On the Solvability of the Mixed Problem for the Semilinear Wave Equation with a Nonlinear Boundary Condition

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In the plane of independent variables x and t in the domain $D_T:\ 0<x<l,\ 0<t<T$ consider the mixed problem of finding the solution $u(x,t)$ of semilinear wave equation of the form

$$u_{tt}-u_{xx}+g(u)=f(x,t),\ \ (x,t)\in D_T, \tag{1}$$

satisfying the initial

$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x),\ \ 0\le x\le l, \tag{2}$$

and boundary conditions

$$u_x(0,t)=F[u(0,t)],\quad u_x(l,t)=\alpha(t)u(l,t),\ \ 0\le t\le T, \tag{3}$$

where f, φ, ψ, α, g and F are given, and u is an unknown real functions.

Let the following conditions of smoothness

$$\begin{gathered} f\in C^1(\overline{D}_T),\ \ g,F\in C^1(\mathbb{R}),\\ \varphi\in C^2([0,l]),\ \ \psi\in C^1([0,l]),\ \ \alpha\in C^1([0,T]) \end{gathered} \tag{4}$$

be fulfilled. It is assumed that the second order conditions of agreement are fulfilled at the points $(0,0)$ and $(l,0)$.

Note that nonlinear boundary condition of the form given in (3) arises, for example, in description of the process of longitudinal oscillations of a spring in case of elastic fixing of one of its ends when the tension does not comply with linear Hooke's law and is nonlinear function of shift, and also in description of processes in the distributed self-oscillatory systems.

Consider the conditions

$$\begin{gathered} \int\limits_0^s g(s_1)\,ds_1\ge -M_1s^2-M_2,\quad \int\limits_0^s F(s_1)\,ds_1\ge -M_3\ \ \forall\, s\in\mathbb{R},\\ \alpha(t)\le 0,\ \ \alpha'(t)\ge 0,\ \ 0\le t\le T, \end{gathered} \tag{5}$$

where $M_i:=const\ge 0$, $1\le i\le 3$.

The following theorem is valid.

Theorem. *Let the conditions* (4), (5) *be fulfilled. Then there exists a unique classical solution of the problem* (1)–(3).

Remark 1. In the case when at least one of the conditions (5), imposed on nonlinear functions g and F, is violated, as the following particular case shows, the solution u of considering problem can be explosive, i.e. there exists a number $T^*>0$ such that the problem (1)–(3) has a unique solution, besides

$$\lim_{T\to T^*-0}\|u\|_{C(\overline{D}_T)}=\infty. \tag{6}$$

Thus, in particular, it follows that the problem under consideration does not have a solution in the domain D_T for $T\ge T^*$.

Indeed, consider the case of the problem (1)–(3) when functions f, g, α equal zero, besides $\varphi \in C^2([0,l])$, $\varphi(0) > 0$, $\psi \in C^1([0,l])$ and $F(s) = -\delta|s|^\lambda s$, $\delta := const > 0$, $\lambda := const > 0$, $s \in \mathbb{R}$, and the corresponding conditions of agreement are fulfilled. Then in the case $\psi = -\varphi'$ the solution u of this problem in the domain D_T for $T = T^*$ is given by the formula

$$u(x,t) = \begin{cases} \varphi(x-t), & (x,t) \in \Delta_1 \cap \{t < T^*\}, \\ \mu_1(t-x), & (x,t) \in \Delta_2 \cap \{t < T^*\}, \\ \varphi(2l-x-t) - \varphi(l) + \varphi(x-t), & (x,t) \in \Delta_3 \cap \{t < T^*\}, \\ \mu_1(t-x) + \varphi(2l-x-t) - \varphi(x+t-l), & (x,t) \in \Delta_4 \cap \{t < T^*\}. \end{cases} \tag{7}$$

Here

$$\mu_1(t) = \frac{\varphi(0)}{[1-\delta\lambda\varphi^\lambda(0)t]^{\frac{1}{\lambda}}}, \quad 0 \le t < T^* := \frac{1}{\delta\lambda\varphi^\lambda(0)} < l, \tag{8}$$

and

$$\Delta_1 := \Delta OO_1C, \quad \Delta_2 := \Delta OO_1A, \quad \Delta_3 := \Delta CO_1B, \quad \Delta_4 := \Delta O_1AB$$

are right-angled triangles, where

$$O = O(0,0), \quad A = A(0,l), \quad B = B(l,l), \quad C = C(l,0), \quad O_1 = O_1\Big(\frac{l}{2}, \frac{l}{2}\Big).$$

From (7), (8) it follows that the solution of problem (1)–(3) is explosive, i.e. the equality (6) holds. Therefore, in this case, at the problem statement we should require that $T < T^*$.

Remark 2. In fact, the formula (7) allows continuation of the solution of considering problem from the domain D_{T^*} into the domain $D_l \cap \{t < x + T^*\}$, besides, this solution $u(x,t)$ will rise indefinitely at approaching of the point (x,t) from the domain $D_l \cap \{t < x + T^*\}$ to the characteristics $t - x = T^*$, to which adjoins this domain with a part of its boundary.

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