Bounded Solutions to Systems of Nonlinear Functional Differential Equations

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Consider the system of functional differential equations

$$x'(t)=F(x)(t) \tag{1}$$

where $F: C_{loc}(\mathbb{R};\mathbb{R}^n)\to L_{loc}(\mathbb{R};\mathbb{R}^n)$ is a continuous operator satisfying the local Carathéodory conditions, i.e., there exists a function $\psi:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}_+$ nondecreasing in the second argument such that $\psi(\cdot,r)\in L_{loc}(\mathbb{R};\mathbb{R})$ for $r\in\mathbb{R}_+$ and for any $x\in C_0(\mathbb{R};\mathbb{R}^n)$ the inequality

$$\|F(x)(t)\|\leq\psi\big(t,\|x\|\big)\quad\text{for a.e. } t\in\mathbb{R}$$

is fulfilled.

By a solution to the system (1) we understand a vector-valued function $x\in AC_{loc}(\mathbb{R};\mathbb{R}^n)$ satisfying the equality (1) almost everywhere in $\mathbb{R}$. By a bounded solution to the system (1) it is understood a solution x to the system (1) that satisfies

$$\sup\big\{\|x(t)\|:\ t\in\mathbb{R}\big\}<+\infty.$$

To formulate our results, we need to introduce the following definition (the complete list of notation and symbols is given at the end of this text). Let $\sigma\in\{-1,1\}$ and put

$$I_\sigma(t)=\begin{cases}]-\infty,t] & \text{if } \sigma=1,\\ [t,+\infty[& \text{if } \sigma=-1\end{cases}\quad\text{for } t\in\mathbb{R}.$$

A linear continuous operator $\ell: C_{loc}(\mathbb{R};\mathbb{R})\to L_{loc}(\mathbb{R};\mathbb{R})$ is called a σ-Volterra operator if for arbitrary $t\in\mathbb{R}$ and $v\in C_{loc}(\mathbb{R};\mathbb{R})$ such that $v(s)=0$ for $s\in I_\sigma(t)$, the equality $\ell(v)(s)=0$ for a.e. $s\in I_\sigma(t)$ is fulfilled.

Theorem 1. *Let the inequality*

$$\mathcal{D}(\sigma)\operatorname{Sgn}(v(t))\big[F(v)(t)-\mathcal{D}(h(t))v(t)+g_0(v)(t)\big]\leq p(|v|)(t)+\eta(t,\|v\|)\quad\textit{for a.e. } t\in\mathbb{R} \tag{2}$$

be fulfilled for any $v\in C_0(\mathbb{R};\mathbb{R}^n)$, where $\sigma\in\mathbb{R}^n$, $\sigma_i\in\{-1,1\}$ $(i=1,\dots,n)$, $h\in L_{loc}(\mathbb{R};\mathbb{R}^n)$,

$$\begin{gathered} g_0(v)(t)\overset{def}{=}(g_{0i}(v_i)(t))_{i=1}^n\quad\textit{for a.e. } t\in\mathbb{R},\ \ v\in C_{loc}(\mathbb{R};\mathbb{R}^n)\\ \mathcal{D}(\sigma)g_0\in\mathcal{P}_n(\mathbb{R}),\ \ p\in\mathcal{P}_n(\mathbb{R}), \end{gathered} \tag{3}$$

each g_{0i} is a σ_i-Volterra operator, and $\eta \in K_{loc}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R}^n_+)$ satisfies

$$\lim_{r \to +\infty} \frac{1}{r} \int_a^b \|\eta(s,r)\| \, ds = 0 \tag{4}$$

for every interval $[a,b]$. Let, moreover, there exist functions $\beta, \gamma \in AC_{loc}(\mathbb{R}; \mathbb{R}^n)$ such that

$$\begin{gathered}
\beta(t) > 0, \;\; \gamma(t) > 0 \quad \textit{for } t \in \mathbb{R}, \;\; \|\gamma\| < +\infty, \\
\mathcal{D}(\sigma)\big[\beta'(t) - \mathcal{D}(h(t))\beta(t) + g_0(\beta)(t)\big] \le 0 \quad \textit{for a.e. } t \in \mathbb{R}, \\
\mathcal{D}(\sigma)\big[\gamma'(t) - \mathcal{D}(h(t))\gamma(t) - \mathcal{D}(\sigma)p(\gamma)(t)\big] \ge 0 \quad \textit{for a.e. } t \in \mathbb{R}.
\end{gathered}$$

Let, in addition, for every $i \in \{1, \dots, n\}$,

$$G_i(t,r) \overset{def}{=} \lim_{\tau \to -\sigma_i\infty} \sigma_i \int_\tau^t \exp\bigg(\int_s^t h_i(\xi)\, d\xi\bigg) \eta_i(s,r)\, ds < +\infty \quad \textit{for } t \in \mathbb{R}, \;\; r \in \mathbb{R}_+, \tag{5}$$

$$H_i(t) \overset{def}{=} \lim_{\tau \to -\sigma_i\infty} \gamma_i(\tau) \exp\bigg(\int_\tau^t h_i(s)\, ds\bigg) > 0 \quad \textit{for } t \in \mathbb{R}, \tag{6}$$

and

$$\limsup_{r \to +\infty} \frac{G_i(t,r)}{r H_i(t)} < \frac{1}{\|\gamma\|} \quad \textit{uniformly for } t \in \mathbb{R}. \tag{7}$$

Then (1) has at least one bounded solution.

Theorem 2. *Let the inequality*

$$\begin{aligned}
\mathcal{D}(\sigma)\operatorname{Sgn}(v(t))\big[F(v)(t) - \mathcal{D}(h(t))v(t) - \ell_0(v)(t) + g_0(v)(t)\big]& \\
\le p(|v|)(t) + \eta(t, \|v\|) \quad \textit{for a.e. } t \in \mathbb{R}&
\end{aligned}$$

be fulfilled for any $v \in C_0(\mathbb{R}; \mathbb{R}^n)$, where $\sigma \in \mathbb{R}^n$, $\sigma_i \in \{-1, 1\}$ $(i = 1, \dots, n)$, $h \in L_{loc}(\mathbb{R}; \mathbb{R}^n)$, (3) and

$$\mathcal{D}(\sigma)\ell_0 \in \mathcal{P}_n(\mathbb{R}), \quad \mathcal{D}(\sigma)\big[\ell_0 - g_0\big] \in \mathcal{P}^\sigma_n(\mathbb{R}; h)$$

hold, and $\eta \in K_{loc}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R}^n_+)$ satisfies (4) for every interval $[a,b]$. Let, moreover, there exist a function $\gamma \in AC_{loc}(\mathbb{R}; \mathbb{R}^n)$ such that

$$\begin{gathered}
\gamma(t) > 0 \quad \textit{for } t \in \mathbb{R}, \;\; \|\gamma\| < +\infty, \\
\mathcal{D}(\sigma)\big[\gamma'(t) - \mathcal{D}(h(t))\gamma(t) - \ell_0(\gamma)(t) - \mathcal{D}(\sigma)p(\gamma)(t)\big] \ge 0 \quad \textit{for a.e. } t \in \mathbb{R}.
\end{gathered}$$

Let, in addition, (6)–(7) be fulfilled for every $i \in \{1, \dots, n\}$. Then (1) has at least one bounded solution.

Consider the nonlinear differential system with argument deviation

$$\begin{aligned}
x_i'(t) = h_i(t)x_i(t) + \sum_{j=1}^n p_{ij}(t)x_j(\tau_{ij}(t)) - \sum_{j=1}^n g_{ij}(t)x_j(\mu_{ij}(t))& \\
+ f_i(t, x(t), x(\nu_1(t)), \dots, x(\nu_m(t))) \;\; (i = 1, \dots, n),&
\end{aligned} \tag{8}$$

where $h = (h_i)_{i=1}^n \in L_{loc}(\mathbb{R}; \mathbb{R}^n)$, $P = (p_{ij})_{i,j=1}^n \in L_{loc}(\mathbb{R}; \mathbb{R}^{n \times n})$, $G = (g_{ij})_{i,j=1}^n \in L_{loc}(\mathbb{R}; \mathbb{R}^{n \times n})$, $f = (f_i)_{i=1}^n \in K_{loc}(\mathbb{R} \times \mathbb{R}^{(m+1)n}; \mathbb{R}^n)$, $t_{ij}, \mu_{ij}, \nu_k : \mathbb{R} \to \mathbb{R}$ $(i, j = 1, \dots, n;\ k = 1, \dots, m)$ are locally essentially bounded functions, and $x = (x_i)_{i=1}^n$. Then Theorems 1 and 2 imply in particular the following corollaries.

Corollary 1. *Let the inequality*

$$\operatorname{Sgn}(v(t))f(t,v(t),v(\nu_1(t)),\dots,v(\nu_m(t))) \le q(t) \quad \text{for a.e. } t\in\mathbb{R} \tag{9}$$

be fulfilled for any $v\in C_0(\mathbb{R};\mathbb{R}^n)$, $q\in L_{loc}(\mathbb{R};\mathbb{R}^n_+)$. *Let, moreover,*

$$P(t)\ge\Theta, \quad G(t)\ge\Theta \quad \text{for a.e. } t\in\mathbb{R}, \tag{10}$$

$$g_{ij}(t)=0 \quad \text{for a.e. } t\in\mathbb{R} \;\; (i\ne j;\, i,j=1,\dots,n), \tag{11}$$

$$g_{ii}(t)\big[\mu_{ii}(t)-t\big]\le 0 \quad \text{for a.e. } t\in\mathbb{R} \;\; (i=1,\dots,n), \tag{12}$$

and

$$\int\limits_{\mu_{ii}(t)}^{t} g_{ii}(s)\exp\bigg(-\int\limits_{\mu_{ii}(s)}^{s} h_i(\xi)\,d\xi\bigg)\,ds \le \frac{1}{e} \quad \text{for a.e. } t\in\mathbb{R}, \;\; (i=1,\dots,n),$$

$$\int\limits_{t}^{\tau_{ij}(t)} \widetilde{p}(s)\,ds \le \frac{1}{e} \quad \text{for a.e. } t\in\mathbb{R} \;\; (i,j=1,\dots,n), \tag{13}$$

where

$$\widetilde{p}(t) \overset{def}{=} \max\bigg\{\sum_{k=1}^{n} p_{ik}(t)\exp\bigg(\int\limits_{t}^{\tau_{ik}(t)} \widetilde{h}(s)\,ds\bigg) : \; i=1,\dots,n\bigg\} \quad \text{for a.e. } t\in\mathbb{R}, \tag{14}$$

$$\widetilde{h}(t) \overset{def}{=} \max\big\{h_i(t) : i=1,\dots,n\big\} \quad \text{for a.e. } t\in\mathbb{R}. \tag{15}$$

Let, in addition,

$$\sup\bigg\{\int\limits_{0}^{t}\big[\widetilde{h}(s)+e\widetilde{p}(s)\big]\,ds : \; t\in\mathbb{R}\bigg\}<+\infty, \quad \int\limits_{-\infty}^{0}\widetilde{p}(s)\,ds<+\infty, \tag{16}$$

$$\int\limits_{-\infty}^{+\infty} q(s)\exp\bigg(-\int\limits_{0}^{s} h_i(\xi)\,d\xi\bigg)\,ds<+\infty \;\; (i=1,\dots,n). \tag{17}$$

Then (8) *has at least one bounded solution.*

Corollary 2. *Let the inequality* (9) *be fulfilled for any* $v\in C_0(\mathbb{R};\mathbb{R}^n)$, $q\in L_{loc}(\mathbb{R};\mathbb{R}^n_+)$. *Let, moreover,* (10) *hold,*

$$p_{ik}(t)\exp\bigg(\int\limits_{\mu_{ik}(t)}^{\tau_{ik}(t)} h_k(s)\,ds\bigg)\ge g_{ik}(t), \quad g_{ik}(t)\big[\tau_{ik}(t)-\mu_{ik}(t)\big]\ge 0 \quad \text{for a.e. } t\in\mathbb{R} \;\; (i,k=1,\dots,n),$$

and let (13) *be fulfilled, where* $\widetilde{p}$ *is given by* (14) *and* (15). *Let, in addition,* (16) *and* (17) *hold. Then* (8) *has at least one bounded solution.*

Corollary 3. *Let the inequality*

$$\mathcal{D}(\sigma)\operatorname{Sgn}(v(t))f\big(t,v(t),v(\nu_1(t)),\dots,v(\nu_m(t))\big)\le q(t) \quad \text{for a.e. } t\in\mathbb{R} \tag{18}$$

be fulfilled for any $v \in C_0(\mathbb{R};\mathbb{R}^n)$, $q \in L_{loc}(\mathbb{R};\mathbb{R}^n_+)$, *where* $\sigma \in \mathbb{R}^n$, $\sigma_i \in \{-1,1\}$ $(i=1,\dots,n)$. *Let, moreover,*

$$\mathcal{D}(\sigma)P(t) \geq \Theta, \quad \mathcal{D}(\sigma)G(t) \geq \Theta \quad \text{for a.e. } t \in \mathbb{R}, \tag{19}$$

(11) *and* (12) *hold, and*

$$\int\limits_{-\infty}^{\infty} |g_{ii}(s)| \exp\bigg(- \int\limits_{\mu_{ii}(s)}^{s} h_i(\xi)\,d\xi \bigg)\,ds < 1 \ \ (i=1,\dots,n).$$

Furthermore, let there exist $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}_+^{n\times n}$ *such that* $r(A) < 1$ *and*

$$\int\limits_{-\infty}^{+\infty} |p_{ij}(s)| \exp\bigg(\int\limits_{0}^{\tau_{ij}(s)} h_j(\xi)\,d\xi - \int\limits_{0}^{s} h_i(\xi)\,d\xi \bigg)\,ds \leq a_{ij} \ \ (i,j=1,\dots,n). \tag{20}$$

Let, in addition,

$$\sup\bigg\{ \int\limits_{0}^{t} h_i(s)\,ds : \ t \in \mathbb{R} \bigg\} < +\infty \ \ (i=1,\dots,n) \tag{21}$$

and (17) *hold. Then* (8) *has at least one bounded solution.*

Corollary 4. *Let* (18) *be fulfilled for any* $v \in C_0(\mathbb{R};\mathbb{R}^n)$, $q \in L_{loc}(\mathbb{R};\mathbb{R}^n_+)$, *where* $\sigma \in \mathbb{R}^n$, $\sigma_i \in \{-1,1\}$ $(i=1,\dots,n)$. *Let* (19) *hold and, moreover,*

$$\sigma_i p_{ik}(t) \exp\bigg(\int\limits_{\mu_{ik}(t)}^{\tau_{ik}(t)} h_k(s)\,ds \bigg) \geq \sigma_i g_{ik}(t), \quad \sigma_i\sigma_k g_{ik}(t)\big[\tau_{ik}(t) - \mu_{ik}(t)\big] \geq 0 \ \ (i,k=1,\dots,n)$$

for a.e. $t \in \mathbb{R}$. *Furthermore, let there exist* $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}_+^{n\times n}$ *such that* $r(A) < 1$ *and* (20) *hold. Let, in addition,* (21) *and* (17) *hold. Then* (8) *has at least one bounded solution.*

Notation

If $x = (x_i)_{i=1}^n \in \mathbb{R}^n$, then

$$\mathcal{D}(x) = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{pmatrix}, \quad \operatorname{Sgn}(x) = \mathcal{D}(\operatorname{sgn} x), \ \ \text{where} \ \ \operatorname{sgn} x = (\operatorname{sgn} x_i)_{i=1}^n.$$

Θ is a zero matrix, $r(X)$ is a spectral radius of the matrix X.

$C_{loc}(\mathbb{R};\mathbb{R}^n)$ is a space of continuous functions $x : \mathbb{R} \to \mathbb{R}^n$ with a topology of uniform convergence on every compact interval.

$C_0(\mathbb{R};\mathbb{R}^n)$ is a Banach space of bounded continuous functions $x : \mathbb{R} \to \mathbb{R}^n$ endowed with a norm

$$\|x\| = \sup\big\{ \|x(t)\| : \ t \in \mathbb{R} \big\}.$$

$AC_{loc}(\mathbb{R};\mathbb{R}^n)$ is a set of locally absolutely continuous functions $x : \mathbb{R} \to \mathbb{R}^n$.

$L_{loc}(\mathbb{R};\mathbb{R}^n)$ is a space of locally Lebesgue integrable vector-valued functions $p : \mathbb{R} \to \mathbb{R}^n$ with a topology of convergence in mean on every compact interval.

$L_{loc}(\mathbb{R};\mathbb{R}^{n\times n})$ is a space of locally Lebesgue integrable matrix-valued functions $P:\mathbb{R}\to\mathbb{R}^{n\times n}$.

$\mathcal{P}_n(\mathbb{R})$ is a set of linear continuous operators $\ell : C_{loc}(\mathbb{R};\mathbb{R}^n)\to L_{loc}(\mathbb{R};\mathbb{R}^n)$ that transforms non-negative functions into the set of non-negative functions.

$\mathcal{P}_n^\sigma(\mathbb{R};h)$, where $h\in L_{loc}(\mathbb{R};\mathbb{R}^n)$ and $\sigma=(\sigma_i)_{i=1}^n\in\mathbb{R}^n$, $\sigma_i\in\{-1,1\}$ $(i=1,\dots,n)$, is a set of linear continuous operators $\ell : C_{loc}(\mathbb{R};\mathbb{R}^n)\to L_{loc}(\mathbb{R};\mathbb{R}^n)$ such that

$$\ell(x)(t)\geq 0 \quad \text{for a.e. } \ t\in\mathbb{R},$$

whenever $x\in AC_{loc}(\mathbb{R};\mathbb{R}^n)$ satisfies

$$x(t)\geq 0 \quad \text{for } \ t\in\mathbb{R}, \quad \mathcal{D}(\sigma)\big[x'(t)-\mathcal{D}(h(t))x(t)\big]\geq 0 \quad \text{for a.e. } \ t\in\mathbb{R}.$$

$K([a,b]\times A;B)$, where $A\subseteq\mathbb{R}^m$ and $B\subseteq\mathbb{R}^n$, is a set of functions $f:[a,b]\times A\to B$ satisfying the Carathéodory conditions, i.e.,

(i) $f(\cdot,x):[a,b]\to B$ is a measurable function for every $x\in A$,

(ii) $f(t,\cdot):A\to B$ is a continuous function for almost all $t\in[a,b]$,

(iii) for every $r>0$ there exists a function $q_r\in L([a,b];\mathbb{R}_+)$ such that

$$\|f(t,x)\|\leq q_r(t) \quad \text{for a.e. } \ t\in[a,b], \ \ x\in A, \ \ \|x\|\leq r.$$

$K_{loc}(\mathbb{R}\times A;B)$, where $A\subseteq\mathbb{R}^m$ and $B\subseteq\mathbb{R}^n$, is a set of functions $f:\mathbb{R}\times A\to B$ such that $f\in K([a,b]\times A;B)$ for every compact interval $[a,b]$.