The Cauchy–Nicoletti Weighted Problem for Nonlinear Singular Functional Differential Systems

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Let $-\infty < a < b < +\infty$, and let $J \subset [a,b]$ be the measurable set such that

$$J \neq [a,b], \ \ \operatorname{mes} J = b-a.$$

Consider the functional differential system with deviating arguments

$$\frac{du_i(t)}{dt} = f_i\big(t, u_1(t), \dots, u_n(t), u_1(\tau_1(t)), \dots, u_n(\tau_n(t))\big) \ \ (i=1,\dots,n) \tag{1}$$

with the weighted boundary conditions

$$\limsup_{t\to t_i} \frac{|u_i(t)|}{\varphi_i(t)} < +\infty \ \ (i=1,\dots,n). \tag{2}$$

Here $f_i : J \times \mathbb{R}^{2n} \to \mathbb{R}$ $(i=1,\dots,n)$ are measurable in the first and continuous in the last $2n$ arguments function,

$$t_i \in [a,b] \setminus J \ \ (i=1,\dots,n),$$

while $\varphi_i : [a,b] \to \mathbb{R}$ $(i=1,\dots,n)$ and $\tau_i : J \to [a,b]$ $(i=1,\dots,n)$ are, respectively, absolutely continuous and continuous functions such that

$$\varphi_i(t) > 0 \quad \text{for} \ \ t \neq t_i \ \ (i=1,\dots,n),$$
$$\varphi_i'(t)(t-t_i) \geq 0, \quad \tau_i(t) \neq t_i \quad \text{for} \ \ t \in J \ \ (i=1,\dots,n).$$

A vector function $(u_i)_{i=1}^n : [a,b] \to \mathbb{R}^n$ with absolutely continuous components $u_1,\dots,u_n$ is said to be **a solution of system** (1) if it satisfies that system almost everywhere on J. The solution $(u_i)_{i=1}^n$ of system (1) is said to be **a solution of problem** (1)**,** (2) if it satisfies conditions (2).

Note that the boundary conditions

$$u_i(t_i) = 0 \ \ (i=1,\dots,n) \tag{3}$$

are called Cauchy–Nicoletti conditions, and problem (1), (3) is said to be a Cauchy–Nicoletti problem (see, e.g., [1–3, 5–8], where the Cauchy–Nicoletti problem is investigated both for differential and functional differential systems). Thus it is natural to call the boundary conditions (2) and problem (1), (2) the Cauchy–Nicoletti weighted conditions and the Cauchy–Nicoletti weighted problem, respectively.

We are interested in study of problem (1), (2) in the case where system (1) has non-integrable singularities in the time variable, i.e., where

$$\int\limits_a^b \Big(\sum_{i=1}^n \big| f_i(t, x_1, \dots, x_n, y_1, \dots, y_n) \big| \Big)\, dt = +\infty \quad \text{if} \ \ \sum_{i=1}^n (|x_i| + |y_i|) > 0.$$

For singular systems of ordinary differential equations, the unimprovable conditions for the solvability and unique solvability of the Cauchy–Nicoletti weighted problem are established by I. Kiguradze [2, 4]. In this paper, analogous results are obtained for the singular problem (1), (2).

Below everywhere we use the following notation.

- $I_k = [a,b] \setminus \{t_k\}$ $(k = 1, \dots, n)$.
- $\chi_k(t,\delta,\lambda) = \begin{cases} 0 & \text{if } t \in [t_k - \delta, t_k + \delta], \\ \lambda & \text{if } t \notin [t_k - \delta, t_k + \delta]. \end{cases}$
- $L_{loc}(I_k;\mathbb{R})$ is the set of Lebesgue integrable on each closed interval contained in I_k functions $v : I_k \to \mathbb{R}$.
- $X = (x_{ik})_{i,k=1}^n$ is the $n \times n$ matrix with the components x_{ik} $(i,k = 1, \dots, n)$.
- $r(X)$ is the spectral radius of the matrix X.

Moreover, below everywhere it is assumed that

$$f^*_{\rho,k} \in L_{loc}(I_k;\mathbb{R}) \quad \text{for every } \ \rho > 0 \ \ (k = 1, \dots, n),$$

where

$$f^*_{\rho,k}(t) = \max\bigg\{\Big|f_k\Big(t, \varphi_1(t)x_1, \dots, \varphi_n(t)x_n, \varphi_1(\tau_1(t))y_1, \dots, \varphi_n(\tau_n(t))y_n\Big)\Big| : \ \sum_{i=1}^n (|x_i| + |y_i|) \le \rho\bigg\}.$$

Along with (1) we consider the functional differential system

$$\frac{du_i(t)}{dt} = \chi_i(t,\delta,\lambda) f_i\big(t, u_1(t), \dots, u_n(t), u_1(\tau_1(t)), \dots, u_n(\tau_n(t))\big) \ \ (i = 1, \dots, n), \tag{4}$$

depended on parameters $\lambda \in [0,1]$ and $\delta \in]0,1[$.

Theorem 1 (A principle of a priori boundedness). *Let there exist a positive constant ρ such that for every $\delta \in]0,1[$ and $\lambda \in [0,1]$ any solution $(u_i)_{i=1}^n$ of problem (4), (2) admits the estimates*

$$|u_i(t)| \le \rho\varphi_i(t) \quad \textit{for } \ t \in [a,b] \ \ (i = 1, \dots, n).$$

Then problem (1), (2) has at least one solution.

Theorem 2. *Let on the set $J \times \mathbb{R}^{2n}$ the inequalities*

$$f_i(t, x_1, \dots, x_n, y_1, \dots, y_n) \operatorname{sgn}[(t - t_i)x_i] \\ \le |\varphi_i'(t)| \Big[\sum_{k=1}^n \big(p_{1ik} \frac{|x_k|}{\varphi_k(t)} + p_{2ik} \frac{|y_k|}{\varphi_k(\tau_k(t))}\big) + q\Big] \ \ (i = 1, \dots, n)$$

be fulfilled, where p_{1ik}, p_{2ik} $(i,k = 1, \dots, n)$ and q are nonnegative constants, at that the matrix $\mathcal{P} = (p_{1ik} + p_{2ik})_{i,k=1}^n$ satisfies the inequality

$$r(\mathcal{P}) < 1. \tag{5}$$

Then problem (1), (2) has at least one solution.

Remark 1. Under the conditions of Theorem 2, each function f_i may have the singularity of arbitrary order at the point t_i. Indeed, if $\varphi_i(t) = |t - t_i|$ $(i = 1, \dots, n)$, then the conditions of the above-mentioned theorem are satisfied, for example, by the functions

$$f_i(t, x_1, \dots, x_n, y_1, \dots, y_n) = \exp\Big(\frac{1 + |x_1| + \dots + |x_n| + |y_1| + \dots + |y_n|}{|t - t_i|}\Big)(t_i - t)x_i \\ + \sum_{k=1}^n \Big(p_{1ik} \frac{|x_k|}{|t - t_k|} + p_{2ik} \frac{|y_k|}{|\tau_k(t) - t_k|}\Big) + q \ \ (i = 1, \dots, n).$$

Condition (5) in Theorem 2 is unimprovable and it cannot be replaced by the condition

$$r(\mathcal{P}) \le 1.$$

What is more, the following theorem is valid.

Theorem 3. *Let on the set $J \times \mathbb{R}^{2n}$ the inequalities*

$$f_i(t, x_1, \dots, x_n, y_1, \dots, y_n)\,\mathrm{sgn}(t - t_i) \\ \ge |\varphi_i'(t)| \Big[\sum_{k=1}^n \Big(p_{1ik}\, \frac{|x_k|}{\varphi_k(t)} + p_{2ik}\, \frac{|y_k|}{\varphi_k(\tau_k(t))} \Big) + q \Big] \quad (i = 1, \dots, n)$$

be fulfilled, where $p_{1ik} \ge 0$, $p_{2ik} \ge 0$ $(i, k = 1, \dots, n)$, $q > 0$, and the matrix $\mathcal{P} = (p_{1ik} + p_{2ik})_{i,k=1}^n$ satisfies the inequality

$$r(\mathcal{P}) \ge 1.$$

Then problem $(1), (2)$ has no solution.

Along with $(1), (2)$ let us consider the perturbed problem

$$\frac{dv_i(t)}{dt} = f_i\big(t, v_1(t), \dots, v_n(t), v_1(\tau_1(t)), \dots, v_n(\tau_n(t))\big) + h_i(t) \quad (i = 1, \dots, n), \tag{6}$$

$$\limsup_{t \to t_i} \frac{|v_i(t)|}{\varphi_i(t)} < +\infty \quad (i = 1, \dots, n), \tag{7}$$

and introduce

Definition. Problem $(1), (2)$ is said to be **well-posed** if:

(i) it has a unique solution $(u_i)_{i=1}^n$;

(ii) there exists a positive constant ρ such that for arbitrary integrable functions $h_k : J \to \mathbb{R}$ $(k = 1, \dots, n)$, satisfying the conditions

$$\nu_k(h_k) = \sup \bigg\{ \frac{1}{\varphi_k(t)} \bigg| \int_{t_k}^t |h_k(s)|\, ds \bigg| :\ t \in I_k \bigg\} < +\infty \quad (k = 1, \dots, n),$$

problem $(6), (7)$ is solvable and its every solution satisfies the inequalities

$$|v_i(t) - u_i(t)| \le \rho \Big[\sum_{k=1}^n \nu_k(h_k) \Big] \varphi_i(t) \quad \text{for } t \in [a, b] \quad (i = 1, \dots, n).$$

Theorem 4. *Let on the set $J \times \mathbb{R}^{2n}$ the inequalities*

$$f_i(t, x_1, \dots, x_n, y_1, \dots, y_n)\,\mathrm{sgn}[(t - t_i) x_i] \\ \le |\varphi_i'(t)| \sum_{k=1}^n \Big(p_{1ik}\, \frac{|x_k|}{\varphi_k(t)} + p_{2ik}\, \frac{|y_k|}{\varphi_k(\tau_k(t))} \Big) \quad (i = 1, \dots, n)$$

be fulfilled, where p_{1ik}, p_{2ik} $(i, k = 1, \dots, n)$ are nonnegative constants, and the matrix $\mathcal{P} = (p_{1ik} + p_{2ik})_{i,k=1}^n$ satisfies inequality (5). Then problem $(1), (2)$ is well-posed.

Theorems 3 and 4 yield the following result.

Corollary 1. *Let on the set* $J \times \mathbb{R}^{2n}$ *the equalities*

$$f_i(t, x_1, \dots, x_n, y_1, \dots, y_n) = \varphi_i'(t) \sum_{k=1}^{n} \Big(p_{1ik} \, \frac{|x_k|}{\varphi_k(t)} + p_{2ik} \, \frac{|y_k|}{\varphi_k(\tau_k(t))} \Big) \;\; (i = 1, \dots, n)$$

hold, where p_{1ik}, p_{2ik} $(i, k = 1, \dots, n)$ *are nonnegative constants. Then for problem* (1), (2) *to be well-posed it is necessary and sufficient that the matrix* $\mathcal{P} = (p_{1ik} + p_{2ik})_{i,k=1}^{n}$ *to satisfy inequality* (5)*.*

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