On Asymptotic Behavior of Singular $P_{t_*}(Y_0, Y_1, \lambda_0)$-Solutions of Second-Order Differential Equations

L. I. Kusik

Odessa National Marine University, Odessa, Ukraine
E-mail: ludakusik@mail.ru

Consider the differential equation

$$y'' = f(t, y, y'), \tag{1}$$

where $f : [a, \omega[\, \times \Delta_{Y_0} \times \Delta_{Y_1} \to \mathbf{R}$ is continuous function, $-\infty < a < \omega \le +\infty$, Δ_{Y_i} $(i \in \{0,1\})$ is a one-side neighborhood of Y_i and Y_i $(i \in \{0,1\})$ is either 0 or $\pm\infty$. We assume that the numbers μ_i $(i = 0, 1)$ given by the formula

$$\mu_i = \begin{cases} 1 & \text{if either } \ Y_i = +\infty, \ \text{ or } \ Y_i = 0 \ \text{ and } \ \Delta_{Y_i} \ \text{ is right neighborhood of the point 0,} \\ -1 & \text{if either } \ Y_i = -\infty, \ \text{ or } \ Y_i = 0 \ \text{ and } \ \Delta_{Y_i} \ \text{ is left neighborhood of the point 0,} \end{cases}$$

satisfy the relations

$$\mu_0 \mu_1 > 0 \ \text{ for } \ Y_0 = \pm\infty \ \text{ and } \ \mu_0 \mu_1 < 0 \ \text{ for } \ Y_0 = 0. \tag{2}$$

Conditions (2) are necessary for the existence of solutions of Eq. (1) defined in a left neighborhood of ω and satisfying the conditions

$$y^{(i)}(t) \in \Delta_{Y_i} \ \text{ for } \ t \in [t_0, \omega[\,, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1).$$

In monograph [1] definitions of singular solutions of first and second kinds are introduced. Here we study Eq. (1) on class singular $P_{t_*}(Y_0, Y_1, \lambda_0)$-solutions, that are defined as follows.

Definition 1. Let $t_* < \omega$. A solution y of Eq. (1) on interval $[t_0, t_*[\, \subset [a, \omega[$ is called singular $P_{t_*}(Y_0, Y_1, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it satisfies the conditions

$$y^{(i)}(t) \in \Delta_{Y_i} \ \text{ for } \ t \in [t_0, t_*[\,, \quad \lim_{t \uparrow t_*} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t \uparrow t_*} \frac{[y'(t)]^2}{y(t) y''(t)} = \lambda_0.$$

Note that the singular $P_{t_*}(Y_0, Y_1, \lambda_0)$-solution of Eq. (1) is noncontinuable to the right solution. Depending on the values of λ_0 the set of all such solutions of Eq.(1) is disconnected into 4 disjoint subsets respective to the values of λ_0: $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$, $\lambda_0 = 0$, $\lambda_0 = 1$, $\lambda_0 = \pm\infty$. Here we'll formulate the properties of singular $P_{t_*}(Y_0, Y_1, \lambda_0)$-solutions that correspond to the value $\lambda_0 = \pm\infty$. With this aim, we impose a restriction on the function f.

Definition 2. We say that a function f satisfies condition $(\mathrm{RN})^*_\infty$ if there exists a number $\alpha_0 \in \{-1, 1\}$, a positive number A_* and continuous functions $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i = 0, 1)$ of orders σ_i $(i = 0, 1)$ regular varying [1] as $z \to Y_i$ $(i = 0, 1)$ such that for arbitrary continuously differentiable functions $z_i : [a, \omega[\ \Delta_{Y_i}$ $(i = 0, 1)$ satisfying the conditions

$$\lim_{t \uparrow t_*} z_i(t) = Y_i \ \ (i = 0, 1),$$

$$\lim_{t \uparrow t_*} \frac{(t - t_*) z_0'(t)}{z_0(t)} = 1, \quad \lim_{t \uparrow t_*} \frac{(t - t_*) z_1'(t)}{z_1(t)} = 0,$$

[1]Definition of regular varying function see in [2].

one has representation

$$f(t,z_0(t),z_1(t))=\alpha_0A_*\varphi_0(z_0(t))\varphi_1(z_1(t))[1+o(1)] \ \text{ as } \ t\uparrow t_*.$$

For each singular $P_{t_*}(Y_0,Y_1,\lambda_0)$-solution assuming that the function f satisfies condition $(\mathrm{RN})^*_\infty$ with condition (2) we have

$$\alpha_0\mu_1>0 \ \text{ for } \ Y_1=\pm\infty \ \text{ and } \ \alpha_0\mu_1<0 \ \text{ for } \ Y_1=0. \tag{3}$$

Definition 3. We say that a slowly varying as $z\to Y_i$ $(z\in\Delta_{Y_i})$ $(i\in\{0,1\})$ function $L:\Delta_{Y_i}\to\,]0;+\infty[$ satisfies the condition S if for any continuous differentiable function $l:\Delta_{Y_i}\to]0;+\infty[$, such that

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}}\frac{zl'(z)}{l(z)}=0,$$

the following condition takes place

$$L(zl(z))=L(z)(1+o(1)) \ \text{ as } \ z\to Y_i \ \ (z\in\Delta_{Y_i}).$$

We introduce an auxiliary function $\overline{I_\infty}$ by the formula

$$\overline{I_\infty}(t)=\int\limits_{A_\infty}^{t}(t_*-\tau)^{-1}L_0(\mu_0(t_*-\tau))\,d\tau,$$

where the integration limit $A_\infty\in\{a_\infty;t_*\}$ $(a_\infty>a)$ is chosen so as the integrals $\overline{I_\infty}$ tends either to zero or to $\pm\infty$ as $t\uparrow t_*$, $L_0(z)=\varphi_0(z)|z|^{-\sigma_0}$.

Theorem 1. [1] *Let the function f satisfy condition* $(\mathrm{RN})_{\lambda_0}$, *the function φ_0 satisfy condition S. Moreover, let the orders σ_i $(i=0,1)$ of the functions φ_i $(i=0,1)$ regularly varying as $y^{(i)}\to Y_i$ $(i=0,1)$ satisfy the condition $\sigma_0+\sigma_1\neq1$. Then, for the existence of singular $P_{t_*}(Y_0,Y_1,\lambda_0)$-solutions of the differential equation* (1), *it is necessary and sufficient that together with conditions* (2), (3) *the conditions*

$$\sigma_0=-1,\quad \sigma_1\neq2,\quad Y_0=0,\quad Y_1=\mu_1\lim_{t\uparrow t_*}|\overline{I_\infty}(t)|^{\frac{1}{2-\sigma_1}},$$
$$\mu_0\mu_1<0,\quad \alpha_0\mu_1(2-\sigma_1)\overline{I_\infty}(t)>0 \ \ \textit{as} \ \ t\in]a_\infty,t_*[$$

hold. Moreover, each solution of this kind admits the asymptotic representations

$$\frac{y'(t)^2}{\varphi_1(y'(t))}=\alpha_0\mu_1(2-\sigma_1)A_*\overline{I_\infty}(t)[1+o(1)],\quad \frac{y'(t)}{y(t)}=\frac{(1+o(1))}{(t-t_*)} \ \ \textit{as} \ \ t\uparrow\omega$$

and such solutions form a one-parameter family if $\alpha_0\mu_1(2-\sigma_1)>0$.

Theorem 2. *Let the function f satisfy condition* $(\mathrm{RN})_{\lambda_0}$, *the functions φ_0, φ_1 satisfy condition S, $\sigma_0+\sigma_1\neq1$. Then each singular $P_{t_*}(Y_0,Y_1,\lambda_0)$-solutions (in case of the existence) of the differential equation* (1) *admits the asymptotic representations*

$$y(t)=\mu_0(t_*-t)\Big(|2-\sigma_1|A_*|\overline{I_\infty}(t)|L_1\big(\mu_1A_*|\,|\overline{I_\infty}(t)|^{\frac{1}{2-\sigma_1}}\big)\Big)^{\frac{1}{2-\sigma_1}}(1+o(1)),$$
$$y'(t)=\mu_1\Big(|2-\sigma_1|A_*|\overline{I_\infty}(t)|L_1\big(\mu_1A_*||\overline{I_\infty}(t)|^{\frac{1}{2-\sigma_1}}\big)\Big)^{\frac{1}{2-\sigma_1}}(1+o(1)) \ \ \textit{as} \ \ t\uparrow t_*.$$

[1]Theorem 1, Theorem 2 are obtained as corollaries from theorems of [3].

To illustrate Theorem 1, we give the result of Eq. (1) of special form

$$y'' = \frac{\sum\limits_{k=1}^{m} \alpha_k A_{*k} \varphi_{k0}(y)\varphi_{k1}(y')}{\sum\limits_{k=m+1}^{m+n} \alpha_k A_{*k} \varphi_{k0}(y)\varphi_{k1}(y')}\,, \tag{4}$$

where $\alpha_k \in \{-1,1\}$ $(k=1,\dots,m{+}n)$, $A_{*k} = const > 0$ $(k=1,\dots,m{+}n)$ and $\varphi_{ki} : \Delta_{Y_i} \to]0,+\infty[$ $(k=1,\dots,n+m;\ i=0,1)$ are regular varying as $z \to Y_i$ continuous functions of σ_{ki}-th orders.

Theorem 3. *Let for any $i \in \{1,\dots,m\}$, $j \in \{m+1,\dots,m+n\}$ inequalities*

$$\sigma_{i0} - \sigma_{j0} + \sigma_{i1} - \sigma_{j1} \neq 1, \quad \sigma_{i0} - \sigma_{k0} < 0 \ \textit{ for } \ k \in \{1,\dots,m\} \setminus \{i\},$$
$$\sigma_{j0} - \sigma_{k0} < 0 \ \textit{ for } \ k \in \{m+1,\dots,m+n\} \setminus \{j\}$$

hold and function $\frac{\varphi_{i0}}{\varphi_{j0}}$ satisfy condition S. Then, for the existence of singular $P_{t_*}(Y_0,Y_1,\lambda_0)$-solutions of the differential equation (4), it is necessary and sufficient that together with conditions (2), (3) the conditions

$$\mu_0\mu_1 < 0, \quad \alpha_i\alpha_j\mu_1(2-\sigma_{i1}-\sigma_{j1})\overline{I_{\infty ij}}(t) > 0 \ \textit{ as } \ t \in]a_\infty, t_*[\,,$$
$$\sigma_{i0} - \sigma_{j0} = -1, \quad \sigma_{i1} - \sigma_{j1} \neq 2, \quad Y_0 = 0, \quad Y_1 = \mu_1 \lim_{t \uparrow t_*} |\overline{I_{\infty ij}(t)}|^{\frac{1}{2-\sigma_{i1}-\sigma_{j1}}},$$

where

$$\overline{I_{\infty ij}}(t) = \int\limits_{A_\infty}^{t} (t_* - \tau)^{-1}\, \frac{L_{0i}(\mu_0(t_*-\tau))}{L_{0j}(\mu_0(t_*-\tau))}\, d\tau, \quad L_{0k}z = \varphi_{0k}(z)|z|^{\sigma_{0k}}, \ \ k = i,j,$$

hold. Moreover, each solution of this kind admits the asymptotic representations

$$\frac{y'(t)^2\varphi_{j1}(y'(t))}{\varphi_{i1}(y'(t))} = \alpha_i\alpha_j\mu_1(2-\sigma_{i1}+\sigma_{j1})\,\frac{A_{*i}}{A_{*j}}\overline{I_{\infty ij}}(t)[1+o(1)], \quad \frac{y'(t)}{y(t)} = \frac{(1+o(1))}{(t-t_*)} \ \ \textit{as} \ \ t \uparrow \omega$$

and such solutions form a one-parameter family if $\alpha_i\alpha_j\mu_1(2-\sigma_{i1}+\sigma_{j1}) > 0$.

References

[1] I. T. Kiguradze and T. A. Chanturia, Asymptotic properties of solutions of nonautonomous ordinary differential equations. (Russian) *Nauka, Moscow*, 1990.

[2] E. Seneta, Regularly varying functions. *Lecture Notes in Mathematics*, Vol. 508. *Springer-Verlag, Berlin–New York*, 1976.

[3] Л. И. Кусик, Признаки существования одного класса нелинейных дифференциальных уравнений второго порядка. *Вестник Одесского нац. ун-та* **17** (2012), вып. 1-2 (13-14), Математика и механика, 80–97.