Non-Lipschitz Lower Sigma-Exponents of Linear Differential Systems

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For investigation of exponential stability and instability of perturbed linear differential systems

$$\dot{y} = A(t)y + Q(t)y, \;\; y \in R^n, \;\; t \geq 0, \tag{1_{A+Q}}$$

with bounded piecewise-constant coefficients, characteristic exponents $\lambda_1(A+Q) \leq \cdots \leq \lambda_n(A+Q)$ and exponentially decreasing sigma-perturbations Q satisfying the condition

$$\lambda[Q] \equiv \overline{\lim_{t\to+\infty}} \frac{1}{t} \ln \|Q(t)\| \leq -\sigma < 0,$$

the use is made of the so-called higher [3,4]

$$\nabla_\sigma(A) \equiv \sup_{\lambda[Q]\leq-\sigma} \lambda_n(A+Q), \;\; \sigma > 0,$$

and lower [5–7]

$$\Delta_\sigma(A) \equiv \inf_{\lambda[Q]\leq-\sigma} \lambda_1(A+Q), \;\; \sigma > 0 \tag{2}$$

sigma-exponents. And if for the first of them the calculation algorithm by the Cauchy matrix $X_A(t,\tau)$ of the initial system (1_A) is constructed [3,4] and fully described [1,2,8] as the function of a parameter $\sigma > 0$ (with the properties of boundedness, concavity and coincidence with the constant σ greater than some $\sigma_0 \geq 0$), then for the second, lower sigma-exponent $\Delta_\sigma(A)$, there is nothing.

In works [6,7] devoted to the investigation of the lower sigma-exponent $\Delta_\sigma(A)$, relying only on its definition (2), the author constructed lower sigma-exponents of linear differential systems (1_A) of general Lipschitz on the interval $(0,+\infty)$ type, more general compared to the higher sigma-exponents. In particular, they are not only convex or only concave functions in the whole domain $(0,+\infty)$ of their definition. Indeed, for every nondecreasing function $f : (0,+\infty) \to R$ coinciding with the constant on some interval $[\sigma_0,+\infty)$ (the lower sigma-exponent of any system (1_A) possesses these obvious properties) and satisfying the Lipschitz condition on the interval $(0,\sigma_0$, the existence of the linear differential system (1_A) with a lower sigma-exponent $\Delta_\sigma(A) \equiv f(\sigma)$, $\sigma > 0$ is proved.

There arises the question whether there exist lower sigma-exponents $\Delta_\sigma(A)$ of linear non-Lipschitz type systems, that is not satisfying in parameter $\sigma > 0$ Lipschits condition on the whole interval $(0,+\infty)$ with a finite Lipschitz constant $L > 0$. The positive answer is contained in the following

Theorem. *Any nondecreasing function*

$$f : [0,+\infty) \to [c_0,c_1] \subset (-\infty,+\infty),$$

coinciding with the constant c_1 on some interval $[\sigma_1,+\infty)$ and satisfying the Lipschitz condition

$$0 \leq f(\xi_2) - f(\xi_1) < L(\sigma_0)(\xi_2-\xi_1), \;\; 0 < \sigma_0 \leq \xi_1 < \xi_2 \leq \sigma_1,$$

on any interval $[\sigma_0,\sigma_1]$ with the Lipschitz constant $L(\sigma_0) \leq const/\sigma_0$, $\sigma_0 > 0$, is a lower sigma-exponent $\Delta_\sigma(A) \equiv f(\sigma)$, $\sigma > 0$, of some linear differential system (1_A) with a piecewise-continuous bounded on the time semi-axis $[0,+\infty)$ matrix of coefficients $A(t)$.

Remark. Such satisfying conditions of the theorem (and not satisfying the Lipschitz on the whole interval $(0,+\infty)$ condition with one finite Lipschitz constant $L>0$) are, for example, the functions

$$f(\sigma)=\begin{cases}\sigma^\alpha, & \sigma\in[0,\sigma_1],\\ \sigma_1^\alpha, & \sigma>\sigma_1,\ \ \alpha\in(0,1).\end{cases}$$

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