On Asymptotic Behavior of Solutions to Second-Order Regular and Singular Emden–Fowler Type Differential Equations with Negative Potential

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1 Introduction

Consider the second-order Emden–Fowler type differential equation

$$y'' - p(x, y, y')|y|^k \operatorname{sgn} y = 0, \quad k > 0, \quad k \neq 1, \tag{1}$$

where the function $p(x, u, v)$ defined on $\mathbb{R} \times \mathbb{R}^2$ is positive, continuous in x, Lipschitz continuous in u, v.

Asymptotic classification of all solutions to equation (1) in the case $p = p(x)$ was described by I. T. Kiguradze and T. A. Chanturia in [13]. Asymptotic classification of non-extensible solutions to similar third- and fourth-order differential equations was obtained by I. V. Astashova (see [1, 3–5]). Asymptotic classification of solutions to equation (1) for the bounded function $p(x, u, v)$ is contained in [8, 9].

Sufficient conditions providing $\lim\limits_{x\to a} |y'(x)| = +\infty$, $a \in \mathbb{R}$, were obtained in [13]. However, the question of separating two cases

$$\lim_{x\to a} |y(x)| = +\infty \ \text{ and } \ \lim_{x\to a} |y(x)| < +\infty \tag{2}$$

remained open. The answer on this question for $p(x, u, v) = \widetilde{p}(x)|v|^\lambda$, $\lambda \neq 1$ was considered in [11].

Asymptotic behavior of non-extensible solutions to equation (1) for unbounded function $p(x, u, v)$ is investigated in [6, 7, 10]. By using methods described in [1, 2], conditions on function $p(x, u, v)$ and initial data providing the existence of a vertical asymptote to related solution (i.e. the first case of (2)) are obtained. Other conditions on $p(x, u, v)$ and initial data sufficient for the second case of (2) are considered. Solutions satisfying the second condition of (2) are called *black hole* solutions (see [12]).

2 Asymptotic classification of solutions to Emden–Fowler type differential equations with bounded negative potential

Let us use the notation

$$\alpha = \frac{2}{k-1}\,, \quad C(\widetilde{p}) = \Big(\frac{\alpha(\alpha+1)}{\widetilde{p}}\Big)^{\frac{1}{k-1}} = \Big(\frac{\widetilde{p}(k-1)^2}{2(k+1)}\Big)^{\frac{1}{1-k}}.$$

Definition 2.1. A solution $y(x)$ to (1) is called *positive Kneser solution on* $(x_0; +\infty)$ if it satisfies the conditions $y(x) > 0$, $y'(x) < 0$ at $x > x_0$.

Definition 2.2. A solution $y(x)$ to (1) is called *negative Kneser solution on* $(x_0; +\infty)$ if it satisfies the conditions $y(x) < 0$, $y'(x) > 0$ at $x > x_0$.

Definition 2.3. A solution $y(x)$ to (1) is called *positive Kneser solution on* $(-\infty; x_0)$ if it satisfies the conditions $y(x) > 0$, $y'(x) > 0$ at $x < x_0$.

Definition 2.4. A solution $y(x)$ to (1) is called *negative Kneser solution on* $(-\infty; x_0)$ if it satisfies the conditions $y(x) < 0$, $y'(x) < 0$ at $x < x_0$.

Theorem 2.1. *Suppose $k > 1$. Let the function $p(x, u, v)$ be continuous in x, Lipschitz continuous in u, v and satisfying inequalities*

$$0 < m \le p(x, u, v) \le M < +\infty. \tag{3}$$

Let there also exist the following limits of $p(x, u, v)$:

1) P_+ *as* $x \to +\infty$, $u \to 0$, $v \to 0$,
2) P_- *as* $x \to -\infty$, $u \to 0$, $v \to 0$,

and for any $c \in \mathbb{R}$,

3) P_c^+ *as* $x \to c$, $u \to +\infty$, $v \to \pm\infty$,
4) P_c^- *as* $x \to c$, $u \to -\infty$, $v \to \pm\infty$.

Then all non-extensible solutions to (1) are divided into the following nine types according to their asymptotic behavior:

0. *Defined on the whole axis trivial solution $y_0(x) \equiv 0$.*

1–2. *Defined on $(b, +\infty)$ positive and negative Kneser solutions with power asymptotic behavior near domain boundaries:*

$$y_1(x) = C(P_b^+)(x-b)^{-\alpha}(1+o(1)), \ \ x \to b+0, \quad y_1(x) = C(P_+)x^{-\alpha}(1+o(1)t), \ \ x \to +\infty,$$
$$y_2(x) = -C(P_b^-)(x-b)^{-\alpha}(1+o(1)t), \ \ x \to b+0, \quad y_2(x) = -C(P_+)x^{-\alpha}(1+o(1)), \ \ x \to +\infty.$$

3–4. *Defined on $(-\infty, a)$ positive and negative Kneser solutions with power asymptotic behavior near domain boundaries:*

$$y_3(x) = C(P_a^+)(a-x)^{-\alpha}(1+o(1)), \ \ x \to a-0, \quad y_3(x) = C(P_-)|x|^{-\alpha}(1+o(1)), \ \ x \to -\infty,$$
$$y_4(x) = -C(P_a^-)(a-x)^{-\alpha}(1+o(1)), \ \ x \to a-0, \quad y_4(x) = -C(P_-)|x|^{-\alpha}(1+o(1)), \ \ x \to -\infty.$$

5–6. *Defined on (a, b) positive and negative solutions with power asymptotic behavior near domain boundaries:*

$$y_5(x) = C(P_a^+)(x-a)^{-\alpha}(1+o(1)), \ x \to a+0, \ \ y_5(x) = C(P_b^+)(b-x)^{-\alpha}(1+o(1)), \ x \to b-0,$$
$$y_6(x) = -C(P_a^-)(x-a)^{-\alpha}(1+o(1)), \ x \to a+0, \ \ y_6(x) = -C(P_b^-)(b-x)^{-\alpha}(1+o(1)), \ x \to b-0.$$

7–8. *Defined on (a, b) solutions with different signs and power asymptotic behavior near domain boundaries:*

$$y_7(x) = C(P_a^+)(x-a)^{-\alpha}(1+o(1)), \ x \to a+0, \ \ y_7(x) = -C(P_b^-)(b-x)^{-\alpha}(1+o(1)), \ x \to b-0,$$
$$y_8(x) = -C(P_a^-)(x-a)^{-\alpha}(1+o(1)), \ x \to a+0, \ \ y_8(x) = C(P_b^+)(b-x)^{-\alpha}(1+o(1)), \ x \to b-0.$$

Definition 2.5 (see [5])**.** A solution $y : (a, b) \to \mathbb{R}$ with $-\infty \le a < b \le +\infty$ to any ordinary differential equation is called a *MU-solution* if the following conditions hold:

(i) the equation has no solution equal to y on some subinterval of (a, b) and not equal to y at some point of (a, b);

(ii) either there is no solution defined on another interval containing (a, b) and equal to y on (a, b) or there exist at least two such solutions not equal to each other at points arbitrary close to the boundary of (a, b).

Theorem 2.2. *Suppose $0 < k < 1$. Let the function $p(x, u, v)$ be continuous in x, Lipschitz continuous in u, v and satisfying inequalities* (3)*. Let there also exist the following limits of $p(x, u, v)$:*

1) P_{++} *as* $x \to +\infty$, $u \to +\infty$, $v \to +\infty$;

2) P_{+-} *as* $x \to +\infty$, $u \to -\infty$, $v \to -\infty$;

3) P_{-+} *as* $x \to -\infty$, $u \to +\infty$, $v \to -\infty$;

4) P_{--} *as* $x \to -\infty$, $u \to -\infty$, $v \to +\infty$,

and for any $c \in \mathbb{R}$ denote $P_c = p(c, 0, 0)$.

Then all MU–solutions to equation (1) *are divided into the following eight types according to their asymptotic behavior:*

1–2. *Defined on semi-axis $(b, +\infty)$ positive and negative solutions tending to zero with their derivatives as $x \to b + 0$ with power asymptotic behavior near domain boundaries:*

$$y_1(x) = C(P_b)(x-b)^{-\alpha}(1+o(1)), \ \ x \to b+0, \quad y_1(x) = C(P_{++})x^{-\alpha}(1+o(1)), \ \ x \to +\infty,$$
$$y_2(x) = -C(P_b)(x-b)^{-\alpha}(1+o(1)), \ \ x \to b+0, \quad y_2(x) = -C(P_{+-})x^{-\alpha}(1+o(1)), \ \ x \to +\infty.$$

3–4. *Defined on semi-axis $(-\infty, a)$ positive and negative solutions tending to zero with their derivatives as $x \to a - 0$ with power asymptotic behavior near domain boundaries:*

$$y_3(x) = C(P_a)(a-x)^{-\alpha}(1+o(1)), \ \ x \to a-0, \quad y_3(x) = C(P_{-+})|x|^{-\alpha}(1+o(1)), \ \ x \to -\infty,$$
$$y_4(x) = -C(P_a)(a-x)^{-\alpha}(1+o(1)), \ \ x \to a-0, \quad y_4(x) = -C(P_{--})|x|^{-\alpha}(1+o(1)), \ \ x \to -\infty.$$

5–6. *Defined on the whole axis solutions with same signs and power asymptotic behavior near domain boundaries:*

$$y_5(x) = C(P_{++})x^{-\alpha}(1+o(1)), \ \ x \to +\infty, \quad y_5(x) = C(P_{-+})|x|^{-\alpha}(1+o(1)), \ \ x \to -\infty,$$
$$y_6(x) = -C(P_{+-})x^{-\alpha}(1+o(1)), \ \ x \to +\infty, \quad y_6(x) = -C(P_{--})|x|^{-\alpha}(1+o(1)), \ \ x \to -\infty.$$

7–8. *Defined on the whole axis solutions with different signs and power asymptotic behavior near domain boundaries:*

$$y_7(x) = C(P_{++})x^{-\alpha}(1+o(1)), \ \ x \to +\infty, \quad y_7(x) = -C(P_{--})|x|^{-\alpha}(1+o(1)), \ \ x \to -\infty,$$
$$y_8(x) = -C(P_{+-})x^{-\alpha}(1+o(1)), \ \ x \to +\infty, \quad y_8(x) = C(P_{-+})|x|^{-\alpha}(1+o(1)), \ \ x \to -\infty.$$

3 Asymptotic behavior of solutions to Emden–Fowler type differential equations with unbounded negative potential

Lemma 3.1. *Suppose $k > 1$. Let $p(x, u, v)$ be continuous in x, Lipschitz continuous in u, v, and bounded below by a positive constant. Let $y(x)$ be a nontrivial non-extensible solution to equation (1) satisfying the condition $y(x_0)y'(x_0) \geq 0$ or $y(x_0)y'(x_0) \leq 0$ at some point x_0. Then there exists $x^* \in (x_0, +\infty)$ or respectively $x_* \in (-\infty, x_0)$, such that*

$$\lim_{x \to x^*-0} |y'(x)| = +\infty \ \ \text{or respectively} \ \ \lim_{x \to x_*+0} |y'(x)| = +\infty. \tag{4}$$

Lemma 3.2. *Suppose $0 < k < 1$. Let $p(x, u, v)/|v|$ be continuous in x, Lipschitz continuous in u, v, for $v \neq 0$ and bounded below by a positive constant. Let $y(x)$ be a nontrivial non-extensible solution to equation (1) satisfying the condition $y(x_0)y'(x_0) \geq 0$ or $y(x_0)y'(x_0) \leq 0$ but not $y(x_0) = y'(x_0) = 0$ at some point x_0. Then there exists $x^* \in (x_0, +\infty)$ or respectively $x_* \in (-\infty, x_0)$ providing (4).*

Using the substitutions $x \longmapsto -x$, $y(x) \longmapsto -y(x)$ we obtain an equation of the same type as (1). That is why we investigate asymptotic behavior of non-extensible positive solutions to equation (1) near the right domain boundary only.

Theorem 3.1. *Suppose there exist constants $u_0 > 0$, $v_0 > 0$ such that for $u > u_0$, $v > v_0$ the function $p = p(x, u, v)$ has the representation $p = h(u)g(v)$, where the functions $h(u)$, $g(v)$ are continuous and bounded below by a positive constant, and for $0 < k < 1$ function p additionally satisfies the conditions of Lemma 3.2. Then for any non-extensible solution $y(x)$ to equation (1) with initial data $y(x_0) \geq u_0$, $y'(x_0) \geq v_0$ and the first property of (2) the line $x = x^*$ is a vertical asymptote if and only if*

$$\int_{v_0}^{+\infty} \frac{v}{g(v)}\, dv = +\infty. \tag{5}$$

Theorem 3.2. *Suppose for $k > 1$ or $0 < k < 1$ the function $p(x, u, v)$ satisfies the conditions of Lemma 3.1 or respectively Lemma 3.2. Let there exist constants $u_0 > 0$, $v_0 > 0$ such that for $u > u_0$, $v > v_0$ the inequality $p(x, u, v) \leq f(x, u)g(v)$ holds, where the function $f(x, u)$ is continuous, the function $g(v)$ is continuous, bounded below by a positive constant and satisfies the condition*

$$\int_{v_0}^{+\infty} \frac{dv}{g(v)} = +\infty. \tag{6}$$

Then for any non-extensible solution $y(x)$ to equation (1) with initial data satisfying inequalities $y(x_0) \geq u_0$, $y'(x_0) \geq v_0$ and with the first property of (2) the line $x = x^*$ is a vertical asymptote.

Theorem 3.3. *Suppose for $k > 1$ or $0 < k < 1$ the function $p(x, u, v)$ satisfies the conditions of Lemma 3.1 or respectively Lemma 3.2. Let there exist constants $u_0 > 0$, $v_0 > 0$ such that for $u > u_0$, $v > v_0$ the inequality $p(x, u, v) \leq g(v)$ holds, where the function $g(v)$ is continuous and satisfies the condition (6). Then for any non-extensible solution $y(x)$ to equation (1) with initial data $y(x_0) \geq u_0$, $y'(x_0) \geq v_0$ and the first property of (2) the line $x = x^*$ is a vertical asymptote.*

Theorem 3.4. *Suppose for $k > 1$ or $0 < k < 1$ the function $p(x, u, v)$ satisfies the conditions of Lemma 3.1 or respectively Lemma 3.2. Let there exist constants $u_0 > 0$, $v_0 > 0$ such that for $u > u_0$, $v > v_0$ the inequality $p(x, u, v) \geq g(v)$ holds, where the function $g(v)$ is continuous, bounded below* *by a positive constant and doesn't satisfy the condition* (5)*. Then for any non-extensible solution* $y(x)$ *to equation* (1) *with initial data* $y(x_0) \ge u_0$, $y'(x_0) \ge v_0$ *and the first property of* (2) *we have*

$$0 < \lim_{x \to x^*-0} y(x) < +\infty, \quad x^* - x_0 < \frac{1}{y^k(x_0)} \int\limits_{y'(x_0)}^{+\infty} \frac{dv}{g(v)}\,.$$

Theorem 3.5. *Suppose* $k > 0$, $k \neq 1$*. Let the function* $p(x,u,v)$ *be continuous in* x*, Lipschitz continuous in* u*,* v*. Let there exist constants* $u_0 > 0$, $v_0 > 0$ *such that for* $u > u_0$, $v > v_0$ *the inequality* $p(x,u,v) \le C|v|^{-\alpha}$ *holds. Then any non-extensible solution* $y(x)$ *to equation* (1) *with initial data* $y(x_0) \ge u_0$, $y'(x_0) \ge v_0$ *can be extended to* $(x_0, +\infty)$ *and*

$$\lim_{x \to +\infty} y(x) = \lim_{x \to +\infty} y(x) = +\infty.$$

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