Variation Formulas of Solution for One Class of Controlled Functional Differential Equation with Several Delays and the Continuous Initial Condition

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Let $O \subset \mathbb{R}^n$ and $U_0 \subset \mathbb{R}^r$ be open sets. Let $\theta_{i2} > \theta_{i1} > 0$, $i = \overline{1,s}$ be given numbers and n-dimensional function $f(t,x,x_1,\dots,x_s,u)$ satisfy the following conditions: for almost all fixed $t \in I = [a,b]$ the function $f(t,\,\cdot\,) : O^{1+s} \times U_0 \to \mathbb{R}^n$ is continuously differentiable; for each fixed $(x,x_1,\dots,x_s,u) \in O^{1+s} \times U_0$ the functions $f(t,x,x_1,\dots,x_s,u)$, $f_x(t,\,\cdot\,)$ and $f_{x_i}(t,\,\cdot\,)$, $i = \overline{1,s}$, $f_u(t,\,\cdot\,)$ are measurable on I; for compact sets $K \subset O$, $U \subset U_0$ there exist a function $m_{K,U}(t) \in L_1(I,[0,\infty))$ such that

$$\big|f(t,x,x_1,\dots,x_s,u)\big| + |f_x(t,\,\cdot\,)| + \sum_{i=1}^{s} |f_{x_i}(t,\,\cdot\,)| + |f_u(t,\,\cdot\,)| \le m_{K,U}(t)$$

for all $(x,x_1,\dots,x_s,u) \in K^{1+s} \times U$ and for almost all $t \in I$. Furthermore, Φ is the set of continuous initial functions $\varphi : I_1 = [\widehat{\tau},b] \to O$, $\widehat{\tau} = a - \max\{\theta_{12},\dots,\theta_{s2}\}$, and Ω is the set of measurable control functions $u : I \to U$ with $clu(I)$ is a compact set and $clu(I) \subset U$.

To each element

$$\mu = (t_0,\tau_1,\dots,\tau_s,\varphi,u) \in \Lambda = [a,b) \times [\theta_{11},\theta_{12}] \times \cdots [\theta_{s1},\theta_{s2}] \times \Phi \times \Omega$$

we assign the delay controlled functional differential equation

$$\dot{x}(t) = f\big(t,x(t),x(t-\tau_1),\dots,x(t-\tau_s),u(t)\big) \tag{1}$$

with the continuous initial condition

$$x(t) = \varphi(t), \ \ t \in [\widehat{\tau},t_0]. \tag{2}$$

Condition (2) is said to be the continuous initial condition since always $x(t_0) = \varphi(t_0)$.

Definition. Let $\mu = (t_0,\tau_1,\dots,\tau_s,\varphi,u) \in \Lambda$. A function $x(t) = x(t;\mu) \in O$, $t \in [\widehat{\tau},t_1]$, $t_1 \in (t_0,b]$ is called a solution of equation (1) with the initial condition (2) or a solution corresponding to μ and defined on the interval $[\widehat{\tau},t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (1) almost everywhere on $[t_0,t_1]$.

Let us introduce the set of variation:

$$V = \bigg\{\delta\mu = (\delta t_0,\delta\tau_1,\dots,\delta\tau_s,\delta\varphi,\delta u) : \ |\delta t_0| \le \alpha, \ |\delta\tau_i| \le \alpha, \ i = \overline{1,s},$$

$$\delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i, \ \delta u = \sum_{i=1}^{k} \lambda_i \delta u_i, \ |\lambda_i| \le \alpha, \ i = \overline{1,k}\bigg\},$$

where $\delta\varphi_i \in \Phi - \varphi_0$, $\delta u_i \in \Omega - u_0$, $i = \overline{1,k}$. Here $\varphi_0 \in \Phi, u_0 \in \Omega$ are fixed functions and $\alpha > 0$ is a fixed number.

Let $\mu_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0(t), u_0(t)) \in \Lambda$ be a fixed element, where $t_{00}, t_{10} \in (a,b)$, $t_{00} < t_{10}$ and $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = \overline{1,s}$. Let $x_0(t)$ be the solution corresponding to μ_0.

There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times V$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$, and the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to it (see [2, Theorem 1.3]).

By the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$. Therefore, we can assume that the solution $x_0(t)$ is defined on the whole interval $[\widehat{\tau}, t_{10} + \delta_1]$.

Now we introduce the increment of the solution $x_0(t) = x(t;\mu_0)$:

$$\Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), \ \ (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times V.$$

Theorem 1. *Let the following conditions hold:*

1) *the function $\varphi_0(t)$ is absolutely continuous and $\dot{\varphi}_0(t)$, $t \in I_1$, is bounded;*
2) *function $f(w,u)$, $w = (t, x, x_1, \dots, x_s) \in I \times O^{1+s}$ is bounded on $I \times O^{s+1} \times U_0$*
3) *there exist the finite limits*

$$\lim_{t \to t_{00}-} \dot{\varphi}_0(t) = \dot{\varphi}_0^-, \quad \lim_{w \to w_0} f(w, u_0(t)) = f^-, \ \ w \in (a, t_{00}] \times O^{1+s},$$

where $w_0 = (t_{00}, \varphi_0(t_{00}), \varphi_0(t_{00} - \tau_{10}), \dots, \varphi_0(t_{00} - \tau_{s0}))$. Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V^-$, where $V^- = \{\delta\mu \in V : \delta t_0 \le 0\}$, we have

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu). \tag{3}$$

Here

$$\delta x(t; \delta\mu) = Y(t_{00}; t)\big[\dot{\varphi}_0^- - f^-\big]\delta t_0 + \beta(t; \delta\mu), \tag{4}$$

$$\beta(t;\delta\mu) = Y(t_{00};t)\delta\varphi(t_{00}) - \sum_{i=1}^{s}\bigg[\int_{t_{00}}^{t} Y(\xi;t) f_{x_i}[\xi]\dot{x}_0(\xi - \tau_{i0})\,d\xi\bigg]\delta\tau_i$$
$$+ \sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t} Y(\xi + \tau_{i0}; t) f_{x_i}[\xi + \tau_{i0}]\delta\varphi(\xi)\,d\xi + \int_{t_{00}}^{t} Y(\xi;t) f_u[\xi]\delta u(\xi)\,d\xi,$$

where $Y(\xi;t)$ is the $n \times n$-matrix function satisfying the equation

$$Y_\xi(\xi;t) = -Y(\xi;t) f_x[\xi] - \sum_{i=1}^{s} Y(\xi + \tau_{i0}; t) f_{x_i}[\xi + \tau_{i0}], \ \ \xi \in [t_{00}, t]$$

and the condition

$$Y(\xi;t) = \begin{cases} H & for \ \ \xi = t, \\ \Theta & for \ \ \xi > t, \end{cases}$$

H is the identity matrix and Θ is the zero matrix;

$$f_{x_i}[\xi] = f_{x_i}\big(\xi, x_0(\xi), x_0(\xi - \tau_{10}), \dots, x_0(\xi - \tau_{s0}), u_0(\xi)\big).$$

The expression (4) is called the variation formula of solution. The addend

$$-\sum_{i=1}^{s}\bigg[\int_{t_{00}}^{t} Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i$$

in the formula (4) is the effects of perturbations of the delays τ_{i0}, $i=\overline{1,s}$.

The expression

$$Y(t_{00};t)\Big\{\delta\varphi(t_{00})+\big[\dot{\varphi}_0^- - f^-\big]\delta t_0\Big\}+\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t} Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi$$

is the effect of the continuous initial condition and perturbation of the initial moment t_{00} and the initial function $\varphi_0(t)$.

The expression

$$\int_{t_{00}}^{t} Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi$$

is the effect of perturbation of the control function $u_0(t)$.

In [4] variation formulas of solution were proved for the equation

$$\dot{x}(t)=f\big(t,x(t),x(t-\tau),u(t)\big)$$

with the condition (2) in the case when the initial moment and delay variations have the same signs.

In the present paper, the equation with several delays is considered and variation formulas of solution are obtained with respect to wide classes of variations (see V^- and V^+).

Variation formulas of solution for various classes of controlled delay functional differential equations, without perturbations of delays, are proved in [1,3].

Theorem 2. *Let the conditions* 1) *and* 2) *of the Theorem* 1 *hold. Moreover, there exist the finite limits*

$$\lim_{t\to t_{00}+}\dot{\varphi}_0(t)=\dot{\varphi}_0^+,\quad \lim_{w\to w_0} f(w,u_0(t))=f^+,\ \ w\in[t_{00},b).$$

Then for any $\widehat{t}\in(t_{00},t_{10})$ *there exist numbers* $\varepsilon_2\in(0,\varepsilon_1)$ *and* $\delta_2\in(0,\delta_1)$ *such that for arbitrary* $(t,\varepsilon,\delta\mu)\in[\widehat{t},t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^+$, *where* $V^+=\{\delta\mu\in V:\delta t_0\geq 0\}$ *the formula* (3) *holds, where*

$$\delta x(t;\delta\mu)=Y(t_{00};t)\big[\dot{\varphi}_0^+ - f^+\big]\delta t_0+\beta(t;\delta\mu).$$

Theorem 3. *Let the conditions* 1) *and* 2) *of the Theorem* 1 *and the condition* 6) *hold. Moreover,*

$$\dot{\varphi}_0^- - f^- = \dot{\varphi}_0^+ - f^+ := \widehat{f}.$$

Then for any $\widehat{t}\in(t_{00},t_{10})$ *there exist numbers* $\varepsilon_2\in(0,\varepsilon_1)$ *and* $\delta_2\in(0,\delta_1)$ *such that for arbitrary* $(t,\varepsilon,\delta\mu)\in[\widehat{t},t_{10}+\delta_2]\times(0,\varepsilon_2)\times V$ *the formula* (3) *holds, where*

$$\delta x(t;\delta\mu)=Y(t_{00};t)\widehat{f}\delta t_0+\beta(t;\delta\mu).$$

References

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