# The Plane Rotatability Indicators of a Differential System

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In a Euclidean space $\mathbb{R}^n$ with $n > 1$, consider the set $\mathcal{M}^n$ of linear systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \ \ t \in \mathbb{R}^+ \equiv [0, \infty), \tag{1}$$

with continuous operator-functions $A : \mathbb{R}^+ \to \operatorname{End} \mathbb{R}^n$, identified with the systems themselves. Developing the ideas from the papers [1–7], we study the Lyapunov type indicators which are responsible for the oscillation of solutions: in this case, for their rotatability in a specially chosen planes in which it is the most significant.

Let $\mathcal{S}(A)$ be the set of all solutions of system (1), and let $\mathcal{G}^k(A)$ be the set of all its $k$-dimensional subspaces. The asterisk as subscript of a linear space denotes the set with the zero removed.

**Definition 1.** For a given linearly independent solutions $x, y \in \mathcal{S}_*(A)$ of the system $A \in \mathcal{M}^n$ and for a moment $t \in \mathbb{R}^+$ define *the angle of rotation* of function $x$ in direction of function $y$ and, respectively, the *trace variation* of function $x$ in the time from 0 to $t$ by the following formulas

$$\Psi(x, y, t) \equiv \bigg| \int\limits_0^t \big(\dot{e}_{x(\tau)}, R_{y(\tau)} e_{x(\tau)}\big)\, d\tau \bigg|, \quad \mathrm{P}(x, t) \equiv \int\limits_0^t |\dot{e}_{x(\tau)}|\, d\tau, \tag{2}$$

where $e_a \equiv a/|a|$ is a normalized vector $a$, and $R_b a$ is the result of rotation of the vector $a$ by the angle $\pi/2$ to the half-plane which contains the vector $b$ (linearly independent of $a$).

**Definition 2.** For each *plane* (two-dimensional subspace) $G \in \mathcal{G}^2(A)$ of solutions of the system $A \in \mathcal{M}^n$ define the *weak* and, respectively, *strong rotatability indicators* of the plane $G$: *the lower one*

$$\check{\psi}^\circ(G) \equiv \varliminf_{t\to\infty} \inf_{L \in \operatorname{Aut} \mathbb{R}^n} \frac{1}{t}\, \Psi(Lx, Ly, t), \quad \check{\psi}^\bullet(G) \equiv \inf_{L \in \operatorname{Aut} \mathbb{R}^n} \varliminf_{t\to\infty} \frac{1}{t}\, \Psi(Lx, Ly, t) \tag{3}$$

and *the upper one*

$$\hat{\psi}^\circ(G) \equiv \varlimsup_{t\to\infty} \inf_{L \in \operatorname{Aut} \mathbb{R}^n} \frac{1}{t}\, \Psi(Lx, Ly, t), \quad \hat{\psi}^\bullet(G) \equiv \inf_{L \in \operatorname{Aut} \mathbb{R}^n} \varlimsup_{t\to\infty} \frac{1}{t}\, \Psi(Lx, Ly, t), \tag{4}$$

where $x$ and $y$ form a basis in $G$.

**Remark 1.** If one replaces in formulas (3) and (4) for each $t \in \mathbb{R}^+$ the angle of rotation $\Psi(Lx, Ly, t)$ of the function $Lx$ in direction of the function $Ly$ in time from 0 to $t$ by the trace variation $\mathrm{P}(Lx, t)$ of the function $Lx$ in the same time (see eq. (2)), then the resulting formulas will give corresponding *wandering indicators* $\hat{\rho}^\circ(x)$, $\hat{\rho}^\bullet(x)$, $\check{\rho}^\circ(x)$, $\check{\rho}^\bullet(x)$ of the solution $x \in \mathcal{S}_*(A)$ of the system $A \in \mathcal{M}^n$ (see [3] in somewhat different notation).

**Definition 3.** For each solution $x \in \mathcal{S}_*(A)$ of the system $A \in \mathcal{M}^n$ define *weak* and, respectively, *strong plain rotatability indicators* of the solution $x$: *the lower one*

$$\check{\psi}^\circ(x, A) \equiv \sup_{x \in G \in \mathcal{G}^2(A)} \check{\psi}^\circ(G), \quad \check{\psi}^\bullet(x, A) \equiv \sup_{x \in G \in \mathcal{G}^2(A)} \check{\psi}^\bullet(G) \tag{5}$$

and *the upper one*

$$\hat{\psi}^{\circ}(x,A) \equiv \sup_{x\in G\in\mathcal{G}^2(A)} \hat{\psi}^{\circ}(G), \quad \hat{\psi}^{\bullet}(x,A) \equiv \sup_{x\in G\in\mathcal{G}^2(A)} \hat{\psi}^{\bullet}(G). \tag{6}$$

**Definition 4.** If the upper indicator in Definitions 2 and 3 coincides with the similar lower one, then it is called *exact* and its accent (check or hat) is removed, and in case of coincidence of weak indicator with the similar strong one it is called *absolute* and its circle (empty or full) is omitted.

**Definition 5.** For each system $A \in \mathcal{M}^n$, by *the spectrum* of an indicator defined on the set $\mathcal{S}_*(A)$ or $\mathcal{G}^2(A)$ (or perhaps only on a part of these) we mean the set of all its values on that set.

**Remark 2.** The case $n = 2$ is special in that the plane $G \in \mathcal{G}^2(A)$ of solutions of the system $A \in \mathcal{M}^2$ coincides with the whole space $\mathcal{S}(A)$, and hence, indicators (3) and (4) coincide with the corresponding *oriented rotatability indicators* $\check{\theta}^{\circ}(x) = \check{\theta}^{\bullet}(x)$ and $\hat{\theta}^{\circ}(x) = \hat{\theta}^{\bullet}(x)$ of some solution $x \in G_*$ (actually, of any one; see [7] in other notation), and they are the absolute lower $\check{\psi}(G)$ and upper $\hat{\psi}(G)$ rotatability indicators of the plane $G = \mathcal{S}(A)$, respectively, and have one-point spectrum.

The apparent incorrectness of Definition 2, in the part of its possible dependence on the choice of linearly independent solutions $x, y$ in $G$ and of a scalar product in $\mathbb{R}^n$, is eliminated by

**Theorem 1.** *The rotatability indicators of a plane $G \in \mathcal{G}^2(A)$ of solutions of any system $A \in \mathcal{M}^n$, defined by formulas* (3) *and* (4), *are invariant under the choice of a basis $x, y \in G_*$ and the choice of a Euclidean structure in $\mathbb{R}^n$.*

The proof of Theorem 1 is provided by

**Lemma 1.** *For any plane $G \in \mathcal{G}^2(A)$ of any system $A \in \mathcal{M}^n$, there are a system $B \in \mathcal{M}^2$ and a continuously differentiable family of orthogonal transformations*

$$U(t) : G(t) \to G(0) \equiv \mathbb{R}^2, \quad t \in \mathbb{R}^+, \ \ U(0) = I,$$

*sending any linearly independent solutions $x, y \in G_*$ into solutions $u, v \in \mathcal{S}(B)$ such that*

$$u \equiv Ux, \quad v \equiv Uy, \quad \Psi(x,y,t) = \Psi(u,v,t), \ \ t \in \mathbb{R}^+.$$

According to the notation given in Definition 3 for the plane rotatability indicator of a solution of a system, it is not uniquely determined by that solution alone and may depend on the other solutions of the system, which is justified by

**Theorem 2.** *There exist an autonomous system $A \in \mathcal{M}^3$ and a non-autonomous system $B \in \mathcal{M}^3$, having a common solution $x \in \mathcal{S}_*(A) \cap \mathcal{S}_*(B)$ with exact, absolute, but different plane rotatability indicators*

$$\psi(x,A) > \psi(x,B).$$

There exists a usual order in the set of plane indicators [3]: the lower indicators do not exceed the upper ones and the weak indicators do not exceed the strong ones. In addition, the seminorm

$$\|A\|_I \equiv \varlimsup_{t\to\infty} \frac{1}{t} \int_0^t \|A(\tau)\|\, d\tau < \infty, \quad \|A(\tau)\| \equiv \sup_{|e|=1} |A(\tau)e|, \tag{7}$$

in the space $\mathcal{M}^n$ gives the upper bound for all the wandering indicators and hence for all the indicators introduced in Definitions 2 and 3, since the following assertion holds.

**Theorem 3.** *For any solution $x \in G_*$ from any plane $G \in \mathcal{G}^2(A)$ of solutions of any system $A \in \mathcal{M}^n$ the following estimates hold*

$$0 \le \check{\psi}^\circ(G) \le \check{\psi}^\circ(x,A) \le \check{\rho}^\circ(x), \quad \check{\psi}^\bullet(G) \le \check{\psi}^\bullet(x,A) \le \check{\rho}^\bullet(x),$$
$$\hat{\psi}^\circ(G) \le \hat{\psi}^\circ(x,A) \le \hat{\rho}^\circ(x), \quad \hat{\psi}^\bullet(G) \le \hat{\psi}^\bullet(x,A) \le \hat{\rho}^\bullet(x) \le \|A\|_I.$$

The inequalities in Theorem 3 between the plane rotatability indicators and the wandering indicators are not equalities in general, already for solutions of two-dimensional systems (but non-autonomous, according to Theorem 10 below) as shown by

**Theorem 4.** *There exists a system $A \in \mathcal{M}^2$ such that the plane rotatability indicators of all solutions $x \in \mathcal{S}_*(A)$ are exact, absolute, and the same but do not coincide with the wandering indicators, which are also exact, absolute, and the same:*

$$\psi(x,A) < \rho(x).$$

If in Definition 2 instead of the exact lower bounds over all automorphisms of the phase space the upper bounds are taken, then so defined indicators are upper estimated neither by the seminorm (7) nor by anything else, as shown by

**Theorem 5.** *For any $\varepsilon > 0$ there exists a system $A \in \mathcal{M}^3$ satisfying the conditions*

$$\|A(t)\| \le \begin{cases} \varepsilon, & t \in [0,1], \\ 0, & t \ge 1, \end{cases} \qquad \|A\|_I = 0,$$

*such that all the indicators of some plane $G \in \mathcal{G}^2(A)$ obtained from formulas* (3) *and* (4) *by replacement of all the exact lower bounds by the upper ones equal $\infty$.*

If in Definition 3 instead of the exact upper bounds over all planes of solution space (containing the given solution) the lower bounds are taken, then so defined indicators are too less informative, already for three-dimensional autonomous systems as shown by

**Theorem 6.** *All the indicators of all solutions $x \in \mathcal{S}_*(A)$ of any autonomous $A \in \mathcal{M}^3$ obtained from formulas* (5) *and* (6)*, with the exact upper bounds replaced by the lower ones, equal* 0*.*

In the case of an autonomous system $A \in \mathcal{M}^n$ all the spectra of various indicators from Definitions 2–4 are closely related to the *spectrum* $|\operatorname{Im}\operatorname{Sp}(A)|$ – the set of *absolute values of imaginary parts* of the eigenvalues of the operator $A \in \operatorname{End}\mathbb{R}^n$. This relationship is described by the next three theorems.

**Theorem 7.** *For any autonomous system $A \in \mathcal{M}^n$ the spectrum of the exact absolute rotatability indicator of a plane includes the spectrum $|\operatorname{Im}\operatorname{Sp}(A)|$.*

**Theorem 8.** *There exists an autonomous system $A \in \mathcal{M}^n$ with the spectrum of the exact absolute rotatability indicator of a plane not included in the spectrum $|\operatorname{Im}\operatorname{Sp}(A)|$.*

**Theorem 9.** *For any autonomous system $A \in \mathcal{M}^n$ the spectrum of the exact weak, as well as strong, plane rotatability indicator of a solution coincides with the spectrum $|\operatorname{Im}\operatorname{Sp}(A)|$.*

As an example confirming the validity of Theorem 8, it suffices to take a four-dimensional autonomous system with eigenvalues $\pm i, \pm 2i$: its exact absolute rotatability indicators for at least one of planes equal zero. The proof of Theorem 9 is provided by

**Theorem 10.** *For each solution $x \in \mathcal{S}_*(A)$ of any autonomous system $A \in \mathcal{M}^n$ the weak and strong plane rotatability indicators are exact and coincide with the similar wandering indicators*

$$\psi^\circ(x,A) = \rho^\circ(x), \quad \psi^\bullet(x,A) = \rho^\bullet(x). \tag{8}$$

To prove Theorem 10 it is enough, in its turn, to make sure that the next assertion is true.

**Lemma 2.** *For each solution $x \in \mathcal{S}_*(A)$ of any autonomous system $A \in \mathcal{M}^n$ there exists a linearly independent with $x$ solution $y \in \mathcal{S}_*(A)$ satisfying the condition*

$$\Psi(Lx, Ly, t) = \mathrm{P}(Lx, t), \quad L \in \operatorname{Aut}\mathbb{R}^n, \ \ t \in \mathbb{R}^+.$$

In Lemma 2, in the case when the initial value $x(0)$ of a solution $x$ is an eigenvector for $A \in \operatorname{End}\mathbb{R}^n$ corresponding to a real eigenvalue, any nonzero solution is suitable as a solution $y$ related to the solution $x$, otherwise there is a suitable one, for example, the function $y = Ax$.

**Remark 3.** Applying Theorem 10 and the results of the papers [3, 4] to each of the indicators (8), we can describe the distribution of its values over the space $\mathcal{S}_*(A)$, namely, on the steps of some flag of subspaces in $\mathcal{S}(A)$ it takes constant values ranging in some special order over all the numbers of the spectrum $|\operatorname{Im}\operatorname{Sp}(A)|$.

Theorems 9 and 10 justify the introduction of the plain rotatability indicators of a solution in Definition 3. But equalities (8) do not extend to non-autonomous systems $A \in \mathcal{M}^n$: by Theorem 4 already for $n = 2$ and by Theorem 2 even when the function $x$ is a solution of some autonomous system.

## References


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