Periodic Reflecting Function of Linear Differential System with Incommensurable Periods of Homogeneous and Nonhomogeneous Parts

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Consider the differential system

$$\dot{x} = X(t,x), \;\; t \in \mathbb{R}, \;\; x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n, \tag{1}$$

with continuous in all the variables and continuously differentiable right part over x. Let $\varphi(t;\tau,x)$ denote the general solution in the form of Cauchy system (1), that is $\varphi(t;\tau,x)$ – the solution of (1) with the initial condition $\varphi(\tau;\tau,x) = x$. Let I_x be maximum symmetrical with respect to zero interval of existence of solution $\varphi(t;0,x)$. Let $D(X) := \{(t,\varphi(t;0,x)) \in \mathbb{R}^{n+1} : \; t \in I_x, \; x \in \mathbb{R}^n\}$. From the theorem on continuous dependence of solutions on the initial value and the definition of $D(X)$ it follows that $D(X)$ is the open domain in $\mathbb{R} \times \mathbb{R}^n$ which contains the hyperplane $t = 0$. Reflecting function of system (1) is called [3], [4, p. 11], [5, p. 62] the vector function $F : D(X) \to \mathbb{R}^n$, acting according to the rule $(t,x) \longmapsto \varphi(-t;t,x)$. In other words, for any solution $x(t)$ of this system, which exists on a symmetric interval $(-\xi,\xi)$, the identity $F(t,x(t)) \overset{t}{\equiv} x(-t)$ is valid for all $t \in (-\xi,\xi)$. This property can be taken [4, p. 16] for the definition of a reflecting function. From the definition of the reflecting function and the differentiability theorem on the initial value it follows that the reflecting function $F(t,x)$ of system (1) has partial derivatives F_t and F_x in the region $D(X)$.

Fundamentally important result of the theory of reflecting function is the following criterion [3], [4, pp. 11, 12], [5, pp. 63, 64]: the vector function $F = F(t,x) : D(X) \to \mathbb{R}^n$ is a reflecting function of system (1) if and only if it satisfies the initial condition $F(0,x) \equiv x$ and the system of equations in the partial derivatives

$$F_t + F_x X(t,x) + X(-t,F) = 0. \tag{2}$$

Equation (2) is called [4, p. 12], [5, p. 63] basic equation (the ratio) for the reflecting function. Methods have been developed which in some cases make it possible to find the reflecting function of system (1) without finding its solutions. Moreover, if we know only some of the properties of the reflecting function of the system, it is possible to investigate the behavior of its solutions without resorting to the construction of reflecting function [4–9].

Two systems are equivalent in the sense of the coincidence of reflecting functions [5, p. 75], if their reflecting functions are equal in a domain containing the hyperplane $t = 0$. Since the solutions of equivalent systems have a number of similar properties, the task of constructing classes of equivalent systems, and the choice of simple (for example, integrated into the final form) systems-representatives of these classes will be important and relevant.

In this article, the linear differential systems defined for all $t \in \mathbb{R}$ are discussed, and for them the domain $D(X)$ determination of reflecting function coincides with the extended phase space $\mathbb{R} \times \mathbb{R}^n$, then for such systems it is natural to study the conditions of coincidence of their reflecting functions in all extended phase space. Therefore, further as the equivalence of linear systems in the sense of the coincidence of their reflecting functions the coincidence of the reflecting functions of these systems throughout the extended phase space is understood.

In this article, the quasi-periodic two-frequency linear differential systems are discussed such that their homogeneous and nonhomogeneous parts are periodic with incommensurable periods, and the conditions of existence of the periodic reflecting functions in such systems are clarified.

Theorem 1. *For the linear nonhomogeneous differential system*

$$\dot{x} = A(t)x + f(t), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n \tag{3}$$

with continuous $n \times n$-matrix $A(t)$ and vector-function $f(t)$, to have the same reflecting function as the system

$$\dot{x} = f(t), \tag{4}$$

necessary and sufficient conditions are:

1) *matrix-valued function $A(t)$ is odd;*

2) *there is the identity*

$$A(t) \int\limits_{t}^{-t} f(s)\, ds = 0 \ \text{ for all } \ t \in \mathbb{R}. \tag{5}$$

At the same time, reflecting function $F(t,x)$ of these systems, is the vector-function

$$F(t,x) = x + \int\limits_{t}^{-t} f(s)\, ds. \tag{6}$$

Proof. Sufficiency. The general solution in the form of the Cauchy system (4) is given by $\varphi(t;\tau,x) = x + \int\limits_{\tau}^{t} f(s)\, ds$. As a consequence of this presentation by the definition of the reflecting function we easily find that reflecting function $F(t,x)$ of system (4) is given by equation (6).

We will show that under the conditions 1) and 2) function (6) is the reflecting function of system (3). It's enough to make sure that function (6) satisfies the fundamental ratio (2) for reflecting function of system (3). Substituting in it function (6), after obvious equivalent transformations we obtain the identity:

$$A(t)x + A(-t)x + A(-t) \int\limits_{t}^{-t} f(t)\, dt \overset{t,x}{\equiv} 0. \tag{7}$$

Since under the conditions 1) and 2) of the theorem identity (7) is obviously true, then function (6) is the reflecting function of system (3). The sufficiency is proved.

Necessity. Let systems (3) and (4) are equivalent in the sense of coincidence of the reflecting functions. As it is shown above, system (4) has a reflecting function (6). Since function (6) is also the reflecting function of system (3), then for system (3) and this function the main identity (2) is satisfied. Hence we obtain identity (7). This identity is satisfied for all t and x. Assuming in it $x = 0$ and replacing $-t$ onto t, one obtains the condition 2). Thus, the identity must be satisfied

$$(A(t) + A(-t))x \overset{t,x}{\equiv} 0. \tag{8}$$

Identity (8) means that the linear operator $A(t) + A(-t)$ is null, that is $A(t) = -A(-t)$ for all $t \in \mathbb{R}$.

Thus, the function $A(t)$ – odd, and as proved above, satisfies the condition 2). The necessity, and thus the theorem is proved. □

Corollary 1. *If matrix $A(t)$ is nonsingular for all $t \in \mathbb{R}$, then systems* (3) *and* (4) *have the same reflecting function if and only if the matrix-valued function $A(\,\cdot\,)$ and the vector function $f(\,\cdot\,)$ are odd. In this case, reflecting function of systems* (3) *and* (4) *will be the function $F(t,x) = x$.*

If the set of those $t \in \mathbb{R}$, in which matrix $A(t)$ is non-singular, not coincides with the $\mathbb{R}$, then condition 2) of the theorem does not necessarily mean oddness of the vector-function $f(\,\cdot\,)$ which is confirmed by the following example.

Example 1. Consider the system

$$\dot{x} = A(t)x + f(t), \ \ t \in \mathbb{R}, \ \ x \in \mathbb{R}^2,$$

in which matrix of coefficients $A(t)$ is odd and has zero determinant for all $t \in \mathbb{R}$. Let

$$A(t) = \begin{pmatrix} a_1(t) & a_2(t) \\ a_3(t) & a_4(t) \end{pmatrix}, \quad f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}.$$

We will assume that $a_1^2(t) + a_2^2(t) \neq 0$ for any $t \in \mathbb{R}$. According to Theorem 1, the given system has the same reflecting function as the system $\dot{x} = f(t)$ if and only if identity (5) is satisfied. From this identity we obtain

$$a_1(t) \int\limits_t^{-t} f_1(s)\,ds \equiv -a_2(t) \int\limits_t^{-t} f_2(s)\,ds, \quad a_3(t) \int\limits_t^{-t} f_1(s)\,ds \equiv -a_4(t) \int\limits_t^{-t} f_2(s)\,ds. \tag{9}$$

We will find all vector-functions $f(t) = (f_1(t), f_2(t))^\top$, for which these identities are satisfied. Since $\det A(t) = 0$ for all $t \in \mathbb{R}$ and the first row of the matrix $A(t)$ is nonzero then its second row is proportional to the first one, and then, for the validity of these identities it is necessary and sufficient the first of them to be valid.

Since the vector $(a_1(t), a_2(t))^\top$ is nonzero, then the first identity in (9) is performed, if and only if for some function $h(t)$ satisfies the identities

$$\int\limits_t^{-t} f_1(s)\,ds \equiv -a_2(t)h(t), \quad \int\limits_t^{-t} f_2(s)\,ds \equiv a_1(t)h(t). \tag{10}$$

In order identities (10) to be carried out, it is necessary the function $h(t)$ to be even (as left sides in (10) and functions $a_1(t)$, $a_2(t)$ are odd) and that the functions $a_1(t)h(t)$ and $a_2(t)h(t)$ have been continuously differentiable (as left sides in (10) – continuously differentiable functions).

We will show that these conditions are sufficient for the existence of functions $f_1(t)$, $f_2(t)$, which satisfy (10). Fix some even function $h(t)$, for which the right sides in (10) – continuously differentiable functions. Denote $-a_2(t)h(t)$ through $g_1(t)$. Then the first identity in (10) takes the form $\int\limits_t^{-t} f_1(s)\,ds \equiv g_1(t)$. Differentiating it on t, we obtain

$$f_1(t) + f_1(-t) \equiv -\dot{g}_1(t). \tag{11}$$

The function $\dot{g}_1(t)$ is even, as a derivative of an odd function, and it is continuous. We will seek solution of the functional equation (11) in the form of

$$f_1(t) = -\frac{\dot{g}_1(t)}{2} + r_1(t), \tag{12}$$

where $r_1(t)$ is an unknown continuous function. Replacing in (11) the function $f_1(t)$ by the given representation, we obtain the identity $r_1(t) + r_1(-t) \equiv 0$ in view of parity of $\dot{g}_1(t)$, that is $r_1(t)$ – an odd function. Conversely, it is easy to see that the function of the form (12) with an odd continuous function $r_1(t)$ satisfies the first identity in (10). Indeed,

$$\int\limits_t^{-t} f_1(s)ds \equiv \int\limits_t^{-t} (-\frac{\dot{g}_1(s)}{2} + r_1(s))ds = g_1(t) + \int\limits_t^{-t} r_1(s)ds = g_1(t) = -a_2(t)h(t).$$

Similarly, if we denote the function $a_1(t)h(t)$ via $g_2(t)$, a solution of the second functional equation in (10) we find in the form of

$$f_2(t) = -\frac{\dot{g}_2(t)}{2} + r_2(t), \tag{13}$$

where $g_2(t) \equiv a_1(t)h(t)$, and $r_2(t)$ – arbitrary odd function. Thus, the solution of the problem on the description of the set of vector-functions $f(t) = (f_1(t), f_2(t))^\top$, $t \in \mathbb{R}$, satisfy (9) and it is reduced to the problem of the description of the set of even functions $h(t)$, $t \in \mathbb{R}$, for which both functions $a_1(t)h(t)$ and $a_2(t)h(t)$ would be continuously differentiable.

As we see, the vector function $f(t) = (f_1(t), f_2(t))^\top$, the components of which are built up, and given by equalities (12), (13), generally speaking, is not odd, whatever the elements of a degenerate odd matrix $A(t)$ would be , the first row of which for all $t \in \mathbb{R}$ is nonzero ($a_1^2(t) + a_2^2(t) \neq 0$ for all $t \in \mathbb{R}$).

Remark 1. Considered example gives a partial solution for the following problem, formulated by E. A. Barabanov: for a linear homogeneous differential system $\dot{x} = A(t)x$ in terms of its coefficient matrix $A(t)$ to describe all those its nonhomogeneous perturbations $f(t)$, at which the reflecting functions of systems $\dot{y} = A(t)y + f(t)$ and $\dot{z} = f(t)$ coincide.

Corollary 2. *Let the matrix $A(t)$ have period ω_1, and the vector function $f(t)$ – period ω_2. For system* (3) *to have an ω_2-periodic on t reflecting function* (6) *it is necessary and sufficient the fulfillment of conditions* 1) *and* 2) *of Theorem* 1 *and the equality*

$$\int\limits_0^{\omega_2} f(s)\,ds = 0. \tag{14}$$

Remark 2. In the case 3 when numbers ω_1 and ω_2 are incommensurable, Corollary 2 gives sufficient condition for the existence of ω_2-periodic on t reflecting function in a quasi-periodic system (3).

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