Green–Samoilenko Function and Existence of Integral Sets of Linear Extensions of Differential Equations with Impulses

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We consider the following system of differential equations with impulsive perturbations [7, 9]

$$\frac{d\varphi}{dt} = a(t,\varphi), \quad \frac{dx}{dt} = P(t,\varphi)x + f(t,\varphi), \ \ t \neq \tau_i, \Delta x\big|_{t=\tau_i} = B_i(\varphi)x + I_i(\varphi), \tag{1}$$

where $t \in R$, $x \in R^n$, $\varphi \in \Im^m$, $\Im^m$ is an m-dimensional torus; $a(t,\varphi)$, $f(t,\varphi)$, $P(t,\varphi)$ are continuous (piecewise continuous with first-kind discontinuities at $t = \tau_i$) with respect to t, continuous and 2π-periodic with respect to φ_ν, $\nu = \overline{1,m}$, bounded for all $t \in R$, $\varphi \in \Im^m$ vector and matrix functions, respectively. Functions $B_i(\varphi)$ and $I_i(\varphi)$ are uniformly bounded with respect to $i \in Z$ matrices and vectors, $\det(E + B_i(\varphi)) \neq 0$ for any $\varphi \in \Im^m$. The sequence of the moments of impulsive perturbations $\{\tau_i\}$ is such that $\tau_i \to -\infty$ for $i \to -\infty$ and $\tau_i \to +\infty$ for $\tau_i \to +\infty$. We assume that there exists $\theta > 0$ such that for any $i \in Z$,

$$\tau_{i+1} - \tau_i \geq \theta > 0. \tag{2}$$

Function $a(t,\varphi)$ satisfies the Lipschitz condition with respect to φ and

$$\sup_{t\in R} \big\|a(t,\varphi_1) - a(t,\varphi_2)\big\| \leq l\big\|\varphi_1 - \varphi_2\big\| \tag{3}$$

holds uniformly with respect to $t \in R$. Additionally assume that functions $f(t,\varphi)$ and $I_i(\varphi)$ satisfy the following condition

$$\sup_{t\in R} \max_{\varphi\in\Im_m} \|f(t,\varphi)\| + \sup_{i\in Z} \max_{\varphi\in\Im_m} \|I_i(\varphi)\| = M < \infty.$$

The problems of the existence of bounded solutions and integral sets for the system of the type (1) were considered in [1,2]. The problems of the persistence of integral sets under the perturbations of the right-hand side were considered in [3, 6]. In this paper, analogously to [4, 5, 8], we introduce the notion of Green–Samoilenko function of the problem on integral sets of differential equations with impulses and provide sufficient conditions for the existence of integral sets.

Consider the non-autonomous system of differential equations defined on the surface of the torus $\Im^m$

$$\frac{d\varphi}{dt} = a(t,\varphi) \tag{4}$$

and denote by $\varphi_t(\tau,\varphi)$ a solution of this system satisfying the initial condition $\varphi_\tau(\tau,\varphi) = \varphi$. From the compactness of the phase space of system (4) and the assumptions regarding function $a(t,\varphi)$, for any initial condition $\varphi_\tau(\tau,\varphi) = \varphi$, $\tau \in R$, $\varphi \in \Im^m$ the corresponding solution $\varphi_t(\tau,\varphi)$ exists and can be prolonged to the entire real axis R.

Consider the following non-homogenous system of differential equations with impulsive perturbations

$$\begin{aligned} \frac{dx}{dt} &= P(t,\varphi_t(\tau,\varphi))x + f(t,\varphi_t(\tau,\varphi)), \ \ t \neq \tau_i, \\ \Delta x\big|_{t=\tau_i} &= B_i(\varphi_{\tau_i}(\tau,\varphi))x + I_i(\varphi_{\tau_i}(\tau,\varphi)) \end{aligned} \tag{5}$$

and the corresponding homogeneous system

$$\begin{aligned} \frac{dx}{dt} &= P(t, \varphi_t(\tau, \varphi))x, \ \ t \neq \tau_i, \\ \Delta x\big|_{t=\tau_i} &= B_i(\varphi_{\tau_i}(\tau, \varphi))x, \end{aligned} \tag{6}$$

and denote by $\Omega_s^t(\tau, \varphi)$ the fundamental matrix of (6). Due to continuous dependance of $\varphi_t(\tau, \varphi)$ on parameters $\tau \in R$ and $\varphi \in \Im^m$, the fundamental matrix $\Omega_s^t(\tau, \varphi)$ depends on these parameters also continuously.

Lemma. *For any $t, s, \tau, \sigma \in R$ and $\varphi \in \Im^m$ the following relation holds*

$$\Omega_s^t(\tau, \varphi_\tau(\sigma, \varphi)) = \Omega_s^t(\sigma, \varphi).$$

Let $C(t, \varphi)$ be continuous 2π-periodic with respect to each of the component φ_ν, $\nu = \overline{1, m}$, piecewise continuous with respect to $t \in R$, with first-kind discontinuities at the points $\{\tau_i\}$ matrix function. Denote

$$G(t, s, \varphi) = \begin{cases} \Omega_s^t(t, \varphi)C(s, \varphi_s(t, \varphi)), & s \le t, \\ -\Omega_s^t(t, \varphi)\big[E - C(s, \varphi_s(t, \varphi))\big], & s > t \end{cases} \tag{7}$$

and call $G(t, s, \varphi)$ Green–Samoilenko function of the system

$$\frac{d\varphi}{dt} = a(t, \varphi), \quad \frac{dx}{dt} = P(t, \varphi)x, \ \ t \neq \tau_i, \\ \Delta x\big|_{t=\tau_i} = B_i(\varphi)x,$$

if there exists $K > 0$ such that for all $t, s \in R$, $\varphi \in \Im^m$

$$\int\limits_{-\infty}^{\infty} \|G(t, s, \varphi)\|\, ds + \sum_{i=\infty-}^{+\infty} \|G(t, \tau_i + 0, \varphi)\| \le K. \tag{8}$$

Next, we recall the basic properties of Green–Samoilenko function $G(t, s, \varphi)$. From its definition it follows that Green–Samoilenko function is continuous for all $t, s \in R$, $t \neq s$, $\varphi \in \Im^m$, 2π-periodic with respect to φ_ν, $\nu = \overline{1, m}$, and

$$G(s + 0, s, \varphi) - G(s - 0, s, \varphi) = E.$$

Taking the above lemma into account, we get

$$G(t, s, \varphi_t(\tau, \varphi)) = \begin{cases} \Omega_s^t(t, \varphi)C(s, \varphi_s(\tau, \varphi)), & s \le t, \\ -\Omega_s^t(t, \varphi)\big[E - C(s, \varphi_s(\tau, \varphi))\big], & s > t. \end{cases} \tag{9}$$

For $s = \tau$, we obtain

$$G(t, \tau, \varphi_t(\tau, \varphi)) = \begin{cases} \Omega_\tau^t(t, \varphi)C(\tau, \varphi), & \tau \le t, \\ -\Omega_\tau^t(t, \varphi)[E - C(\tau, \varphi)], & \tau > t. \end{cases}$$

Matrix $G(t, \tau, \varphi_t(\tau, \varphi))$ consists from solutions to the homogeneous system (6) for $t \ge \tau$ and $t < \tau$, respectively.

Consider the relation

$$\int\limits_{-\infty}^{+\infty} G(t, s, \varphi) f(s, \varphi_s(t, \varphi))\, ds + \sum_{i=-\infty}^{+\infty} G(t, \tau_i + 0, \varphi) I_i(\varphi_{\tau_i}(t, \varphi)).$$

From (2) and (8) we get

$$\Big\| \int\limits_{-\infty}^{+\infty} G(t,s,\varphi) f(s,\varphi_s(t,\varphi))\,ds + \sum_{i=-\infty}^{+\infty} G(t,\tau_i+0,\varphi) I_i(\varphi_{\tau_i}(t,\varphi)) \Big\| \\ \le \frac{2K}{\gamma} \sup_{t\in R} \max_{\varphi\in\Im_m} \|f(t,\varphi)\| + \frac{2K}{1-e^{-\gamma\theta}} \sup_{i\in Z} \max_{\varphi\in\Im_m} \|I_i(\varphi)\|.$$

Finally denote

$$u(t,\varphi) = \int\limits_{-\infty}^{+\infty} G(t,s,\varphi) f(s,\varphi_s(t,\varphi))\,ds + \sum_{i=-\infty}^{+\infty} G(t,\tau_i+0,\varphi) I_i(\varphi_{\tau_i}(t,\varphi)). \tag{10}$$

Theorem 1. *Let functions $a(t,\varphi)$, $f(t,\varphi)$, $P(t,\varphi)$ from system* (1) *be continuous with respect to t, continuous and 2π-periodic with respect to φ_ν, $\nu=\overline{1,m}$, bounded for all $t\in R$, $\varphi\in\Im^m$ vector and matrix functions, respectively. Let function $a(t,\varphi)$ satisfy condition* (3)*, functions $B_i(\varphi)$ and $I_i(\varphi)$ be uniformly bounded with respect to i matrices and vectors, $\det(E+B_i(\varphi))\neq 0$ for any $\varphi\in\Im^m$. Let for the sequence of impulsive perturbations $\{\tau_i\}$ estimate* (2) *hold. Let also there exist Green–Samoilenko function $G(t,s,\varphi)$. Then formula* (10) *defines an integral set of system* (1) *and*

$$\sup_{t\in R} \max_{\varphi\in\Im_m} \|u(t,\varphi)\| \le \frac{2K}{\gamma} \sup_{t\in R} \max_{\varphi\in\Im_m} \|f(t,\varphi)\| + \frac{2K}{1-e^{-\gamma\theta}} \sup_{i\in Z} \max_{\varphi\in\Im_m} \|I_i(\varphi)\|. \tag{11}$$

Now assume that the fundamental matrix $\Omega_s^t(\tau,\varphi)$ of system (6) satisfies the estimate

$$\|\Omega_s^t(\tau,\varphi)\| \le Ke^{-\gamma(t-s)} \tag{12}$$

for any $t\ge s\in R$, $\tau\in R$, $\varphi\in\Im^m$ and some $K\ge 1$, $\gamma>0$. In this case there exists Green–Samoilenko function of the following form

$$G(t,s,\varphi) = \begin{cases} \Omega_s^t(t,\varphi), & s<t, \\ 0, & s\ge t. \end{cases} \tag{13}$$

The corresponding integral set of system (1) gets the representation

$$x = u(t,\varphi) = \int\limits_{-\infty}^{t} G(t,s,\varphi) f(s,\varphi_s(t,\varphi))\,ds + \sum_{\tau_i<t} G(t,\tau_i+0,\varphi) I_i(\varphi_{\tau_i}(t,\varphi)), \quad t\in R, \quad \varphi\in\Im^m. \tag{14}$$

Theorem 2. *Let system* (1) *satisfy the condition of Theorem* 1*. Let also the fundamental matrix $\Omega_s^t(\tau,\varphi)$ of system* (6) *satisfy inequality* (12)*. Then system* (1) *has an asymptotically stable integral set* (14) *and this set satisfies the following estimate*

$$\sup_{t\in R} \max_{\varphi\in\Im_m} \|u(t,\varphi)\| \le K_0 \Big[\sup_{t\in R} \max_{\varphi\in\Im_m} \|f(t,\varphi)\| + \sup_{i\in Z} \max_{\varphi\in\Im_m} \|I_i(\varphi)\| \Big],$$

where

$$K_0 = \frac{K}{\gamma} + K \sup_{t\in R} \sum_{\tau_i<t} e^{-\gamma(t-\tau_i)}.$$

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