On Power-Law Asymptotic Behavior of Solutions to Weakly Superlinear Emden–Fowler Type Equations

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1 Introduction

Consider the equation

$$y^{(n)} = |y|^k \operatorname{sgn} y \tag{1.1}$$

with $k > 1$. Hereafter, we put $\gamma = \frac{k-1}{n}$ and $m = n - 1$.

Definition 1.1. A solution $y(x)$ of equation (1.1) will be said to be *n-positive* if it is maximally extended in both directions and eventually satisfies the inequalities

$$y(x) > 0, \;\; y'(x) > 0, \dots, y^{(m)}(x) > 0.$$

Note that if the above inequalities are satisfied by a solution of (1.1) at some point x_0, then they are also satisfied at any point $x > x_0$ in the domain of the solution. Moreover, such a solution, if maximally extended, must be a so-called blow-up solution (having a vertical asymptote at the right endpoint of its domain).

Immediate calculations show that equation (1.1) has n-positive solutions with exact power-law behavior, namely,

$$y(x) = C(x^* - x)^{-1/\gamma}, \;\text{ where }\; C^{k-1} = \prod_{j=0}^{m} \Big(j + \frac{1}{\gamma}\Big), \tag{1.2}$$

defined on $(-\infty, x^*)$ with arbitrary $x^* \in \mathbb{R}$. For $n = 1$ all n-positive solutions of (1.1) are defined by (1.2). For $n \in \{2, 3, 4\}$ it is known that any n-positive solution of (1.1) and even of more general equations is asymptotically equivalent, near the right endpoint of its domain, to the solution defined by (1.2) with appropriate x^* (see [5] for $n = 2$, and [1–3] for $n \in \{3, 4\}$).

The natural hypothesis generalizing this statement for all $n > 4$ appears to be wrong (see [6] for sufficiently large n and [4] for $n \in \{12, 13, 14\}$).

However, a weaker version of this statement for higher-order equations can be proved.

2 Main result

Theorem 2.1. *For any integer $n > 4$ there exists $K > 1$ such that for any real $k \in (1, K)$, any n-positive solution of equation* (1.1) *is asymptotically equivalent, near the right endpoint of its domain, to a solution with exact power-law behavior.*

To prove this result, an auxiliary dynamical system is investigated on the m-dimensional sphere. To define it note that if a function $y(x)$ is a solution of equation (1.1), the same is true for the function

$$z(x) = Ay(A^{\gamma} x + B) \tag{2.1}$$

with any constants $A > 0$ and B. Any non-trivial solution $y(x)$ of equation (1.1) generates in $\mathbb{R}^n \setminus \{0\}$ the curve given parametrically by

$$(y(x), y'(x), y''(x), \dots, y^{(m)}(x)).$$

We can define an equivalence relation on $\mathbb{R}^n \setminus \{0\}$ such that all solutions obtained from $y(x)$ by (2.1) with $A > 0$ generate equivalent curves, i.e. curves passing through equivalent points (maybe for different x). We assume the points $(y_0, y_1, y_2, \dots, y_m)$ and $(z_0, z_1, z_2, \dots, z_m)$ in $\mathbb{R}^n \setminus \{0\}$ to be equivalent if and only if there exists a constant $\lambda > 0$ such that

$$z_j = \lambda^{n+j(k-1)} y_j, \ \ j \in \{0, 1, \dots, m\}.$$

The quotient space obtained is homeomorphic to the m-dimensional sphere

$$S^m = \big\{ y \in \mathbb{R}^n : \ y_0^2 + y_1^2 + y_2^2 + \dots + y_m^2 = 1 \big\}$$

having exactly one representative of each equivalence class since the equation

$$\lambda^{2n} y_0^2 + \lambda^{2(n+2(k-1))} y_1^2 + \dots + \lambda^{2(n+m(k-1))} y_m^2 = 1$$

has exactly one positive root λ for any $(y_0, y_1, y_2, \dots, y_m) \in \mathbb{R}^n \setminus \{0\}$. Equivalent curves in $\mathbb{R}^n \setminus \{0\}$ generate the same curves in the quotient space. The last ones are trajectories of an appropriate dynamical system, which can be described, in different charts covering the quotient space, by different formulae using different independent variables. A unique common independent variable can be obtained from those ones by using a partition of unity.

Within the chart that covers the points corresponding to positive values of solutions and has the coordinate functions

$$u_j = y^{(j)} y^{-1-\gamma j}, \ \ j \in \{1, \dots, m\}, \tag{2.2}$$

the dynamical system can be written as

$$\begin{cases} \dfrac{du_1}{dt} = u_2 - (1+\gamma) u_1^2, \\ \dfrac{du_j}{dt} = u_{j+1} - (1+\gamma j) u_1 u_j, \quad j \in \{2, \dots, m-1\}, \\ \dfrac{du_m}{dt} = 1 - (1+\gamma m) u_1 u_m \end{cases} \tag{2.3}$$

with the independent variable

$$t = \int\limits_{x_0}^{x} y(\xi)^\gamma \, d\xi.$$

The dynamical system described has some equilibrium points corresponding to the solutions of equation (1.1) with exact power-law behavior. One of them, which corresponds to the n-positive solutions with exact power-law behavior, can be found in terms of its u_j coordinates denoted by $(a_1, \dots, a_m)$:

$$\begin{cases} a_{j+1} = (1+\gamma j) a_1 a_j = a_1^{j+1} \prod\limits_{l=1}^{j} (1+\gamma l), \quad j \in \{1, \dots, m-1\}, \\ a_1 = \Big(\prod\limits_{l=1}^{m} (1+\gamma l) \Big)^{-1/n}. \end{cases} \tag{2.4}$$

Instead of system (2.3) it is more convenient for our current purposes to use another one obtained by the substitution $\tau = a_1 t$, $u_j = a_j v_j$, $j \in \{1, \dots, m\}$:

$$\begin{cases} \dfrac{dv_1}{d\tau} = (1+\gamma)(v_2 - v_1^2), \\ \dfrac{dv_j}{d\tau} = (1+\gamma j)(v_{j+1} - v_1 v_j), \quad j \in \{2, \dots, m-1\}, \\ \dfrac{dv_m}{d\tau} = (1+\gamma m)(1 - v_1 v_m). \end{cases} \tag{2.5}$$

The above equilibrium point has in the new chart all coordinates equal to 1.

Lemma 2.1. *There exist $\gamma_1 > 0$ and $r > 0$ such that for any real $\gamma \in [0, \gamma_1]$, the Jacobian matrix of system* (2.5) *at the point* $(1, \dots, 1)$ *has m different eigenvalues with real parts less than $-r$.*

Proof. First, consider the mentioned Jacobian $m \times m$ matrix for $\gamma = 0$:

$$\begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 0 \\ -1 & -1 & 1 & \dots & 0 & 0 \\ -1 & 0 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & 0 & \dots & -1 & 1 \\ -1 & 0 & 0 & \dots & 0 & -1 \end{pmatrix}.$$

We prove by mathematical induction that its characteristic polynomial is equal to

$$P_m(\lambda) = \frac{(1+\lambda)^{m+1} - 1}{(-1)^m \lambda}. \tag{2.6}$$

For $m = 1$ this is proved immediately:

$$P_1(\lambda) = -2 - \lambda = -\frac{(1+\lambda)^2 - 1}{\lambda} = \frac{(1+\lambda)^{1+1} - 1}{(-1)^1 \lambda}.$$

If (2.6) is proved for some m, then $P_{m+1}(\lambda)$ can be calculated by expanding along the last row as follows:

$$\begin{aligned} P_{m+1}(\lambda) &= (-1) \cdot (-1)^m + (-1-\lambda) P_m(\lambda) \\ &= (-1)^{m+1} - (1+\lambda) \cdot \frac{(1+\lambda)^{m+1} - 1}{(-1)^m \lambda} = \frac{(1+\lambda)^{m+2} - 1}{(-1)^{m+1} \lambda}. \end{aligned}$$

Now (2.6) is proved for $m+1$, too.

The roots of the polynomial $P_m(\lambda)$ are equal to

$$\lambda_j = -1 + \cos \frac{2\pi j}{n} + i \sin \frac{2\pi j}{n}, \quad j \in \{1, \dots, m\},$$

with $j = 0$ excluded because of the denominator in (2.6). The real parts of the roots are less then or equal to $-2 \sin^2 \frac{\pi}{n}$. Since all roots of the polynomial are different and therefore simple, they depend continuously on the coefficients of the polynomial. Hence for sufficiently small $\gamma > 0$ the Jacobian matrix of system (2.5) at the point $(1, \dots, 1)$ has all eigenvalues with real part less than $-\sin^2 \frac{\pi}{n}$. □

Lemma 2.2. *If $\gamma = 0$, then any trajectory of system* (2.5) *passing through a point with positive v_j coordinates tends to the equilibrium point* $(1, \dots, 1)$.

Proof. Trajectories of (2.5) passing through a point with positive v_j coordinates correspond to n-positive solutions of equation (1.1). Trajectories of (2.5) with $\gamma = 0$ correspond to solutions of the linear equation $y^{(n)} = y$, which are all known exactly. They are

$$y(x) = C_0\, e^x + \sum_{j=1}^{\lfloor m/2 \rfloor} C_j e^{r_j x} \sin(\omega_j x + \varphi_j) + \widetilde{C} e^{-x}$$

with $r_j = \cos \frac{2\pi j}{n} < 1$, $\omega_j = \sin \frac{2\pi j}{n}$, and arbitrary constants C_j, φ_j, $\widetilde{C}$, though the last one must equal 0 whenever n is odd. Such a solution is n-positive if and only if the constant C_0 is greater than 0. But in this case, all v_j coordinates of the related trajectory, which are equal to $y^{(j)}/y$ whenever $\gamma = 0$, tend to 1. $\square$

Up to the moment, we actually considered, for each $\gamma > 0$, its own dynamical system defined on its own quotient space homeomorphic to the m-dimensional sphere. In what follows, we need one sphere with a γ-parameterized dynamical system having an equilibrium point common for all γ in consideration. Thus, the points $(y_0, y_1, \dots, y_m) \in \mathbb{R} \setminus \{0\}$ obtained while treating solutions of (1.1) with different γ will generate the same point on the sphere S^m if their corresponding coordinates have the same sign and the tuples

$$\left(|y| : \left| \frac{y'}{a_1} \right|^{\frac{1}{1+\gamma}} : \ldots : \left| \frac{y^{(j)}}{a_j} \right|^{\frac{1}{1+\gamma j}} : \ldots : \left| \frac{y^{(m)}}{a_m} \right|^{\frac{1}{1+\gamma m}} \right),$$

if considered as sets of projective coordinates, define the same point in the projective space $\mathbb{R}P^m$. In particular, for points corresponding to n-positive solutions this means that they have the same v_j coordinates in the related charts. Hereafter, the domain consisting of all points with positive v_j coordinates is denoted by S^m_+. The only equilibrium point in S^m_+, which has all v_j coordinates equal to 1, is denoted by v^*.

Lemma 2.3. *There exist $\gamma_2 > 0$ and an open neighborhood U of the point v^* such that for any positive $\gamma < \gamma_2$, any trajectory of the global dynamical system passing through the closure $\overline{U}$ tends to v^*. If such a trajectory does not coincide with v^*, then it passes transversally, at some time, through the boundary ∂U.*

Proof. Now, once more, we choose other local coordinates to describe the dynamical system on S^m_+. First, we use a translation continuous in γ to put the equilibrium point to 0. Then a linear complex transformation also continuous in γ is used to make the linear part of the right-hand side to be a diagonal matrix. If the new complex coordinates are w_j, then our dynamical system can be written as

$$\frac{dw_j}{d\tau} = \lambda_j(\gamma) w_j + q_j(w, \gamma), \ \ j \in \{1, \dots, m\}, \tag{2.7}$$

with some functions $q_j(w, \gamma)$ quadratic in w and continuous in γ. There exists a constant $Q > 0$ such that $|q_j(w, \gamma)|^2 \le Q|w|^2$ for all $j \in \{1, \dots, m\}$, all $w \in \mathbb{C}^m$, and all positive $\gamma \le \gamma_1$, where $|w|^2 = \sum\limits_{j=1}^{m} |w_j|^2$ and the constant γ_1 is taken from Lemma 2.1.

Now consider the quadratic function $|w|^2$ and note that

$$\frac{d|w|^2}{d\tau} = 2 \sum_{j=1}^{m} \operatorname{Re} \big(\lambda_j(\gamma) |w_j|^2 + q_j(w, \gamma) \overline{w}_j \big) < 2|w|^2 \big(-r + Q|w| \big)$$

with the constant $r > 0$ from Lemma 2.1.

Hence $\frac{d\log|w|^2}{d\tau} < -r$ if $|w| < -\frac{r}{2Q}$. Now, the equilibrium point v^* has the neighborhood U defined by the last inequality. For any trajectory passing through $\overline{U}$ we have $\log|w|^2 \to -\infty$ as $t \to \infty$, which means that all such trajectories tend to v^*. Since the function $\log|w|^2$ is defined for all points of $\overline{U} \setminus \{v^*\}$, the above estimate of $\frac{d\log|w|^2}{d\tau}$ proves the last statement of the current lemma. $\square$

To complete the proof of the Theorem 2.1, consider the set difference of the closure $\overline{S^m_+}$ and the neighborhood U from Lemma 2.3. This compact set will be denoted by B. Further, consider the function f defined on B and equal, for each point $b \in B$, to the time needed for the trajectory of the dynamical system with $\gamma = 0$ to reach ∂U starting from b. This time is well-defined due to Lemma 2.2.

By the implicit function theorem, f is a C^1 function. Its derivative along the trajectories with $\gamma = 0$ is equal to -1. Since the dynamical system depends continuously on γ, and B is compact, there exists $\gamma_3 > 0$ such that for all $\gamma \in [0, \gamma_3)$, the derivative of f along all trajectories with such γ is less than to $-\frac{1}{2}$. This means that any trajectory with such γ passing through B must reach ∂U. Hence, due to Lemma 2.3, any trajectory with $\gamma \in [0, \min\{\gamma_2, \gamma_3\})$ starting from any point $b \in S^*_+$ must tend to the equilibrium point v^*, which corresponds to the n-positive solutions of equation (1.1) with exact power-law behavior (1.2). Putting $K = 1 + n\min\{\gamma_2, \gamma_3\}$ we complete the proof of Theorem 2.1. $\square$

References

[1] I. V. Astashova, Asymptotic behavior of solutions of certain nonlinear differential equations. (Russian) *Reports of the extended sessions of a seminar of the I. N. Vekua Institute of Applied Mathematics, Vol. I, no. 3 (Russian) (Tbilisi, 1985)*, 9–11, *Tbilis. Gos. Univ., Tbilisi*, 1985.

[2] I. V. Astashova, Application of dynamical systems to the investigation of the asymptotic properties of solutions of higher-order nonlinear differential equations. (Russian) *Sovrem. Mat. Prilozh.* No. 8 (2003), 3–33; translation in *J. Math. Sci. (N.Y.)* **126** (2005), no. 5, 1361–1391.

[3] I. V. Astashova, Qualitative properties of solutions to quasilinear ordinary differential equations. (Russian) In: Astashova I. V. (ed.) *Qualitative Properties of Solutions to Differential Equations and Related Topics of Spectral Analysis: Scientific Eedition, UNITY-DANA, Moscow*, 2012, 22–290.

[4] I. Astashova, On power and non-power asymptotic behavior of positive solutions to Emden–Fowler type higher-order equations. *Adv. Difference Equ.* **2013**, 2013:220, 15 pp.

[5] I. T. Kiguradze and T. A. Chanturia, Asymptotic properties of solutions of nonautonomous ordinary differential equations. *Mathematics and its Applications (Soviet Series)*, 89. *Kluwer Academic Publishers Group, Dordrecht*, 1993.

[6] V. A. Kozlov, On Kneser solutions of higher order nonlinear ordinary differential equations. *Ark. Mat.* **37** (1999), no. 2, 305–322.