On Non-Negative Periodic Solutions of Second-Order Differential Equations with Mixed Sub-Linear and Super-Linear Non-Linearities

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Consider the periodic problem

$$\boxed{u'' = p(t)u + q(t,u); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega),} \tag{1}$$

where $p \in L([0,\omega])$ and $q\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Under a *solution* of problem (1), as usually, we understand a function $u\colon [0,\omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies given equation almost everywhere and verifies periodic conditions.

We are interested in the existence and uniqueness of a **non-trivial non-negative** solution of problem (1) in the case when the function q may contain both sub-linear and super-linear non-linearities. In particular, it follows from Corollary 4 stated below that for an arbitrary $p \in L([0,\omega])$, the problem

$$u'' = p(t)u + \sqrt[3]{u} - u^3; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)$$

has at least one non-trivial non-negative solution.

Definition 1. We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}^+(\omega)$ (resp. $\mathcal{V}^-(\omega)$) if for any function $u \in AC^1([0,\omega])$ satisfying

$$u''(t) \geq p(t)u(t) \quad \text{for a.e. } t \in [0,\omega], \quad u(0) = u(\omega), \quad u'(0) = u'(\omega),$$

the inequality

$$u(t) \geq 0 \quad \text{for } t \in [0,\omega] \quad \big(\text{resp. } u(t) \leq 0 \quad \text{for } t \in [0,\omega]\big)$$

is fulfilled.

Definition 2. We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}_0(\omega)$ if the problem

$$u'' = p(t)u; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)$$

has a nontrivial sign-constant solution.

Introduce the hypothesis

$$\left.\begin{aligned}
&q(t,x)\le q_0(t,x) \quad \text{for a.e. } t\in[0,\omega] \text{ and all } x\ge x_0,\\
&x_0\ge 0, \quad q_0\colon [0,\omega]\times[x_0,+\infty[\,\to[0,+\infty[\text{ is a Carathéodory function,}\\
&q_0(t,\,\cdot\,)\colon [x_0,+\infty[\,\to[0,+\infty[\text{ is non-decreasing for a.e. } t\in[0,\omega],\\
&\lim_{x\to+\infty}\frac{1}{x}\int_0^\omega q_0(s,x)\,\mathrm{d}s=0.
\end{aligned}\right\}\tag{H_1}$$

A general existence result reads as follows.

Theorem 1. *Let hypothesis* (H_1) *be fulfilled and*

$$q(t,0)\le 0 \quad \text{for a.e. } t\in[0,\omega].\tag{2}$$

Let, moreover, there exist functions $\alpha,\beta\in AC^1([0,\omega])$ *satisfying*

$$\begin{gathered}
\alpha(t)>0, \quad \beta(t)>0 \quad \text{for } t\in[0,\omega],\\
\alpha''(t)\ge p(t)\alpha(t)+q(t,\alpha(t)) \quad \text{for a.e. } t\in[0,\omega], \quad \alpha(0)=\alpha(\omega), \quad \alpha'(0)\ge\alpha'(\omega),\\
\beta''(t)\le p(t)\beta(t)+q(t,\beta(t)) \quad \text{for a.e. } t\in[0,\omega], \quad \beta(0)=\beta(\omega), \quad \beta'(0)\le\beta'(\omega).
\end{gathered}$$

Then problem (1) *has at least one solution* u *such that*

$$u(t)\ge 0 \quad \text{for } t\in[0,\omega], \quad u\not\equiv 0,\tag{3}$$

and

$$\min\big\{\alpha(t_u),\beta(t_u)\big\}\le u(t_u)\le\max\big\{\alpha(t_u),\beta(t_u)\big\} \quad \text{for some } t_u\in[0,\omega].\tag{4}$$

Corollary 1. *Let inequality* (2) *hold, hypothesis* (H_1) *be fulfilled,*

$$\left.\begin{aligned}
&q(t,x)\le -xg(t,x) \quad \text{for a.e. } t\in[0,\omega] \text{ and all } x>\kappa,\\
&\kappa\ge 0, \quad g\colon [0,\omega]\times\,]\kappa,+\infty[\,\to\mathbb{R} \text{ is a locally Carathéodory function,}\\
&g(t,\,\cdot\,)\colon\,]\kappa,+\infty[\,\to\mathbb{R} \text{ is non-decreasing for a.e. } t\in[0,\omega],
\end{aligned}\right\}\tag{H_2}$$

and

$$\left.\begin{aligned}
&q(t,x)\ge xg_1(t,x)-g_2(t,x) \quad \text{for a.e. } t\in[0,\omega] \text{ and all } x\in\,]0,\delta],\\
&\delta>0, \quad g_1,g_2\colon [0,\omega]\times\,]0,\delta]\to\mathbb{R} \text{ are locally Carathéodory functions,}\\
&g_1(t,\,\cdot\,)\colon\,]0,\delta]\to\mathbb{R} \text{ is non-increasing for a.e. } t\in[0,\omega],\\
&g_2(t,\,\cdot\,)\colon\,]0,\delta]\to\mathbb{R} \text{ is non-decreasing for a.e. } t\in[0,\omega],\\
&\lim_{x\to0+}\frac{1}{x}\int_0^\omega |g_2(s,x)|\,\mathrm{d}s=0.
\end{aligned}\right\}\tag{H_3}$$

Let, moreover, there exist a non-negative function $\ell\in L([0,\omega])$ *and numbers* $r_1\in\,]0,\delta]$, $r_2>\kappa$ *such that*

$$p+g_1(\,\cdot\,,r_1)\in\mathcal{V}^-(\omega), \quad p+\ell-g(\,\cdot\,,r_2)\in\operatorname{Int}\mathcal{V}^+(\omega).$$

Then problem (1) *has at least one solution* u *satisfying condition* (3).

Now we provide efficient conditions guaranteeing the existence of a non-trivial non-negative solution of problem (1).

Corollary 2. *Let inequality* (2) *hold, hypotheses* (H_1), (H_2), *and* (H_3) *be fulfilled, and*

$$\lim_{x\to\kappa+} g(t,x)\leq 0 \quad \text{for a.e. } t\in[0,\omega], \qquad \lim_{x\to+\infty}\int_0^\omega g(s,x)\,\mathrm{d}s=+\infty. \tag{5}$$

Let, moreover, at least one of the following conditions be satisfied:

(a) $p\in\mathcal{V}^-(\omega)$ *and*

$$g_1(t,\delta)\geq 0 \quad \text{for a.e. } t\in[0,\omega]; \tag{6}$$

(b) $p\in\mathcal{V}_0(\omega)$, *inequality* (6) *holds, and* $g_1(\,\cdot\,,\delta)\not\equiv 0$;

(c) $p\in\mathcal{V}^+(\omega)$, *inequality* (6) *holds, and*

$$\lim_{x\to 0+}\int_0^\omega g_1(s,x)\,\mathrm{d}s=+\infty; \tag{7}$$

(d)

$$\lim_{x\to 0+}\int_E g_1(s,x)\,\mathrm{d}s=+\infty \quad \text{for every } E\subseteq[0,\omega], \ \ \operatorname{meas}E>0. \tag{8}$$

Then problem (1) *has at least one solution* u *satisfying condition* (3).

Further, we present some consequences of the general results for the following particular cases of (1):

$$\boxed{u''=p(t)u+h(t)\ln(1+|u|)-f(t)\ln(1+|u|)u; \quad u(0)=u(\omega), \ \ u'(0)=u'(\omega)} \tag{9}$$

and

$$\boxed{u''=p(t)u+h(t)|u|^\lambda\operatorname{sgn}u-f(t)|u|^\mu\operatorname{sgn}u; \quad u(0)=u(\omega), \ \ u'(0)=u'(\omega),} \tag{10}$$

where $h,f\in L([0,\omega])$ and $\lambda,\mu>0$, $(1-\lambda)(\mu-1)>0$.

Corollary 3. *Let*

$$f(t)\geq 0 \quad \text{for a.e. } t\in[0,\omega], \ \ f\not\equiv 0, \tag{11}$$

and

$$h(t)\geq 0 \quad \text{for a.e. } t\in[0,\omega].$$

Then problem (9) *has a positive solution if and only if* $p+h\in\mathcal{V}^-(\omega)$.

Concerning problem (10), we first recall a known result in the case, when $0<\mu<1<\lambda$.

Proposition 1. *Let* $0<\mu<1<\lambda$ *and*

$$h(t)\geq 0, \quad f(t)\geq 0 \quad \text{for a.e. } t\in[0,\omega], \ \ h\not\equiv 0, \ \ f\not\equiv 0. \tag{12}$$

If, moreover, $p\in\mathcal{V}^-(\omega)$, *then problem* (10) *has a positive solution.*

Definition 3. We say that the function $p\in L([0,\omega])$ belongs to the set $\mathcal{D}(\omega)$ if the problem

$$u''=\widetilde{p}(t)u; \quad u(a)=0, \ \ u(b)=0$$

has no non-trivial solution for any $a,b\in\mathbb{R}$ satisfying $0<b-a<\omega$, where $\widetilde{p}$ is the ω-periodic extension of the function p to the whole real axis.

For the case, when $0<\lambda<1<\mu$, we get the following statement.

Corollary 4. *Let* $0<\lambda<1<\mu$, *relation* (11) *hold, and one of the following conditions be satisfied:*

(1) $h(t)>0$ *for a.e.* $t\in[0,\omega]$;

(2) $h(t)\geq 0$ *for a.e.* $t\in[0,\omega]$, $h\not\equiv 0$, *and* $p\in\mathcal{D}(\omega)$.

Then problem (10) *has at least one non-trivial non-negative solution.*

Finally, we discuss the question of the positivity of solutions of problem (10), where $0<\lambda<1<\mu$. We start with the following proposition, which provides a sufficient condition guaranteeing that any non-trivial sign-constant solution of problem (10) has no zero, i. e., it is either positive or negative.

Proposition 2. *Let* $p\in\operatorname{Int}\mathcal{D}(\omega)$. *Then there exists* $\varrho>0$ *such that for any* $\lambda\in\,]0,1[\,$, $\mu>1$, *and* $h,f\in L([0,\omega])$ *satisfying conditions* (12) *and*

$$\Big(\frac{\omega}{4}\Big)^{\frac{\mu-1}{1-\lambda}}\,\mathrm{e}^{\frac{\omega(\mu-1)}{8(1-\lambda)}\,\|[p]_+\|_L}\,\|h\|_L^{\frac{\mu-1}{1-\lambda}}\,\|f\|_L\leq\varrho, \tag{13}$$

any non-trivial non-negative solution of problem (10) *is positive.*

In some particular cases, the number ϱ appearing in Proposition 2 can be estimated from below. For example, the following statement holds.

Corollary 5. *Let* $0<\lambda<1<\mu$, *condition* (12) *hold, and*

$$\|[p]_-\|_L<\frac{4}{\omega}\,,$$

$$\Big(\frac{\omega}{4}\Big)^{\frac{\mu-1}{1-\lambda}}\,\mathrm{e}^{\frac{\omega(\mu-1)}{8(1-\lambda)}\,\|[p]_+\|_L}\,\|h\|_L^{\frac{\mu-1}{1-\lambda}}\,\|f\|_L\leq\frac{4}{\omega}-\|[p]_-\|_L\,. \tag{14}$$

Then problem (10) *has at least one positive solution. Moreover, every non-trivial non-negative solution of problem* (10) *is positive.*

The assertion of the previous corollary remains true if $p\in\mathcal{V}^+(\omega)$ and the point-wise condition (15) is satisfied instead of (14).

Corollary 6. *Let* $0<\lambda<1<\mu$, $p\in\mathcal{V}^+(\omega)$, *condition* (12) *hold, and*

$$\Big(\frac{\omega}{4}\,\|h\|_L\,\mathrm{e}^{\frac{\omega}{8}\,\|[p]_+\|_L}\Big)^{\frac{\mu-\lambda}{1-\lambda}}f(t)\leq h(t)\quad\text{for a.e. } t\in[0,\omega]. \tag{15}$$

Then problem (10) *has at least one positive solution. Moreover, every non-trivial non-negative solution of problem* (10) *is positive.*

Remark 1. The inclusion $p\in\mathcal{V}^+(\omega)$ holds, for example, if

$$\|[p]_+\|_L\leq\|[p]_-\|_L\leq\frac{4}{\omega}\,,\quad p\not\equiv 0.$$

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