Asymptotic Behaviour of Solutions of One Class of Third-Order Ordinary Differential Equations

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We consider the differential equation

$$y''' = \alpha_0 p(t) y L(y), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$, $L : \Delta_{Y_0} \to]0, +\infty[$ is a continuous function slowly varying as $y \to Y_0$, Y_0 is equal to either 0 or $\pm\infty$, and Δ_{Y_0} is a one-sided neighborhood of Y_0.

In the case where $L(y) \equiv 1$, Eq. (1) is a linear third-order differential equation. The asymptotic behavior of its solutions as $t \to +\infty$ (the case $\omega = +\infty$) is investigated in details (see, for example, the monograph [2, Ch. I, § 6, pp. 175–194]).

In the paper [1], the conditions for the existence and asymptotic representations as $t \uparrow \omega$ of all possible types of $P_\omega(Y_0, \lambda_0)$-solutions were established for the second-order differential equation with the same kind of right-hand side.

Definition. We say that a solution y of Eq. (1) is a $P_\omega(Y_0, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on the interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the conditions

$$y : [t_0, \omega[\to \Delta_{Y_0}, \quad \lim_{t \uparrow \omega} y(t) = Y_0,$$

$$\lim_{t \uparrow \omega} y^{(k)}(t) = \begin{cases} \text{either } \ 0, \\ \text{or } \ \ \pm\infty \end{cases} \quad (k = 1, 2), \quad \lim_{t \uparrow \omega} \frac{[y''(t)]^2}{y'''(t) y'(t)} = \lambda_0.$$

Further, without loss of generality, we assume that

$$\Delta_{Y_0}(b) = \begin{cases} [b, Y_0[\,, & \text{if } \Delta_{Y_0} \text{ – left neighborhood } Y_0, \\ ]Y_0, b], & \text{if } \Delta_{Y_0} \text{ – right neighborhood } Y_0, \end{cases}$$

where a number $b \in \Delta_{Y_0}$ is chosen such that the inequalities

$$|b| < 1 \ \text{ when } \ Y_0 = 0, \quad b > 1 \ \ (b < -1) \ \text{ when } \ Y_0 = +\infty \ \ (Y_0 = -\infty),$$

are fulfilled and introduce numbers by setting

$$\mu_0 = \operatorname{sign} b, \ \ \mu_1 = \begin{cases} 1, & \text{if } \Delta_{Y_0} \text{ – left neighborhood } Y_0, \\ -1, & \text{if } \Delta_{Y_0} \text{ – right neighborhood } Y_0, \end{cases}$$

respectively, defining the signs of the $P_\omega(Y_0, \lambda_0)$-solution and its first derivative at some left neighborhood ω.

Besides, we introduce the following auxiliary functions

$$\Phi_1(y)=\int\limits_{B_1}^{y}\frac{ds}{sL(s)}\,,\quad \Phi_2(y)=\int\limits_{B_2}^{y}\frac{ds}{sL^{\frac13}(s)}\,,$$

$$I_1(t)=\int\limits_{A_1}^{t}p(\tau)\,d\tau,\quad I_2(t)=\frac{\alpha_0(\lambda_0-1)^2}{\lambda_0}\int\limits_{A_2}^{t}\pi_\omega^2(\tau)p(\tau)\,d\tau,\quad I_3(t)=\frac{\alpha_0(2\lambda_0-1)^{\frac23}}{\lambda_0^{\frac13}}\int\limits_{A_3}^{t}p^{\frac13}(\tau)\,d\tau,$$

where each of the limits of integration $B_i\in\{Y_0;b\}$ $(i=1,2)$ $(A_i\in\{\omega;a\}$ $(i=1,2,3))$ is chosen so that the corresponding integral tends either to zero or to $\pm\infty$ at $y\to Y_0$ (respectively, at $t\uparrow\omega$), as well as the numbers

$$\mu_i^*=\begin{cases}1, & \text{if } B_i=b,\\ -1, & \text{if } B_i=Y_0\end{cases}\quad (i=1,2).$$

Since the functions Φ_i $(i=1,2)$ are strictly monotone on the interval Δ_{Y_0} and the area of their values are intervals

$$\Delta_{Z_i}=\begin{cases}[c_i,Z_i[\,, & \text{if } \mu_0>0,\\ ]Z_i,c_i], & \text{if } \mu_0<0,\end{cases}\quad \text{where } c_i=\Phi_i(b),\quad Z_i=\lim_{y\to Y_0}\Phi_i(y)\ \ (i=1,2),$$

so there exist continuously differentiable and strictly monotone inverse functions for them $\Phi_i^{-1}:\Delta_{Z_i}\to\Delta_{Y_0}$, for which $\lim\limits_{z\to Z_i}\Phi_i^{-1}(z)=Y_0$ $(i=1,2)$.

By the properties of slowly varying functions (see [3]), there exists a continuously differentiable function $L_1:\Delta_{Y_0}\to]0,+\infty[$ slowly varying as $y\to Y_0$ such that

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{L(y)}{L_1(y)}=1\ \text{ and }\ \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{yL_1'(y)}{L_1(y)}=0. \tag{2}$$

We also say that a function L slowly varying as $y\to Y_0$ satisfies the S_1 if the function $L(\mu_0\exp z)$ is a regularly varying function when $z\to Z_0$ of any index γ, where $Z_0=+\infty$ in the case when $Y_0=\pm\infty$, and $Z_0=-\infty$ in the case when $Y_0=0$, so it can be represented in the form

$$L(\mu_0\exp z)=|z|^\gamma L_0(z),$$

where L_0 is continuous in the neighborhood of Z_0 and slowly varying function as $z\to Z_0$.

Theorem 1. *Let the function $L(\Phi_1^{-1}(z))$ be regularly varying as $z\to Z_1$ of index γ and $\lambda_0\in\mathbb{R}\setminus\{0,1\}$. Then for the existence of $P_\omega(Y_0,\lambda_0)$-solutions of the equation* (1) *it is necessary and, if*

$$(2\lambda_0^2+2\lambda_0-1)\big[(2\lambda_0^2+2\lambda_0-1)(\gamma+1)+\lambda_0\big]\neq 0,$$

it is sufficient that following conditions

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)p(t)}{I_1(t)}=-2,\qquad \frac{\lambda_0^2}{(\lambda_0-1)^2}\lim_{t\uparrow\omega}I_2(t)=Z_1,\qquad \lim_{t\uparrow\omega}\pi_\omega^3(t)p(t)L\big(\Phi_1^{-1}(I_2(t))\big)=\frac{\alpha_0\lambda_0(2\lambda_0-1)}{(\lambda_0-1)^3}\,,$$

and inequalities

$$\alpha_0\lambda_0\mu_0\mu_1>0,\quad \mu_0\mu_1\mu_1^*I_2(t)>0\ \ \text{as}\ \ t\in[a,\omega[$$

are satisfied. Moreover, each of these solutions admit the following asymptotic representations

$$\Phi_1(y(t)) = I_2(t)[1+o(1)] \ \text{ as } \ t\uparrow\omega,$$
$$\frac{y'(t)}{y(t)} = \frac{\alpha_0(\lambda_0-1)^2}{\lambda_0}\,\pi_\omega^2(t)p(t)L\big(\Phi_1^{-1}(I_2(t))\big)[1+o(1)] \ \text{ as } \ t\uparrow\omega,$$
$$\frac{y''(t)}{y'(t)} = \frac{\lambda_0}{(\lambda_0-1)\pi_\omega(t)}\,[1+o(1)] \ \text{ as } \ t\uparrow\omega.$$

Theorem 2. *Let the function $L(\Phi_2^{-1}(z))$ be regularly varying as $z\to Z_2$ of index γ and $\lambda_0\in\mathbb{R}\setminus\{0;\frac12;1\}$. Then for the existence of $P_\omega(Y_0,\lambda_0)$-solutions of the equation* (1) *it is necessary and, if*

$$(2\lambda_0^2+2\lambda_0-1)\Big[2\lambda_0^2+2\lambda_0-1+\frac{\gamma}{3}(2\lambda_0^2-\lambda_0-1)\Big]\neq 0,$$

it is sufficient that following conditions

$$\lim_{t\uparrow\omega}\pi_\omega(t)p^{\frac13}(t)L^{\frac13}\big(\Phi_2^{-1}(I_3(t))\big) = \frac{\alpha_0[\lambda_0(2\lambda_0-1)]^{\frac13}}{\lambda_0-1}\,, \quad \frac{|\lambda_0|^{\frac13}}{(2\lambda_0-1)^{\frac23}}\lim_{t\uparrow\omega}I_3(t) = Z_2$$

and inequalities

$$\alpha_0\lambda_0\mu_0\mu_1>0, \quad \mu_0\mu_1\mu_2^*I_3(t)>0 \ \text{ as } \ t\in\,]a,\omega[$$

are satisfied. Moreover, each of these solutions admit the following asymptotic representations

$$\Phi_2(y(t)) = I_3(t)[1+o(1)] \ \text{ as } \ t\uparrow\omega,$$
$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{(3-k)\lambda_0+k-2}{(\lambda_0-1)\pi_\omega(t)}\,[1+o(1)] \ \text{ as } \ t\uparrow\omega \ \ (k=1,2),$$

Theorem 3. *Let the function $L(\Phi_2^{-1}(z))$ be regularly varying as $z\to Z_2$ of index γ. Then for the existence of $P_\omega(Y_0,1)$-solutions of the equation* (1) *it is necessary and, if function $p:[a,\omega[\,\to\,]0,+\infty[$ – is continuously differentiable and there is the finite or equal $\pm\infty$*

$$\lim_{t\uparrow\omega}\frac{\Big(p^{\frac13}(t)L_1^{\frac13}\big(\Phi_2^{-1}\big(\frac{\lambda_0^{\frac13}}{(2\lambda_0-1)^{\frac23}}I_3(t)\big)\big)\Big)'}{p^{\frac23}(t)L_1^{\frac23}\big(\Phi_2^{-1}\big(\frac{\lambda_0^{\frac13}}{(2\lambda_0-1)^{\frac23}}I_3(t)\big)\big)}\,,$$

where $L_1:\Delta_{Y_0}\to\,]0,+\infty[$ is continuously differentiable and slowly varying function as $y\to Y_0$ with properties (2)*, it is sufficient, that*

$$\lim_{t\uparrow\omega}\pi_\omega(t)p^{\frac13}(t)L^{\frac13}\bigg(\Phi_2^{-1}\Big(\frac{\lambda_0^{\frac13}}{(2\lambda_0-1)^{\frac23}}I_3(t)\Big)\bigg) = \infty, \quad \frac{\lambda_0^{\frac13}}{(2\lambda_0-1)^{\frac23}}\lim_{t\uparrow\omega}I_3(t) = Z_2$$

and the following inequalities

$$\alpha_0\mu_0\mu_1>0, \quad \alpha_0\lambda_0\mu_2^*I_3(t)>0 \ \text{ as } \ t\in\,]a,\omega[$$

are satisfied. Moreover, each of these solutions admit the following asymptotic representations

$$\Phi_2(y(t)) = \frac{\lambda_0^{\frac13}}{(2\lambda_0-1)^{\frac23}}\,I_3(t)[1+o(1)] \ \text{ as } \ t\uparrow\omega,$$
$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \alpha_0p^{\frac13}(t)L^{\frac13}\bigg(\Phi_2^{-1}\Big(\frac{\lambda_0^{\frac13}}{(2\lambda_0-1)^{\frac23}}\,I_3(t)\Big)\bigg)[1+o(1)] \ \text{ as } \ t\uparrow\omega \ \ (k=1,2).$$

Theorem 4. *Let L satisfy the S_1. Then for the existence of $P_\omega(Y_0,\pm\infty)$-solutions of the equation* (1) *it is necessary and sufficient that*

$$\mu_0\mu_1\pi_\omega(t)>0 \ \ when \ \ t\in\,]a,\omega[\,, \quad \mu_0\lim_{t\uparrow\omega}|\pi_\omega(t)|=Y_0, \tag{3}$$

$$\lim_{t\uparrow\omega}p(t)\pi_\omega^3(t)L(\mu_0\pi_\omega^2(t))=0, \quad \int\limits_{a_1}^{\omega}p(\tau)\pi_\omega^2(\tau)L(\mu_0\pi_\omega^2(\tau))\,d\tau=+\infty, \tag{4}$$

where $a_1\in[a,\omega[$ such that $\mu_0\pi_\omega^2(t)\in\Delta_{Y_0}$ when $t\in[a_1,\omega[\,$. Moreover, each of solutions admits the following asymptotic representations

$$\ln|y(t)|=2\ln|\pi_\omega(t)|+\frac{\alpha_0}{2}\int\limits_{a_1}^{t}p(\tau)\pi_\omega^2(\tau)L(\mu_0\pi_\omega^2(\tau))\,d\tau[1+o(1)] \ \ as \ \ t\uparrow\omega, \tag{5}$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)}=\frac{3-k}{\pi_\omega(t)}\,[1+o(1)] \ \ as \ \ t\uparrow\omega \ \ (k=1,2). \tag{6}$$

Theorem 5. *Let L satisfies the S_1. Then for the existence of $P_\omega(Y_0,0)$-solutions of the equation* (1) *for which there is the finite or equal to $\pm\infty$, $\lim\limits_{t\uparrow\omega}\frac{\pi_\omega(t)y'''(t)}{y''(t)}$, it is necessary and sufficient that*

$$\mu_0\mu_1\pi_\omega(t)>0 \ \ when \ \ t\in\,]a,\omega[\,, \quad \mu_0\lim_{t\uparrow\omega}|\pi_\omega(t)|=Y_0, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)p(t)}{I_1(t)}=-2, \tag{7}$$

$$\lim_{t\uparrow\omega}p(t)\pi_\omega^3(t)L\big(\mu_0|\pi_\omega(t)|\big)=0, \quad \int\limits_{a_1}^{\omega}p(\tau)\pi_\omega^2(\tau)L\big(\mu_0|\pi_\omega(\tau)|\big)\,d\tau=+\infty, \tag{8}$$

where $a_1\in[a,\omega[$ such that $\mu_0|\pi_\omega(t)|\in\Delta_{Y_0}$ when $t\in[a_1,\omega[\,$. Moreover, each of solutions admits the following asymptotic representations

$$\ln|y(t)|=\ln|\pi_\omega(t)|-\alpha_0\int\limits_{a_1}^{t}p(\tau)\pi_\omega^2(\tau)L\big(\mu_0|\pi_\omega(\tau)|\big)\,d\tau\,[1+o(1)] \ \ as \ \ t\uparrow\omega, \tag{9}$$

$$\frac{y'(t)}{y(t)}=\frac{1+o(1)}{\pi_\omega(t)}\,, \quad \frac{y''(t)}{y'(t)}=-\alpha_0p(t)\pi_\omega^2(t)L\big(\mu_0|\pi_\omega(t)|\big)\,[1+o(1)] \ \ as \ \ t\uparrow\omega. \tag{10}$$

References

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