# Asymptotic Properties of Special Classes of Solutions of Second-Order Differential Equations with Nonlinearities in Some Sense Near to Regularly Varying

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The differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')f(y,y') \tag{1}$$

is considered, where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[ \to ]0,+\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to ]0,+\infty[$ are continuous functions, $f : \Delta_{Y_0} \times \Delta_{Y_1} \to ]0,+\infty[$ is a continuously differentiable function, $Y_i \in \{0,\pm\infty\}$ $(i=0,1)$, $\Delta_{Y_i}$ is a one-sided neighborhood of $Y_i$. We suppose also that each of the functions $\varphi_i(z)$ $(i=0,1)$ is a regularly varying function as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of order $\sigma_i$, $\sigma_0+\sigma_1 \ne 1$, $\sigma_1 \ne 0$ and the function $f$ satisfies the condition

$$\lim_{\substack{v_k \to Y_k \\ v_k \in \Delta_{Y_k}}} \frac{v_k \cdot \frac{\partial f}{\partial v_k}(v_0,v_1)}{f(v_0,v_1)} = 0 \ \text{ uniformly in } \ v_j \in \Delta_{Y_j}, \ \ j \ne k, \ \ k,j = 0,1.$$

A lot of works (see, e.g., [1,3]) were devoted to the establishing of asymptotic representation of solutions of equations of the form (1), in which $f \equiv 1$. In this research the right part of (1) was either in explicit form or asymptotically represented as the product of features, each of which depends only on $t$, or only on $y$, or only on $y'$. Let us notice that it played an important role in the research. Therefore, the general case of equation (1) can contain nonlinearities of another types, for example, $e^{|\gamma \ln|y|+\mu \ln|y'||^{\alpha}}$, $0<\alpha<1$, $\gamma,\mu \in \mathbb{R}$.

**Definition.** The solution $y$ of equation (1) is called $P_\omega(Y_0,Y_1,\lambda_0)$ solution if it is defined on $[t_0,\omega[ \subset [a,\omega[$ and

$$\lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i=0,1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y(t)y''(t)} = \lambda_0.$$

The $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) are regularly varying functions as $t\uparrow\omega$ of index $\frac{\lambda_0}{\lambda_0-1}$ if $\lambda_0 \in R \setminus \{0,1\}$. The asymptotic properties and necessary and sufficient conditions of the existence of such solutions are obtained (see, [2]).

The cases $\lambda_0 \in \{0,1\}$ and $\lambda_0 = \infty$ are special. $P_\omega(Y_0,Y_1,1)$-solutions of equation (1) are rapidly varying functions as $t\uparrow\omega$. The cases $\lambda_0 = 0$ and $\lambda_0 = \infty$ are most difficult for establishing because in these cases such solutions or their derivatives are slowly varying functions as $t\uparrow\omega$. Some results about asymptotic properties end existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) in special cases are presented in this work. Now we need the next definition.

We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $\theta : \Delta_Y \to ]0;+\infty[$ satisfies the condition $S$ if for any continuous differentiable function $L : \Delta_{Y_i} \to ]0;+\infty[$ such that

$$\lim_{\substack{z \to Y_i \\ z \in \Delta_{Y_i}}} \frac{zL'(z)}{L(z)} = 0,$$

the following condition takes place

$$\Theta(zL(z)) = \Theta(z)(1+o(1)) \ \text{ as } \ z \to Y \ \ (z \in \Delta_Y).$$

We need the following subsidiary notations.

$$\pi_\omega(t) = \begin{cases} t & \text{as } \ \omega = +\infty, \\ t-\omega & \text{as } \ \omega < +\infty, \end{cases} \quad \Theta_i(z) = \varphi_i(z)|z|^{-\sigma_i} \ \ (i=0,1),$$

$$I(t) = \alpha_0 \int\limits_{A_\omega}^{t} p(\tau)\,d\tau, \quad A_\omega = \begin{cases} a & \text{if } \displaystyle\int\limits_a^\omega p(\tau)\,d\tau = +\infty, \\ \omega & \text{if } \displaystyle\int\limits_a^\omega p(\tau)\,d\tau < +\infty, \end{cases}$$

$$J_1(t) = \int\limits_{B_\omega^1}^{t} |I(\tau)|^{\frac{1}{1-\sigma_1}}\,d\tau, \quad B_\omega^1 = \begin{cases} b_1 & \text{if } \displaystyle\int\limits_{b_1}^\omega |I(\tau)|^{\frac{1}{1-\sigma_1}}\,d\tau = +\infty, \\ \omega & \text{if } \displaystyle\int\limits_{b_1}^\omega |I(\tau)|^{\frac{1}{1-\sigma_1}}\,d\tau < +\infty, \end{cases}$$

$$J_2(t) = \int\limits_{B_\omega^2}^{t} |I(\tau)|^{\frac{1}{\sigma_0}}\,d\tau, \quad B_\omega^2 = \begin{cases} b_2 & \text{if } \displaystyle\int\limits_{b_2}^\omega |I(\tau)|^{\frac{1}{\sigma_0}}\,d\tau = +\infty, \\ \omega & \text{if } \displaystyle\int\limits_{b_2}^\omega |I(\tau)|^{\frac{1}{\sigma_0}}\,d\tau < +\infty, \end{cases}$$

$$J_3(t) = \int\limits_{B_\omega^1}^{t} \left| I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\right|^{\frac{1}{1-\sigma_1}}\,d\tau,$$

$$B_\omega^3 = \begin{cases} b_3 & \text{if } \displaystyle\int\limits_{b_3}^\omega \left| I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\right|^{\frac{1}{1-\sigma_1}}\,d\tau = +\infty, \\ \omega & \text{if } \displaystyle\int\limits_{b_3}^\omega \left| I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\right|^{\frac{1}{1-\sigma_1}}\,d\tau < +\infty, \end{cases}$$

$$I_0(t) = \alpha_0 \int\limits_{A_\omega^0}^{t} p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\Theta_0\big(|\pi_\omega(\tau)|y_0^0\big)\,d\tau,$$

$$A_\omega^0 = \begin{cases} b & \text{if } \displaystyle\int\limits_{b}^\omega p(t)|\pi_\omega(t)|^{\sigma_0}\Theta_0\big(|\pi_\omega(t)|y_0^0\big)\,dt = +\infty, \\ \omega & \text{if } \displaystyle\int\limits_{b}^\omega p(t)|\pi_\omega(t)|^{\sigma_0}\Theta_0\big(|\pi_\omega(t)|y_0^0\big)\,dt < +\infty, \end{cases}$$

where $b \in [a,\omega[$ is chosen so that $|\pi_\omega(t)| \operatorname{sign} y_0^0 \in \Delta_{Y_0}$ as $t \in [b,\omega[$ .

**Theorem 1.** *Let $\sigma_1 \neq 1$. Then for the existence of $P_\omega(Y_0, Y_1, 1)$-solutions of equation* (1) *the following conditions are necessary*

$$y_0^0\alpha_0 > 0, \quad y_1^0 I(t)(1-\sigma_0-\sigma_1) > 0 \ \text{ as } \ t \in [a, \omega[\,, \tag{2}$$

$$\lim_{t\uparrow\omega} y_0^0 |J_1(t)|^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}} = Y_0, \quad \lim_{t\uparrow\omega} y_1^0 |J_1(t)|^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}} = Y_1, \quad \lim_{t\uparrow\omega} \frac{J_1(t)I'(t)}{J_1'(t)I(t)} = 1-\sigma_1. \tag{3}$$

*If*

$$\sigma_1 \neq 2 \ \text{ or } \ (\sigma_1-1)(\sigma_0+\sigma_1-1) > 0,$$

*conditions* (2), (3) *are sufficient for the existence of such solutions of equation* (1).

*For $P_\omega(Y_0, Y_1, 1)$-solutions of equation* (1) *the following asymptotic representations take place as $t \uparrow \omega$*

$$\frac{y(t)|y(t)|^{-\frac{\sigma_0}{1-\sigma_1}}}{\big(f(y(t), y'(t))\Theta_0(y(t))\Theta_1(y'(t))\big)^{\frac{1}{1-\sigma_1}}} = J_1(t)\,\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}\,|1-\sigma_1-\sigma_0|^{\frac{1}{1-\sigma_1}}[1+o(1)],$$
$$\frac{y(t)}{y'(t)} = \frac{J_1(t)(1-\sigma_0-\sigma_1)}{J_1'(t)(1-\sigma_1)}\,[1+o(1)].$$

**Theorem 2.** *Let $\sigma_1 = 1$. Then for the existence of $P_\omega(Y_0, Y_1, 1)$-solutions of equation* (1) *the following conditions are necessary*

$$y_0^0\alpha_0 > 0, \quad \sigma_0 y_1^0 I(t) < 0 \ \text{ as } \ t \in [a, \omega[\,, \tag{4}$$

$$\lim_{t\uparrow\omega} y_0^0 |J_2'(t)|^{-1} = Y_0, \quad \lim_{t\uparrow\omega} y_1^0 |J_2(t)|^{-1} = Y_1, \quad \lim_{t\uparrow\omega} \frac{J_2(t)I'(t)}{J_2'(t)I(t)} = \sigma_0. \tag{5}$$

*If $\sigma_0 I(t) < 0$, conditions* (4), (5) *are sufficient for the existence of such solutions of equation* (1). *For $P_\omega(Y_0, Y_1, 1)$-solutions of equation* (1) *the following asymptotic representations take place as $t \uparrow \omega$*

$$|y'(t)|\big(f(y(t), y'(t))\Theta_0(y(t))\Theta_1(y'(t))\big)^{\frac{1}{\sigma_0}} = |\sigma_0|^{-\frac{1}{\sigma_0}}|J_2(t)|^{-1}[1+o(1)],$$
$$\frac{y(t)}{y'(t)} = -\frac{J_2(t)}{J_2'(t)}\,[1+o(1)].$$

**Theorem 3.** *Let in equation* (1) *the function $f$ be of the type $f(y, y') = \exp(R(|\ln|yy'||))$, the function $R : ]0, +\infty[\, \to \,]0, +\infty[$ be continuously differentiable with monotone derivative and regularly varying on infinity of the order $\mu$, $0 < \mu < 1$. Let, moreover, $\varphi_1(y')$ satisfy the condition $S$ and the following conditions take place*

$$\lim_{t\uparrow\omega} \frac{R(|\ln|\pi_\omega(t)||)J_3(t)}{\pi_\omega(t)\ln|\pi_\omega(t)|J_3'(t)} = 0.$$

*Then for the existence of $P_\omega(Y_0, Y_1, 0)$-solutions of equation* (1) *the following conditions are necessary and sufficient*

$$\lim_{t\uparrow\omega} y_0^0 |J_3(t)|^{\frac{1-\sigma_1}{1-\sigma_0-\sigma_1}} = Y_0, \quad \lim_{t\uparrow\omega} \frac{J_3'(t)}{y_1^0|J(t)|} = Y_1, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)I'(t)}{I(t)} = \sigma_1 - 1,$$
$$\frac{I(t)}{y_1^0(1-\sigma_1)} > 0 \ \text{ as } \ t \in\, ]a, \omega[\,, \quad \frac{y_0^0 y_1^0(1-\sigma_1)J_3(t)}{1-\sigma_0-\sigma_1} > 0 \ \text{ as } \ t \in\, ]b, \omega[\,.$$

*For such solutions the following asymptotic representations take place as $t \uparrow \omega$*

$$\frac{y(t)}{|\exp(R(|\ln|y(t)y'(t)||))\varphi_0(y(t))|^{\frac{1}{1-\sigma_1}}} = \frac{1-\sigma_0-\sigma_1}{1-\sigma_1}\,|1-\sigma_1|^{\frac{1}{1-\sigma_1}} J_3(t)[1+o(1)],$$
$$\frac{y(t)}{y'(t)} = \frac{(1-\sigma_0-\sigma_1)J_3(t)}{(1-\sigma_1)J_3'(t)}\,[1+o(1)].$$

**Theorem 4.** *Let in equation* (1) *the function $f$ be of the type $f(y,y') = \exp(R(|\ln|yy'||))$, the function $R : ]0,+\infty[\, \to\, ]0,+\infty[$ be continuously differentiable with monotone derivative and regularly varying on infinity of the order $\mu$, $0<\mu<1$. Then for the existence of $P_\omega(Y_0,Y_1,0)$-solutions of equation* (1) *the following conditions are necessary*

$$Y_0 = \begin{cases} \pm\infty & \text{if } \omega = +\infty, \\ 0 & \text{if } \omega < +\infty, \end{cases} \qquad \pi_\omega(t) y_0^0 y_1^0 > 0 \ \text{ as } \ t \in [a,\omega[\,. \tag{6}$$

*If $\varphi_0$ satisfies the condition $S$ and*

$$\lim_{t\uparrow\omega} \frac{R'(|\ln|\pi_\omega(t)||)I_0(t)}{\pi_\omega(t)I_0'(t)} = 0,$$

*then along with* (6) *the following conditions are necessary and sufficient for the existence of $P_\omega(Y_0,Y_1,\pm\infty)$-solutions of equation* (1)*:*

$$y_1^0(1-\sigma_0-\sigma_1)I_0(t) > 0 \ \text{ as } \ t\in[b,\omega[\,,$$
$$\lim_{t\uparrow\omega} y_1^0|I_0(t)|^{\frac{1}{1-\sigma_0-\sigma_1}} = Y_1, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_0'(t)}{I_0(t)} = 0.$$

*For such solutions the following asymptotic representations take place as $t \uparrow \omega$*

$$\frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)||))} = (1-\sigma_0-\sigma_1)I_0(t)[1+o(1)], \quad \frac{y'(t)}{y(t)} = \frac{1}{\pi_\omega(t)}\,[1+o(1)].$$

## References

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