On Some Special Classes of Solutions of the Countable Block-Diagonal Differential System

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Let

$$G(\varepsilon_0)=\big\{t,\varepsilon:\ t\in\mathbf{R},\ \varepsilon\in[0,\varepsilon_0],\ \varepsilon_0\in\mathbf{R}^+\big\}.$$

Definition 1. We say that a function $p(t,\varepsilon)$, in general a complex-valued, belongs to the class $S(m;\varepsilon_0)$ $(m\in\mathbf{N}\cup\{0\})$ if

1) $p:G(\varepsilon_0)\to\mathbf{C}$;

2) $p(t,\varepsilon)\in C^m(G(\varepsilon_0))$ with respect to t;

3) $\dfrac{d^kp(t,\varepsilon)}{dt^k}=\varepsilon^kp_k^*(t,\varepsilon)$ $(0\le k\le m)$,

$$\|p\|_{S(m;\varepsilon_0)}\overset{def}{=}\sum_{k=0}^{m}\sup_{G(\varepsilon_0)}|p_k^*(t,\varepsilon)|<+\infty.$$

Definition 2. We say that a function $f(t,\varepsilon,\theta)$ belongs to the class $F(m;\varepsilon_0;\theta)$ $(m\in\mathbf{N}\cup\{0\})$ if this function can be represented as

$$f(t,\varepsilon,\theta)=\sum_{n=-\infty}^{\infty}f_n(t,\varepsilon)\exp(in\,\theta(t,\varepsilon)),$$

and

1) $f_n(t,\varepsilon)\in S(m;\varepsilon_0)$ $(n\in\mathbf{Z})$;

2) $\|f\|_{F(m;\varepsilon_0,\theta)}\overset{def}{=}\displaystyle\sum_{n=-\infty}^{\infty}\|f_n\|_{S(m;\varepsilon_0)}<+\infty$;

3) $\theta(t,\varepsilon)=\displaystyle\int_0^t\varphi(\tau,\varepsilon)\,d\tau$, $\varphi\in\mathbf{R}^+$, $\varphi\in S(m,\varepsilon_0)$, $\displaystyle\inf_{G(\varepsilon_0)}\varphi(t,\varepsilon)=\varphi_0>0$.

The set of functions of the class $F(m;\varepsilon_0;\theta)$ forms a linear space, that turns into a complete normed space by introducing norms $\|\cdot\|_{F(m;\varepsilon_0;\theta)}$. The chain of next inclusions are true: $F(0;\varepsilon_0;\theta)\supset F(1;\varepsilon_0;\theta)\supset\cdots\supset F(m;\varepsilon_0;\theta)$.

Suppose we have two functions of the class $F(m;\varepsilon_0;\theta)$,

$$u(t,\varepsilon,\theta)=\sum_{n=-\infty}^{\infty}u_n(t,\varepsilon)\exp(in\theta(t,\varepsilon)),\quad v(t,\varepsilon,\theta)=\sum_{n=-\infty}^{\infty}v_n(t,\varepsilon)\exp(in\theta(t,\varepsilon)).$$

The product of these functions we define by the formula:

$$(uv)(t,\varepsilon,\theta)=\sum_{n=-\infty}^{\infty}\Big(\sum_{s=-\infty}^{\infty}u_{n-s}(t,\varepsilon)v_s(t,\varepsilon)\Big)\exp(in\theta(t,\varepsilon)).$$

Obviously, $uv\in F(m;\varepsilon_0;\theta)$.

We formulate some properties of the norm $\|\cdot\|_{F(m;\varepsilon_0;\theta)}$. Let $u,v\in F(m;\varepsilon_0;\theta)$, $k=const$. Then

1) $\|ku\|_{F(m;\varepsilon_0;\theta)}=|k|\cdot\|u\|_{F(m;\varepsilon_0;\theta)}$;
2) $\|u+v\|_{F(m;\varepsilon_0;\theta)}\le\|u\|_{F(m;\varepsilon_0;\theta)}+\|v\|_{F(m;\varepsilon_0;\theta)}$;
3) $\|u\|_{F(m;\varepsilon_0;\theta)}=\displaystyle\sum_{k=0}^{m}\Big\|\frac{1}{\varepsilon^k}\,\frac{\partial^k u}{\partial t^k}\Big\|_{F(0;\varepsilon_0;\theta)}$;
4) $\|uv\|_{F(m;\varepsilon_0;\theta)}\le 2^m\|u\|_{F(m;\varepsilon_0;\theta)}\cdot\|v\|_{F(m;\varepsilon_0;\theta)}$.

Definition 3. We say that the infinite vector $x(t,\varepsilon)=\operatorname{col}(x_1(t,\varepsilon),x_2(t,\varepsilon),\dots)$ belongs to the class $S_1(m;\varepsilon_0)$ if $x_j\in S(m;\varepsilon_0)$ $(j=1,2,\dots)$ and

$$\|x\|_{S_1(m;\varepsilon_0)}\overset{def}{=}\sup_j\|x_j\|_{S(m;\varepsilon_0)}<+\infty.$$

Definition 4. We say that the infinite matrix $A(t,\varepsilon)=(a_{jk}(t,\varepsilon))_{j,k=1,2,\dots}$ belongs to the class $S_2(m;\varepsilon_0)$ if $a_{jk}\in S(m;\varepsilon_0)$, and

$$\|A\|_{S_2(m;\varepsilon_0)}\overset{def}{=}\sup_j\sum_{k=1}^{\infty}\|a_{jk}\|_{S(m;\varepsilon_0)}<+\infty.$$

Definition 5. We say that the infinite vector $x(t,\varepsilon,\theta)=\operatorname{col}(x_1(t,\varepsilon,\theta),x_2(t,\varepsilon,\theta),\dots)$ belongs to the class $F_1(m;\varepsilon_0;\theta)$ if $x_j\in F(m;\varepsilon_0)$ $(j=1,2,\dots)$, and

$$\|x\|_{F_1(m;\varepsilon_0,\theta)}\overset{def}{=}\sup_j\|x_j\|_{F(m;\varepsilon_0,\theta)}<+\infty.$$

Definition 6. We say that the infinite matrix $A(t,\varepsilon,\theta)=(a_{jk}(t,\varepsilon,\theta)_{j,k=1,2,\dots}$ belongs to the class $F_2(m;\varepsilon_0,\theta)$ if $a_{jk}\in F(m;\varepsilon_0,\theta)$, and

$$\|A\|_{F_2(m;\varepsilon_0,\theta)}\overset{def}{=}\sup_j\sum_{k=1}^{\infty}\|a_{jk}\|_{F(m;\varepsilon_0,\theta)}<+\infty.$$

Consider the countable system of differential equations:

$$\frac{dx}{dt}=A(t,\varepsilon)x+f(t,\varepsilon,\theta)+\mu X(t,\varepsilon,\theta,x),\tag{1}$$

where $t,\varepsilon\in G(\varepsilon_0)$, $x=\operatorname{col}(x_1,x_2,\dots)\in D\subset l_1$ (l_1 – the space of boundary numerical sequences), $f=\operatorname{col}(f_1,f_2,\dots)\in F_1(m;\varepsilon_0;\theta)$, $A=\operatorname{diag}[A_1,A_2,\dots]$, $A_j=A_j(t,\varepsilon)=(a_{j,\alpha\beta})_{\alpha,\beta=1,2}$ $(j=1,2,\dots)$, $a_{j,\alpha\beta}\in S(m;\varepsilon_0)$ $(j=1,2,\dots;\ \alpha,\beta=1,2)$, eigenvalues of matrix $A_j(t,\varepsilon)$ have a kind $\pm i\omega_j(t,\varepsilon)$, $\omega_j\in\mathbf{R}^+$ $(j=1,2,\dots)$; infinite vector-function $X=\operatorname{col}(X_1,X_2,\dots)\in F_1(m;\varepsilon_0;\theta)$ with respect to t, ε, θ and continuous with respect to $x\in D$; parameter $\mu\in(0,\mu_0)\subset\mathbf{R}^+$.

The purpose of the article is to establish conditions under which the system (1) has a particular solution $x(t,\varepsilon,\theta,\mu)\in F_1(m_1;\varepsilon_1;\theta)$ $(0\le m_1\le m;\ 0<\varepsilon_1\le\varepsilon_0)$.

We assume the next conditions.

1^0. $\inf\limits_{G(\varepsilon_0)} |a_{j,12}(t,\varepsilon)| = a_0 > 0$ $(j = 1,2,\dots)$.

2^0. $\sup\limits_{j} \sup\limits_{G(\varepsilon_0)} \omega_j(t,\varepsilon) = \omega < +\infty$.

3^0. $\forall\, n \in \mathbf{Z}$: $|n| \le (2\omega+1)\varphi_0^{-1}$:

$$\inf_{G(\varepsilon_0)} \big|k\omega_j(t,\varepsilon) - n\varphi(t,\varepsilon)\big| \ge \gamma > 0 \ \ (k = 1,2;\ \ j = 1,2,\dots).$$

4^0. The functions X_j $(j = 1,2,\dots)$ have in D continuous particular derivations with respect to $x_1, x_2, \dots$ up to order $2q+1$ $(q \in \mathbf{N})$, and if $x_1, x_2, \dots \in F(m;\varepsilon_0;\theta)$, then all these derivations belong to the class $F(m;\varepsilon_0;\theta)$ also, and

$$\sup_j \bigg\| \frac{\partial^{2q+1} X_j(x_1, x_2, \dots)}{\partial x_{k_1}^{q_1} \partial x_{k_2}^{q_2} \cdots \partial x_{k_s}^{q_s}} \bigg\|_{F(m;\varepsilon_0;\theta)} < +\infty$$

$(q_1 + q_2 + \dots + q_s = 2q+1;\ k_1, k_2, \dots, k_s \in \mathbf{N})$.

Lemma 1. *Let the countable system of the differential equations*

$$\frac{dx}{dt} = \Big(\Lambda(t,\varepsilon) + \sum_{l=1}^{q} B_l(t,\varepsilon,\theta)\mu^l\Big)x, \tag{2}$$

where $x = \operatorname{col}(x_1, x_2, \dots)$, $\Lambda(t,\varepsilon) = \operatorname{diag}(\lambda_1(t,\varepsilon), \lambda_2(t,\varepsilon), \dots)$, $\lambda_j \in S(m;\varepsilon_0)$, $B_l(t,\varepsilon,\theta) \in F_2(m;\varepsilon_0;\theta)$ $(l = 1,\dots,q)$, $\mu \in (0,\mu_0) \subset \mathbf{R}^+$, *satisfy the condition:* $\forall\, n \in \mathbf{Z}$, $j \ne k$*:*

$$\inf_{G(\varepsilon_0)} \big|\lambda_j(t,\varepsilon) - \lambda_k(t,\varepsilon) - in\varphi(t,\varepsilon)\big| \ge \gamma_1 > 0,$$

where $\varphi(t,\varepsilon)$ *– the function is involved in the definition of the class* $F(m;\varepsilon_0;\theta)$*. Then there exists* $\mu_1 \in (0,\mu_0)$ *such that* $\forall\, \mu \in (0,\mu_1)$ *there exists a non-degenerate transformation*

$$x = \Big(E + \sum_{l=1}^{q} \Phi_l(t,\varepsilon,\theta)\mu^l\Big)y,$$

where $\Phi_l \in F_2(m;\varepsilon_0;\theta)$ $(l = 1,\dots,q)$*, which leads the system* (2) *to the kind:*

$$\frac{dy}{dt} = \Big(\Lambda(t,\varepsilon) + \sum_{l=1}^{q} U_l(t,\varepsilon)\mu^l + \varepsilon \sum_{l=1}^{q} V_l(t,\varepsilon,\theta)\mu^l + \mu^{q+1} W(t,\varepsilon,\theta,\mu)\Big)y,$$

where $U_l(t,\varepsilon)$ *– infinite diagonal matrices whose elements belong to the class* $S(m;\varepsilon_0)$*,* $V_l, W \in F_2(m-1;\varepsilon_0;\theta)$ $(l = 1,\dots,q)$*.*

Lemma 2. *Let the system* (1) *satisfy conditions* 1^0*–*4^0*. Then there exists* $\mu_2 \in (0,\mu_0)$ *such that* $\forall\, \mu \in (0,\mu_2)$ *there exists a transformation of kind*

$$x = \xi(t,\varepsilon,\theta,\mu) + \Psi(t,\varepsilon,\theta,\mu)y, \tag{3}$$

where $\xi(t,\varepsilon,\theta,\mu) \in F_1(m;\varepsilon_0;\theta)$*,* $\Psi(t,\varepsilon,\theta,\mu) \in F_2(m;\varepsilon_0;\theta)$*, which leads the system* (1) *to the kind:*

$$\begin{aligned} \frac{dy}{dt} = \Big(\widetilde{\Lambda}(t,\varepsilon) + \sum_{l=1}^{q} K_l(t,\varepsilon)\mu^l\Big)y + \varepsilon h(t,\varepsilon,\theta,\mu) + \mu^{2q} r(t,\varepsilon,\theta,\mu) \\ + \varepsilon C(t,\varepsilon,\theta,\mu)y + \mu^{q+1} P(t,\varepsilon,\theta,\mu)y + \mu Y(t,\varepsilon,\theta,y,\mu), \end{aligned} \tag{4}$$

where $\widetilde{\Lambda}(t,\varepsilon) = \operatorname{diag}[\Lambda_1(t,\varepsilon),\Lambda_2(t,\varepsilon),\dots]$, $\Lambda_j(t,\varepsilon) = \operatorname{diag}(-i\omega_j(t,\varepsilon), i\omega_j(t,\varepsilon))$ $(j = 1,2,\dots)$, $K_l(t,\varepsilon) = \operatorname{diag}(k_{l,1}(t,\varepsilon), k_{l,2}(t,\varepsilon),\dots) \in S_2(m;\varepsilon_0)$, $h, r \in F_1(m-1;\varepsilon_0;\theta)$, $C, P \in F_2(m-1;\varepsilon_0;\theta)$. *Vector-function* Y *belongs to the class* $F_1(m;\varepsilon_0;\theta)$ *with respect to* (t,ε,θ) *and contains the terms not lower than the second order with respect to the components of vector* y.

Theorem 1. *Let the system* (4) *satisfy the condition: there exists* $q_0 \in \mathbf{N}$ *such that* $|\operatorname{Re} k_{q_0,j}(t,\varepsilon)| \geq \gamma_0 > 0$, *and for all* $l = 1,\dots,q_0-1$ *(if* $q_0 > 1$*):* $\operatorname{Re} k_{l,j}(t,\varepsilon) \equiv 0$ $(j = 1,2,\dots)$. *Then there exists* $\mu_3 \in (0,\mu_0)$, $\varepsilon_1(\mu) \in (0,\varepsilon_0)$ *such that for all* $\mu \in (0,\mu_3)$, $\varepsilon \in (0,\varepsilon_1(\mu))$ *the system* (4) *has a particular solution* $y(t,\varepsilon,\theta,\mu) \in F_1(m-1;\varepsilon_1(\mu))$.

Proof. We make in the system (4) the substitution:

$$y = \frac{\varepsilon + \mu^{2q}}{\mu^{q_0}}\,\widetilde{y},$$

where $\widetilde{y}$ is a new unknown vector. We obtain:

$$\begin{aligned}\frac{d\widetilde{y}}{dt} = \Big(\widetilde{\Lambda}(t,\varepsilon) + \sum_{l=1}^{q} K_l(t,\varepsilon)\mu^l\Big)\widetilde{y} + \frac{\varepsilon\mu^{q_0}}{\varepsilon+\mu^{2q}}\,h(t,\varepsilon,\theta,\mu) + \frac{\mu^{2q+q_0}}{\varepsilon+\mu^{2q}}\,r(t,\varepsilon,\theta,\mu)\\ + \varepsilon C(t,\varepsilon,\theta,\mu)\widetilde{y} + \mu^{q+1}P(t,\varepsilon,\theta,\mu)\widetilde{y} + \frac{\varepsilon+\mu^{2q}}{\mu^{q_0-1}}\,\widetilde{Y}(t,\varepsilon,\theta,\widetilde{y},\mu). \quad (5)\end{aligned}$$

Consider the appropriate linear homogeneous and diagonal system:

$$\frac{d\widetilde{y}^{(0)}}{dt} = \Big(\widetilde{\Lambda}(t,\varepsilon) + \sum_{l=1}^{q} K_l(t,\varepsilon)\mu^l\Big)\widetilde{y}^{(0)} + \frac{\varepsilon\mu^{q_0}}{\varepsilon+\mu^{2q}}\,h(t,\varepsilon,\theta,\mu) + \frac{\mu^{2q+q_0}}{\varepsilon+\mu^{2q}}\,r(t,\varepsilon,\theta,\mu). \quad (6)$$

In the paper [2] it has been found that the conditions of the theorem guarantee the existence of a particular solution $\widetilde{y}^{(0)}(t,\varepsilon,\theta,\mu) \in F_1(m-1;\varepsilon_0;\theta)$ of the system (6), and there exists $M \in (0,+\infty)$ such that

$$\begin{aligned}\|\widetilde{y}^{(0)}\|_{F_1(m-1;\varepsilon_0;\theta)} &\leq \frac{M}{\gamma_0\mu^{q_0}}\Big(\frac{\varepsilon\mu^{q_0}}{\varepsilon+\mu^{2q}}\,\|h\|_{F_1(m-1;\varepsilon_0;\theta)} + \frac{\mu^{2q+q_0}}{\varepsilon+\mu^{2q}}\,\|r\|_{F_1(m-1;\varepsilon_0;\theta)}\Big)\\ &< \frac{M}{\gamma_0}\big(\|h\|_{F_1(m-1;\varepsilon_0;\theta)} + \|r\|_{F_1(m-1;\varepsilon_0;\theta)}\big).\end{aligned}$$

We seek the solution belonging to the class $F_1(m-1;\varepsilon_1(\mu);\theta)$ of the system (5) by the method of succesive approximations, defining the initial approximations $\widetilde{y}^{(0)}$ and the subsequents approximations defining as solutions, belonging to the class $F_1(m-1;\varepsilon_0;\theta)$ of the countable linear, homogeneous and diagonal systems:

$$\begin{aligned}\frac{d\widetilde{y}^{(s+1)}}{dt} = \Big(\widetilde{\Lambda}(t,\varepsilon) + \sum_{l=1}^{q} K_l(t,\varepsilon)\mu^l\Big)\widetilde{y}^{(s+1)} + \frac{\varepsilon\mu^{q_0}}{\varepsilon+\mu^{2q}}\,h(t,\varepsilon,\theta,\mu) + \frac{\mu^{2q+q_0}}{\varepsilon+\mu^{2q}}\,r(t,\varepsilon,\theta,\mu)\\ + \varepsilon C(t,\varepsilon,\theta,\mu)\widetilde{y}^{(s)} + \mu^{q+1}P(t,\varepsilon,\theta,\mu)\widetilde{y}^{(s)} + \frac{\varepsilon+\mu^{2q}}{\mu^{q_0-1}}\,\widetilde{Y}(t,\varepsilon,\theta,\widetilde{y}^{(s)},\mu), \quad s = 0,1,2,\dots. \quad (7)\end{aligned}$$

Let

$$\Omega = \Big\{\widetilde{y} \in F_1(m-1;\varepsilon_0;\theta):\ \|\widetilde{y} - \widetilde{y}^{(0)}\|_{F_1(m-1;\varepsilon_0;\theta)} \leq d\Big\}.$$

By virtue of the condition 4^0, there exists $L(d) \in (0,+\infty)$ such that $\forall\,\widetilde{y},\widetilde{z} \in \Omega$:

$$\big\|\widetilde{Y}(t,\varepsilon,\theta,\widetilde{y},\mu) - \widetilde{Y}(t,\varepsilon,\theta,\widetilde{z},\mu)\big\|_{F_1(m-1;\varepsilon_0;\theta)} \leq L(d)\|\widetilde{y} - \widetilde{z}\|_{F_1(m-1;\varepsilon_0;\theta)}.$$

Using the ordinary technique of the contraction mapping principle [1], it is easy to show that there exists $\mu_3 \in (0, \mu_0)$, $N_1 \in (0, +\infty)$ such that $\forall\, \mu \in (0, \mu_0)$, $\forall\, \varepsilon \in (0, \varepsilon_1(\mu))$, where $\varepsilon_1(\mu) = N_1 \mu^{2q_0 - 1}$, the process (7) converges to the solution $\widetilde{y}(t, \varepsilon, \theta, \mu) \in F_1(m-1; \varepsilon_1(\mu); \theta)$ of the system (5). $\square$

Lemma 2 and Theorem 1 immediately yield the following theorem.

Theorem 2. *Let the system* (1) *satisfy conditions* 1^0–4^0, *and the system* (4), *which is obtained from the system* (1) *by the transformation* (3), *satisfy the conditions of Theorem* 1. *Then there exists* $\mu_4 \in (0, \mu_0)$, $\varepsilon_2(\mu) \in (0, \varepsilon_0)$ *such that* $\forall\, \mu \in (0, \mu_4)$, $\varepsilon \in (0, \varepsilon_2(\mu))$ *the system* (1) *has a particular solution* $x(t, \varepsilon, \theta, \mu) \in F_1(m-1; \varepsilon_2(\mu); \theta)$.

References

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[2] A. V. Kostin and S. A. Shchogolev, On the stability of oscillations represented by Fourier series with slowly varying parameters. (Russian) *Differ. Uravn.* **44** (2008), no. 1, 45–51; translation in *Differ. Equ.* **44** (2008), no. 1, 47–53.