Global Attractor of Impulsive Parabolic System Without Uniqueness

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An autonomous evolution system is called impulsive dynamical system (impulsive DS) if its trajectories have jumps at moments of intersection with certain surface of the phase space. These systems are an important subclass of systems with impulsive perturbations at fixed moments of time whose qualitative theory was developed in [6]. In this paper, using the theory of global attractors for multi-valued semiflows [3], we describe the dynamics of infinite-dimensional impulsive systems without uniqueness of solution of the Cauchy problem. We consider global attractor as a minimal uniformly attracting set for corresponding multi-valued semiflow [4]. Using the results of [1, 2], we construct abstract theory of multi-valued impulsive dynamical systems and apply obtained results to weakly non-linear impulsive parabolic system.

Let (X,ρ) be a metric space, $P(X)$ $(\beta(X))$ be a set of all non-empty (non-empty bounded) subset of X.

Definition 1 ([3])**.** A multi-valued map $G: R_+\times X\to P(X)$ is called multi-valued dynamical system (MDS) if

$$\forall\, x\in X\ \ G(0,x)=x \ \text{ and } \ \forall\, t,s\geq 0\ \ G(t+s,x)\subseteq G(t,G(s,x)).$$

Definition 2 ([4])**.** A non-empty subset $\Theta\subset X$ is called a global attractor of MDS G if

1) Θ is a compact set;

2) Θ is uniformly attracting set, i.e., $\forall\, B\in\beta(X)$ $\operatorname{dist}(G(t,B),\Theta)\longrightarrow 0$, $t\to\infty$;

3) Θ is minimal among all closed uniformly attracting sets.

Lemma 1. *Assume that MDS G satisfies dissipativity condition:*

$$\exists\, B_0\in\beta(X),\ \ \forall\, B\in\beta(X),\ \ \exists\, T=T(B)>0,\ \ \forall\, t\geq T\ \ G(t,B)\subset B_0. \tag{1}$$

Then the following conditions are equivalent:

1) *MDS G has a global attractor Θ;*

2) *MDS G is asymptotically compact, i.e.,*

$$\forall\, t_n\nearrow\infty\ \ \forall\, B\in\beta(X),\ \ \forall\,\xi_n\in G(t_n,B)\ \ \textit{sequence}\ \ \{\xi_n\}\ \ \textit{is precompact in}\ X.$$

Impulsive MDS G consists of a given non-empty closed *impulsive* set $M\subset X$, compact-valued *impulsive* map $I: M\to P(X)$ and a given family K of continuous maps $\varphi:[0,+\infty)\to X$ satisfying the following properties:

K1) $\forall\, x\in X,\ \exists\,\varphi\in K:\ \varphi(0)=x$;

K2) $\forall\,\varphi\in K,\ \forall\, s\geq 0\ \ \varphi(\,\cdot\,+s)\in K$.

We denote
$$K_x = \{\varphi \in K \mid \varphi(0) = x\}.$$

Impulsive MDS describes the following behaviour: a phase point moves along trajectories of K and when it meets the impulsive set M, it jumps onto a new position from the set of *impulsive points* IM.

For "well-posedness" of impulsive problem we assume the following conditions:
$$\begin{gathered} M \cap IM = \varnothing; \\ \forall x \in M, \ \forall \varphi \in K_x, \ \exists \tau = \tau(\varphi) > 0, \ \forall t \in (0,\tau) \ \ \varphi(t) \notin M. \end{gathered} \tag{2}$$

We denote
$$\forall \varphi \in K \ M^+(\varphi) = \bigcup_{t>0} \varphi(t) \cap M.$$

If $M^+(\varphi) \neq \varnothing$, then there exists a moment of time $s := s(\varphi) > 0$ such as
$$\forall t \in (0,s) \ \ \varphi(t) \notin M, \ \ \varphi(s) \in M. \tag{3}$$

Hence, we can define the following function : $K \to (0,+\infty]$:
$$s(\varphi) = \begin{cases} s, & \text{if } M^+(\varphi) \neq \varnothing, \\ +\infty, & \text{if } M^+(\varphi) = \varnothing. \end{cases} \tag{4}$$

Impulsive trajectory $\widetilde{\varphi}$, starting from the point $x \in X$, is a right continuous function
$$\widetilde{\varphi}(t) = \begin{cases} \varphi_n(t - t_n), & \text{if } t \in [t_n, t_{n+1}), \\ x^+_{n+1}, & \text{if } t = t_{n+1}, \end{cases} \tag{5}$$

where $\{x^+_n\}_{n\geq 1} \subset IM$ are impulsive points, $\{s_n\}_{n\geq 0} \subset (0,+\infty)$ are the corresponding moments of time, $\{\varphi_n\}_{n\geq 0} \subset K$, $\varphi_0(0) = x$ and $\forall n \geq 0$ $t_0 := 0$, $t_{n+1} := \sum\limits_{k=0}^{n} s_k$, $n \geq 0$.

By $\widetilde{K}_x$ we denote the set of all impulsive trajectories starting from $x \in X$.

We assume that every impulsive trajectory is defined on $[0,+\infty)$, i.e.,
$$\forall x \in X \ \text{ every } \ \widetilde{\varphi} \in \widetilde{K}_x \ \text{ is defined on } \ [0,+\infty). \tag{6}$$

Definition 3. A multi-valued map $G : \mathbb{R}_+ \times X \to P(X)$
$$G(t,x) = \{\widetilde{\varphi}(t) \mid \widetilde{\varphi} \in \widetilde{K}_x\} \tag{7}$$
is called impulsive MDS.

Lemma 2. *Let conditions* K1)*,* K2)*,* (2)*,* (6) *be satisfied. Then* (7) *defines the MDS* G*.*

To state further results concerning invariance property of the global attractor we have to impose additional constraints on the parameters of our impulsive problem:

K3) $\forall x_n \to x$, $\forall \varphi_n \in K_{x_n}$, $\exists \varphi \in K_x$ such that on some subsequence
$$\forall t \geq 0 \ \ \varphi_n(t) \to \varphi(t);$$

I) the compact-valued map $I : M \to P(X)$ is upper-semicontinuous [3];

S1) if for $x \in X \setminus M$, $x_n \to x$, $\varphi_n \in K_{x_n}$ and $\varphi \in K_x$ we have $\forall\, t \geq 0$ $\varphi_n(t) \to \varphi(t)$, then

$$\begin{cases} s(\varphi) = \infty, & \text{if } s(\varphi_n) = \infty \text{ for infinitely many } n \geq 1, \\ s(\varphi_n) \to s(\varphi), & \text{otherwise.} \end{cases}$$

Lemma 3. *Assume that impulsive MDS G satisfies* K1), K2), (2), (6), K3), I), S1) *and Θ is a global attractor of G. Then the following property holds:*

$$\forall\, t \geq 0, \ \ \forall\, \xi \in \Theta \setminus M \ \ G(t, \xi) \cap (\Theta \setminus M) \neq \varnothing. \tag{8}$$

If, additionally, G is single-valued, then

$$\forall\, t \geq 0 \ \ G(t, \Theta \setminus M) \subseteq \Theta \setminus M. \tag{9}$$

In order to prove the inverse embedding in (9)*, it is necessary to impose the following additional assumptions on K, M, I:*

K4) $\forall\, x_n \to x$, $\forall\, \varphi_n \in K_{x_n}$, $\exists\, \varphi \in K_x$ *such that on some subsequence*

$$\varphi_n \to \varphi \ \ \textit{uniformly on every } [a,b] \subset [0, \infty), \tag{10}$$

S2) *if for $\forall\, x_n \notin M$ $x_n \to x \in M$, $\varphi_n \in K_{x_n}$ and $\varphi \in K_x$ we have $\forall\, t \geq 0$ $\varphi_n(t) \to \varphi(t)$, then either $s(\varphi_n) = \infty$ for an infinite number $n \geq 1$,*

$$\textit{or } \ s(\varphi_n) \to 0, \ \ n \to \infty.$$

Lemma 4. *Assume that impulsive MDS G satisfies* K1), K2), (2), (6), K3), I), S1), K4), S2)*, and Θ is a global attractor of G. Then*

$$\forall\, t \geq 0 \ \ \Theta \setminus M \subseteq G(t, \Theta \setminus M). \tag{11}$$

If $\forall\, x \in X$, $\forall\, t, s \geq 0$ $G(t+s, x) = G(t, G(s,x))$, then in (11) *equality takes place.*

We apply obtained results for impulsive weakly non-linear parabolic problem. Let $\Omega \subset R^n$ be a bounded domain. For unknown functions $u(t,x)$, $v(t,x)$ on $(0, +\infty) \times \Omega$ we consider the following weakly non-linear problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a_1 \Delta u + \varepsilon f_1(u,v), \\[2mm] \dfrac{\partial v}{\partial t} = a_2 \Delta v + b \Delta u + \varepsilon f_2(u,v), \end{cases} \tag{12}$$

where $\varepsilon > 0$ is a small parameter, $a_1, a_2 > 0$, $|b| < 2\sqrt{a_1 a_2}$. Continuous non-linear functions $f_i : R^2 \longmapsto R$, $i = 1, 2$ satisfy the following condition:

$$\exists\, C > 0 \ \ \forall\, u, v \in R \ \ |f_1(u,v)| + |f_2(u,v)| \leq C. \tag{13}$$

It is known that under such conditions for every $\varepsilon > 0$, $z_0 \in X$ there exists at least one solution $\varphi(\,\cdot\,) = \begin{pmatrix} u(\,\cdot\,) \\ v(\,\cdot\,) \end{pmatrix} \in C([0, +\infty), X)$ of the problem (12) with $\varphi(0) = z_0$, where $X = L_2(\Omega) \times L_2(\Omega)$ is a phase space.

Thus the problem (12) generates the family of continuous maps:

$$K^{\varepsilon} = \big\{\varphi : [0, +\infty) \to X \mid \ \varphi \text{ is a solution of } (12)\big\},$$

which satisfies conditions K1), K2). For fixed $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\mu > 0$ we consider the following impulsive perturbation:

$$M = \left\{ z = \begin{pmatrix} u \\ v \end{pmatrix} \in X \mid \ \alpha(u, \psi_1) + \beta(v, \psi_1) = 1, \ |(u, \psi_1)| \leq \gamma \right\}, \tag{14}$$

$$I : M \to P(X) \ \text{ such that for } \ z = \sum_{i=1}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \in M,$$

$$Iz \subseteq \left\{ \begin{pmatrix} c_1' \\ d_1' \end{pmatrix} \psi_1 + \sum_{i=2}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \mid \ |c_1'| \leq \gamma, \ \alpha c_1' + \beta d_1' = 1 + \mu \right\}, \tag{15}$$

where $\{\psi_i\}_{i=1}^{\infty}$ are eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$.

Theorem. *For sufficiently small $\varepsilon > 0$ impulsive problem* (12), (14), (15) *generates an impulsive MDS $G_{\varepsilon} : R_+ \times X \longmapsto P(X)$, which has a global attractor Θ_{ε} and*

$$\operatorname{dist}(\Theta_{\varepsilon}, \Theta) \longrightarrow 0, \ \ \varepsilon \to 0, \tag{16}$$

where Θ is global attractor of impulsive system (12), (14), (15) *with $\varepsilon = 0$.*

Moreover, if $I : M \longmapsto P(X)$ is upper semicontinuous map, then

$$\forall\, t \geq 0 \ \ G_{\varepsilon}(t, \Theta_{\varepsilon} \setminus M) = \Theta_{\varepsilon} \setminus M. \tag{17}$$

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