Linear Stochastic Functional Differential Equations: Stability and N. V. Azbelev's W-Method

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The W-method, in its present form, was proposed by N. V. Azbelev, but according to his comment in [2] it goes back to G. Fubini and F. Tricomi. The method described originally a way to regularize boundary value problems for deterministic differential equations (see e.g. [2,3]). Later on the method has been developed, generalized and applied in the stability theory for determinsitic [1,4,5] and stochastic [6–9] functional differential equations.

Below we describe general principles of the W-method in connection with stochastic functional differential equations.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an increasing, right-continuous family (a filtration) $(\mathcal{F}_t)_{t\geq 0}$ of complete σ-subalgebras of $\mathcal{F}$. By E we denote the expectation on this probability space.

The space k^n consists of all n-dimensional, $\mathcal{F}_0$-measurable random variables, and $k = k^1$ is a commutative ring of all scalar $\mathcal{F}_0$-measurable random variables.

By $Z := (z_1, \dots, z_m)^T$ we denote an m-dimensional semimartingale (see e.g. [11]). A popular example of such Z is the vector Brownian motion (the Wiener process).

We consider the homogeneous stochastic hereditary equation

$$dx(t) = (V_h x)(t)dZ(t), t \geq 0, \tag{1}$$

equipped with two extra conditions

$$x(s) = \varphi(s), \ \ s < 0, \tag{1a}$$

$$x(0) = x_0. \tag{1b}$$

Here V_h is a k-linear Volterra operator (see below), which is defined in certain linear spaces of vector stochastic processes, φ is an $\mathcal{F}_0$-measurable stochastic process, $x_0 \in k^n$.

By k-linearity of the operator V_h we mean the following property:

$$V_h(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 V_h x_1 + \alpha_2 V_h x_2$$

holding for all $\mathcal{F}_0$-measurable, bounded and scalar random values α_1, α_2 and all stochastic processes x_1, x_2 belonging to the domain of the operator V_h.

The solution of the initial value problem $(1), (1a), (1b)$ will be denoted by $x(t, x_0, \varphi)$, $t \in (-\infty, \infty)$. Below the solution is always assumed to exist and be unique for an appropriate choice of $\varphi(s)$, x_0.

The following kinds of stochastic Lyapunov stability are well-known:

Definition 1. For a given real number p $(0 < p < \infty)$ we call the zero solution of the homogeneous equation (1)

- p-stable (w.r.t. the initial data, i.e. w.r.t. x_0 and the "prehistory" function φ) if for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $E|x_0|^p + \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^p < \delta$ implies $E|x(t,x_0,\varphi)|^p \le \varepsilon$ for all $t \ge 0$ and all (admissible) φ, x_0;
- asymptotically p-stable (w.r.t. the initial data) if it is p-stable and, in addition, any φ, x_0 such that $E|x_0|^p + \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^p < \delta$ satisfies $\lim\limits_{t\to+\infty} E|x(t,x_0,\varphi)|^p = 0$;
- exponentially p-stable (w.r.t. the initial data) if there exist positive constants $\overline{c}$, β such that the inequality
$$E|x(t,x_0,\varphi)|^p \le \overline{c}\Big(E|x_0|^p + \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^p\Big)\exp\{-\beta s\}$$
holds true for all $t \ge 0$ and all φ, x_0.

To be able to link stochastic Lyapunov stability and the W-method, we need to represent (1), (1a) as a functional differential equation. Let $x(t)$ be a stochastic process on the real semiaxis $(t \in [0,+\infty))$ and $x_+(t)$ be a stochastic process on the entire real axis $(t \in (-\infty,+\infty))$ coinciding with $x(t)$ for $t \ge 0$ and equalling 0 for $t < 0$, while $\varphi_-(t)$ be a stochastic process on the axis $(t \in (-\infty,+\infty))$ coinciding with $\varphi(t)$ for $t < 0$ and equalling 0 for $t \ge 0$. Then the stochastic process $x_+(t) + \varphi_-(t)$, defined for $t \in (-\infty,+\infty)$ will be a solution of the problem (1), (1a), (1b) if $x(t)$ $(t \in [0,+\infty))$ satisfies the initial value problem

$$dx(t) = \big[(Vx)(t) + f(t)\big]\,dZ(t), \quad t \ge 0, \tag{2}$$
$$x(0) = x_0, \tag{2a}$$

where
$$(Vx)(t) := (V_h x_+)(t), \quad f(t) := (V_h \varphi_-)(t) \ \text{ for } \ t \ge 0.$$
Indeed, by linearity $V_h(x_+ + \varphi_-) = V_h(x_+) + V_h(\varphi_-) = Vx + f$, which gives (2). Note that f is uniquely defined by the stochastic process φ, "the prehistory function". Let us also observe that the initial value problem (2), (2a) is equivalent to the initial value problem (1), (1a), (1b) only for f, which have representation $f = V_h\varphi'$, where φ' is an arbitrary extension of the function φ to the real axis $(-\infty,\infty)$.

In the sequel the following linear spaces of stochastic processes will be used:

- $L^n(Z)$ consists of all predictable $n \times m$-matrix stochastic processes on $[0,+\infty)$, the rows of which are locally integrable w.r.t. the semimartingale Z (see e.g. [11]);
- D^n consists of all n-dimensional stochastic processes on $[0,+\infty)$, which can be represented as
$$x(t) = x(0) + \int\limits_0^t H(s)\,dZ(s),$$
where $x(0) \in k^n$, $H \in L^n(Z)$.

Let B be a linear subspace of the space $L^n(Z)$ equipped with some norm $\|\cdot\|_B$. For a given positive and continuous function $\gamma(t)$ $(t \in [0,\infty))$ we define $B^\gamma = \{f : \ f \in B, \ \gamma f \in B\}$. The latter space becomes a linear normed space if we put $\|f\|_{B^\gamma} := \|\gamma f\|_B$.

We will also need the following linear subspaces of "the space of initial values" k^n and "the space of solutions" D^n:

$$k_p^n = \big\{\alpha : \ \alpha \in k^n, \ E|\alpha|^p < \infty\big\}, \quad M_p^\gamma = \Big\{x : \ x \in D^n, \ \sup_{t\geq 0} E|\gamma(t)x(t)|^p < \infty\Big\}, \ \ M_p^1 = M_p.$$

For $1 \leq p < \infty$ the linear spaces k_p^n, M_p^γ become normed spaces if we define

$$\|\alpha\|_{k_p^n} = \big(E|\alpha|^p\big)^{1/p}, \quad \|x\|_{M_p^\gamma} = \sup_{t\geq 0} \big(E|\gamma(t), x(t)|^p\big)^{1/p}.$$

In the sequel, we will always assume that the operator $V : D^n \to L^n(Z)$ in the equation (2) is a k-linear Volterra operator, $f \in L^n(Z)$ and $x_0 \in k^n$. Recall that $V : D^n \to L^n(Z)$ is said to be *Volterra* if for any (random) stopping time τ, $\tau \in [0, +\infty)$ a.s. and for any stochastic processes $x, y \in D^n$ the equality $x(t) = y(t)$ ($t \in [0, \tau]$ a.s.) implies the equality $(Vx)(t) = (Vy)(t)$ ($t \in [0, \tau]$ a.s.).

A solution of (2), (2a) is a stochastic process from the space D^n satisfying the equation

$$x(t) = x_0 + (Fx)(t), \ \ t \geq 0,$$

where

$$(Fx)(t) = \int\limits_0^t \big[(Vx)(s) + f(s)\big]\, dZ(s)$$

is a k-linear Volterra operator in the space D^n and the integral is understood as a stochastic one w.r.t. the semimartingale Z (see e.g. [11]).

Below $x_f(t, x_0)$ stands for the solution of the initial value problem (2), (2a).

Definition 2. Let $1 \leq p < \infty$. We say that the equation (2) is input-to-state stable (ISS) w.r.t. the pair (M_p^γ, B^γ) if there exists $\overline{c} > 0$, for which $x_0 \in k_p^n$ and $f \in B^\gamma$ imply the relation $x_f(\,\cdot\,, x_0) \in M_p^\gamma$ and the following estimate:

$$\|x_f(\,\cdot\,, x_0)\|_{M_p^\gamma} \leq \overline{c}\big(\|x_0\|_{k_p^n} + \|f\|_{B^\gamma}\big).$$

This definition says that the solutions belong to M_p^γ whenever $f \in B^\gamma$ and $x_0 \in k_p^n$ and that they continuously depend on f and x_0 in the appropriate topologies. The choice of the spaces is closely related to the kind of stability we are interested in.

The following result describes connections between Lyapunov stability of the zero solution of the equation (1) and input-to-state stability of the equation (2) with the operator V which is constructed from the operator V_h in (1).

Theorem 3. *Let $\gamma(t)$ $(t \geq 0)$ be a positive continuous function and $1 \leq p < \infty$. Assume that the equation* (2) *is constructed from* (1), (1a) *and $f(t) \equiv (V_h\varphi_-)(t) \in B^\gamma$ whenever φ satisfies the condition $\operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^p < \infty$, and $\|f\|_{B^\gamma} \leq K \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^p$ for some constant $K > 0$.*

1) *If $\gamma(t) = 1$ $(t \geq 0)$ and the equation* (2) *is ISS w.r.t. the pair (M_p^γ, B^γ), then the zero solution of* (1) *is p-stable.*

2) *If $\gamma(t) = \exp\{\beta t\}$ $(t \geq 0)$ for some $\beta > 0$ and the equation* (2) *is ISS w.r.t. the pair (M_p^γ, B^γ), then the zero solution of* (1) *is exponentially p-stable.*

3) *If $\lim\limits_{t\to+\infty} \gamma(t) = +\infty$, $\gamma(t) \geq \delta > 0$, $t \in [0, +\infty)$ $(t \geq 0)$ for some δ, and the equation* (2) *is ISS w.r.t. the pair (M_p^γ, B^γ), then the zero solution of* (1) *is asymptotically p-stable.*

The main idea of the W-method is to convert the given property of Lyapunov stability – via the property of ISS – into the property of invertibility of a certain regularized operator in a suitable functional space. This operator can be constructed with the help of an auxiliary equation. The latter is similar to the equation (2), but it is "simpler", so that the required ISS property is already established for this equation:

$$dx(t) = \big[(Qx)(t) + g(t)\big]dZ(t), \ \ t \geq 0, \tag{3}$$

where $Q : D^n \to L^n(Z)$ is a k-linear Volterra operator, and $g \in L^n(Z)$. For the equation (3) it is always assumed the existence and uniqueness assumption, i. e. that for any $x(0) \in k^n$ there is the only (up to a P-equivalence) solution $x(t)$ satisfying (3), so that we have the following representation:

$$x(t) = U(t)x_0 + (Wg)(t), \ \ t \geq 0, \tag{4a}$$

where $U(t)$ is the fundamental matrix of the associated homogeneous equation, and W is the corresponding Cauchy operator for the equation (3).

Now, let us rewrite the equation (2) in the following way:

$$dx(t) = \big[(Qx)(t) + ((V-Q)x)(t) + f(t)\big]dZ(t), \ \ t \geq 0,$$

or

$$x(t) = U(t)x(0) + (W(V-Q)x)(t) + (Wf)(t), \ \ t \geq 0.$$

Denoting $W(V-Q) = \Theta$, we obtain the operator equation

$$((I-\Theta)x)(t) = U(t)x(0) + (Wf)(t).$$

Theorem 4. *Given a weight γ (i. e. a positive continuous function defined for $t \geq 0$), let us assume that the equation* (2) *and the reference equation* (3) *satisfy the following conditions:*

1) *the operators V, Q act continuously from M_p^γ to B^γ;*

2) *the reference equation* (3) *is ISS w.r.t. the pair (M_p^γ, B^γ).*

If now the operator $I - \Theta : M_p^\gamma \to M_p^\gamma$ has a bounded inverse in this space, then the equation (2) *is ISS w.r.t. the pair (M_p^γ, B^γ).*

Proof. Under the above assumptions we have that $U(\,\cdot\,)x_0 \in M_p^\gamma$ whenever $x_0 \in k_p^n$ and also that

$$x_f(t, x_0) = \big((I-\Theta)^{-1}(U(\,\cdot\,)x_0)\big)(t) + \big((I-\Theta)^{-1}Wf\big)(t) \ \ (t \geq 0)$$

for an arbitrary $x_0 \in k_p^n$, $f \in B^\gamma$. Taking the norms and using the assumptions put on the reference equation, we, as in the previous theorem, obtain the inequality

$$\|x_f(\,\cdot\,, x_0)\|_{M_p^\gamma} \leq \bar{c}\big(\|x_0\|_{k_p^n} + \|f\|_{B^\gamma}\big),$$

where $x_0 \in k_p^n$, $f \in B^\gamma$. Thus, the equation (2) is ISS w.r.t. the pair (M_p^γ, B^γ). □

The choice of the space B and the weight γ depend on the asymptotic property one is studying.

In the theorem below we use the universal constants c_p $(1 \leq p < \infty)$ from the Burkholder–Davis–Gandy inequalities to estimate stochastic integrals, see e.g. [11].

Theorem 5. *The zero solution of the equation*

$$dx(t) = \Big(a\xi(t)x(t) + b\xi(t)x\Big(\frac{t}{\tau_0}\Big)\Big)dt + c\sqrt{\xi(t)}\,x\Big(\frac{t}{\tau_1}\Big)d\mathcal{B}(t)\ \ (t \geq 0),$$

where $\xi(t) = I_{[0,r]}(t) + tI_{[r,\infty]}(t)$, $t \geq 0$ ($I_A(t)$ *is the indicator of* A), $\mathcal{B}(t)$ *is the standard scalar Brownian motion,* a, b, c, τ_0, τ_1, r *are real numbers* ($\tau_0 > 1$, $\tau_1 > 1$), *is asymptotically* $2p$*-stable* (*with respect to* x_0, *as* φ *is not needed in this case*) *if there exists* $\alpha > 0$ *for which*

$$|a + b + \alpha| + c_p|c|\sqrt{0.5\alpha} + \big(|ab| + b^2\big)\delta_0 + c_p|bc|\sqrt{\delta_0} < \alpha,$$

where

$$\delta_0 = \max\big\{\log\tau_0, (1 - \tau_0^{-1})r\big\}.$$

The proof of the result can be found in [8].

The W-method is also proven to be efficient in the difficult case of stochastic differential equations with impulses, see [10].

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