On the Property of Separateness of the Angle Between Stable and Unstable Lineals of Solutions of Exponentially Dichotomous and Weak Exponentially Dichotomous Systems

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Denote by $\mathcal{M}_n$ the class of linear differential systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geqslant 0, \tag{1}$$

where $n \geqslant 2$, with the piecewise continuous and uniformly bounded on the time half-line $t \geqslant 0$ coefficients matrix $A(\,\cdot\,) : [0, +\infty) \to \operatorname{End} \mathbb{R}^n$. Denote by $\mathcal{X}_A(\,\cdot\,)$ the linear space of solutions of system (1). Its subspaces we call further lineals to distinguish them from linear subspaces in $\mathbb{R}^n$. The angle between lineals $U(\,\cdot\,)$ and $V(\,\cdot\,)$ of the space $\mathcal{X}_A(\,\cdot\,)$ we call the function $\gamma(t)$ of the variable $t \geqslant 0$, which is defined by the equation $\gamma(t) = \angle(U(t), V(t))$, where $\angle(U(t), V(t))$ is the angle between subspaces $U(t)$ $V(t)$ of the space $\mathbb{R}^n$.

It is known [2, p. 236], [3, p. 10], system (1) in $\mathcal{M}_n$ is called an exponentially dichotomous system or a system with exponentially dichotomy on the time half-line if there exist positive constants c_1, c_2 and ν_1, ν_2 and a decomposition of the linear space $\mathcal{X}_A(\,\cdot\,)$ of its solutions into the direct sum $\mathcal{X}_A(\,\cdot\,) = L_A^-(\,\cdot\,) \oplus L_A^+(\,\cdot\,)$ of lineals, so that its solutions $x(\,\cdot\,)$ belonging to these lineals satisfy the following two conditions:

1) if $x(\,\cdot\,) \in L_A^-(\,\cdot\,)$, then $\|x(t)\| \leqslant c_1 \exp\{-\nu_1(t-s)\}\, \|x(s)\|$ for arbitrary $t \geqslant s \geqslant 0$;

2) if $x(\,\cdot\,) \in L_A^+(\,\cdot\,)$, then $\|x(t)\| \geqslant c_2 \exp\{\,\nu_2(t-s)\}\, \|x(s)\|$ for arbitrary $t \geqslant s \geqslant 0$.

In this definition the choice of norm in $\mathbb{R}^n$ does not play any role, because in a finite linear space any two norms are equivalent. The class of exponentially dichotomous n-dimensional systems is denoted by $\mathcal{E}_n$.

Condition of exponential dichotomy of system (1) means, in particular, that in any time segment the norm of any solution in $L_A^-(\,\cdot\,)$ uniformly decreases exponentially, and the norm of any solution in $L_A^+(\,\cdot\,)$ uniformly increases exponentially. If the system is exponentially dichotomous, its lineal $L_A^-(\,\cdot\,)$, called a stable lineal, is uniquely determined (it consists of all solutions, decreasing to zero at infinity), and any of lineals, complementary lineal $L_A^-(\,\cdot\,)$ to the space $\mathcal{X}_A(\,\cdot\,)$ of solutions, may be taken as a lineal $L_A^+(\,\cdot\,)$, called unstable lineal. Although in the above definition the case of zero dimension of one of subspaces is not excluded, i.e. one of the equalities $L_A^-(\,\cdot\,) = \{\mathbf{0}\}$ or $L_A^+(\,\cdot\,) = \{\mathbf{0}\}$ is possible, further we consider that each of the lineals $L_A^-(\,\cdot\,)$ and $L_A^+(\,\cdot\,)$ is nonzero.

We say that the lineals of solutions $U(\,\cdot\,)$ and $V(\,\cdot\,)$ of system (1) are separated if the angle between them is separated from zero: $\inf\{\gamma(t) : t \geqslant 0\} > 0$. It is well known [2, p. 237] that the stable lineal $L_A^-(\,\cdot\,)$ of an exponentially dichotomous system is separated from any of its unstable lineal $L_A^+(\,\cdot\,)$, i.e. for any unstable lineal $L_A^+(\,\cdot\,)$ there is the inequality

$$\inf\big\{\angle(L_A^-(t), L_A^+(t)) :\ t \geqslant 0\big\} > 0. \tag{2}$$

This property of finite-dimensional exponentially dichotomous systems is essential and must be included [2] in the definition of exponential dichotomy, when we extend the concept of exponential dichotomy of the finite-dimensional case to the case of Banach spaces, to preserve basic properties of finite-dimensional exponentially dichotomous system.

Nevertheless, the following theorem shows that the property of separateness from zero of the angle between stable and unstable lineals of exponentially dichotomous systems is not characteristic for such systems. More precisely, the angle between stable and unstable subspaces of exponentially dichotomous system is the same as can generally be the angle between separated subspaces of solutions of an arbitrary system (1) that is not exponentially dichotomous.

Theorem 1. *Let a system in $\mathcal{M}_n$ have separated lineals of solutions $U(\,\cdot\,)$ and $V(\,\cdot\,)$. Then there exists a system $A \in \mathcal{E}_n$ such that for its stable $L_A^-(\,\cdot\,)$ and unstable $L_A^+(\,\cdot\,)$ lineals for all $t \geqslant 0$ the equalities hold*

$$L_A^-(t) = U(t) \;\; \text{and} \;\; L_A^+(t) = V(t).$$

The following statement characterizes more fully the property of the angle between stable and unstable lineals of exponentially dichotomous systems and complements the above statement [2, p. 237] on the separateness of stable and unstable lineals of exponentially dichotomous systems.

Theorem 2. *For any system $A \in \mathcal{E}_n$ there exists a constant $c_A \in (0, \pi/2)$ such that for any of its unstable lineal $L_A^+(\,\cdot\,)$ for all sufficiently large $t \geqslant 0$ the inequality $\angle\{L_A^-(t), L_A^+(t)\} > c_A$ is true, i.e. there is a constant $c_A \in (0, \pi/2)$ such that the inequality*

$$\lim_{\tau\to+\infty} \inf_{t\geqslant\tau} \angle\big\{L_A^-(t), L_A^+(t)\big\} > c_A \tag{3}$$

holds for any unstable lineal $L_A^+(\,\cdot\,)$.

Obviously, inequality (3) enhances inequality (2). Inequality (3), if we denote by $\mathcal{U}_A$ the aggregate of unstable lineals of system $A \in \mathcal{E}n$, can be written as

$$\inf_{L_A^+(\,\cdot\,)\in\mathcal{U}_A} \lim_{\tau\to+\infty} \inf_{t\geqslant\tau} \angle\big\{L_A^-(t), L_A^+(t)\big\} > c_A.$$

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In [1], it is introduced a generalization of the concept of exponentially dichotomous linear differential systems defined in a finite space, that consists in the refusal from the requirement of the uniformness of estimates for the norms of solutions under constants-multipliers in definition of an exponentially dichotomous system. In [1], such systems are referred to as weak exponentially dichotomous. In other words, system (1) in $\mathcal{M}_n$ is called a weak exponentially dichotomous system or a system with a weak exponential dichotomy on the half-line, if there exist positive constants ν_1, ν_2 and a decomposition of the linear space $\mathcal{X}_A(\,\cdot\,)$ of its solutions into the direct sum $\mathcal{X}_A(\,\cdot\,) = L_A^-(\,\cdot\,) \oplus L_A^+(\,\cdot\,)$ of lineals so that its solutions $x(\,\cdot\,)$ belonging to these lineals satisfy the following two conditions:

1′) if $x(\,\cdot\,) \in L_A^-(\,\cdot\,)$, then $\|x(t)\| \leqslant c_1(x)\exp\{-\nu_1(t-s)\}\,\|x(s)\|$ for arbitrary $t \geqslant s \geqslant 0$;

2′) if $x(\,\cdot\,) \in L_A^+(\,\cdot\,)$, then $\|x(t)\| \geqslant c_2(x)\exp\{\nu_2(t-s)\}\,\|x(s)\|$ for arbitrary $t \geqslant s \geqslant 0$,

where $c_1(x)$ and $c_2(x)$ are positive constants which, in general, depend on the choice of the solution $x(\,\cdot\,)$.

As can be seen, if we could choose, in the definition of a weak exponentially dichotomous system, the constants $c_1(x)$ and $c_2(x)$ which are the same for all solutions $x(\,\cdot\,) \in L_A^-(\,\cdot\,)$ and $x(\,\cdot\,) \in L_A^+(\,\cdot\,)$

respectively, then we get the definition of an exponentially dichotomous system. The class of n-dimensional weakly exponentially dichotomous systems is denoted by $W\mathcal{E}_n$. In [1], it is shown that for any $n \geqslant 2$, there is a proper inclusion $\mathcal{E}_n \subset W\mathcal{E}_n$. Just as for exponentially dichotomous systems, lineals $L_A^-(\,\cdot\,)$ and $L_A^+(\,\cdot\,)$ are called stable and unstable lineals of a system $A \in W\mathcal{E}_n$, and, just as in the case of exponentially dichotomous systems, for any system $A \in W\mathcal{E}_n$ its stable lineal $L_A^-(\,\cdot\,)$ is uniquely determined (it consists of all solutions decreasing to zero at infinity), and as a lineal $L_A^+(\,\cdot\,)$ may be taken any algebraic complement $L_A^-(\,\cdot\,)$ to the linear space $\mathcal{X}_A(\,\cdot\,)$ of solutions.

We can ask how significantly the properties of systems of the classes $\mathcal{E}_n$ and $W\mathcal{E}_n$ can differ. In particular, is it true that the unstable and stable lineals of a weak exponentially dichotomous system are separated? If the system $A \in W\mathcal{E}_2$, then, as is easy to show, it is either an exponentially dichotomous or its stable or unstable lineal is zero. That is why weak exponentially dichotomous system with unseparated angle between stable and unstable lineals of solutions should have the dimension of not less than 3. It turns out that for weak exponentially dichotomous system of dimension $n \geqslant 3$ incorrect is not only the property stated in Theorem 2 but also weaker property (2) of separateness of the angle between stable and unstable lineals of solutions as shown by

Theorem 3. *For any natural number $n \geqslant 3$ there exists the system $A \in W\mathcal{E}_n$ and such an unstable lineal $L_A^+(\,\cdot\,)$ of solutions that the angle between it and the stable lineal $L_A^-(\,\cdot\,)$ is not separated from zero, i.e.* $\inf\{\angle(L_A^-(t), L_A^+(t)) :\ t \geqslant 0\} = 0$.

Apparently, Theorem 3 can be enhanced: for any $n \geqslant 3$ there exist such systems in the $W\mathcal{E}_n \setminus \mathcal{E}_n$ that the angle between their stable and any unstable lineals is not separated from zero.

Acknowledgement

The author expresses her gratitude to E. A. Barabanov for the formulation of the problem and his valuable advice.

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