t) f_{0x_i}[\xi]\dot{\varphi}_0(\xi - \tau_{i0})\, d\xi + \int\limits_{t_{00}+\tau_{i0}}^{t} Y(\xi; t) f_{0x_i}[\xi]\dot{x}_0(\xi - \tau_{i0})\, d\xi\Bigg]\delta\tau_i \\ &+ \sum_{i=1}^{s} \int\limits_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}]\delta\varphi(\xi)\, d\xi + \int\limits_{t_{00}}^{t} Y(\xi; t)\delta f[\xi]\, d\xi.\end{aligned}$$

Theorem 4. *Let the conditions* 1)–3) *of the Theorem* 1 *and the condition* (6) *hold. Moreover, there exists the finite limits:*

$$\lim_{(w_{1i}, w_{2i}) \to (w_{1i}^0, w_{2i}^0)} \big[f_0(w_{1i}) - f_0(w_{2i})\big] = f_{0i}^+, \ \ w_{1i}, w_{2i} \in [t_{00} + \tau_{i0}, b) \times O^{1+s}, \ \ i = \overline{1, s},$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V_1^+$, where $V_1^+ = \{\delta\mu \in V : \ \delta t_0 \ge 0, \ \delta\tau_i \ge 0, \ i = \overline{1, s}\}$ the formula (4) *holds. Here*

$$\delta x(t; \delta\mu) = -\Big[Y(t_{00}; t) f_0^+ + \sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t) f_{0i}^+\Big]\delta t_0 - \sum_{i=1}^{s} \big[Y(t_{00} + \tau_{i0}; t) f_{0i}^+\big]\delta\tau_i + \beta_1(t; \delta\mu).$$

References

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