Stabilization of Integro-Differential CNN Model Arising in Nano-Structures

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1 Introduction

Computational Nanotechnology has become an indispensable tool not only in predicting but also in engineering the properties of multi-functional nano-structured materials. The presence of nano-heterogeneities in these materials affects or disturbs their elastic field at the local and the global scale and thus greatly influences their mechanical properties. In this paper we shall study dynamical behaviour of 2D dynamic coupled problem in multifunctional nano-heterogeneous piezoelectric composites. More in detail, we shall present first modeling of two-dimensional anti-plane (SH) wave propagation problem in piezoelectric anisotropic solids containing nano-holes or nano-inclusions. Nano-heterogeneities are considered in two aspects as wave scatters provoking scattered and diffraction wave fields and also as stress concentrators creating local stress concentrations in the considered solid.There are no numerical results for dynamic behavior of bounded piezoelectric domain with heterogeneities under anti-plane load. Validation is done in [1] for infinite piezoelectric plane with a hole, in [3] for isotropic bounded domain with holes and inclusions and in [2] for piezoelectric plane with nano-hole or nano-inclusion.

In Section 2 we shall reduce the model under consideration to a system of integro-differntial equations (IDE) and we shall discretize it by Cellular Nonlinear/Nanoscale Network (CNN) architecture. Simulations and validation will be provided. Section 3 deals with feedback stabilization of the IDE CNN model together with simulations.

We shall state the model of piezoelectric solid with heterogeneities under time-harmonic anti-plane load. Let $G \in \mathbb{R}^2$ is a bounded piezoelectric domain with a set of inhomogeneities $I = \cup I_k \in G$ (holes, inclusions, nano–holes, nano–inclusions) subjected to time–harmonic load on the boundary ∂G. Note that heterogeneities are of macro size if their diameter is greater than $10^{-6}m$, while heterogeneities are of nano–size if their diameter is less than $10^{-7}m$.

The aim is to find the field in every point of $M = G \setminus I$, I and to its dynamic behaviour. Using the methods of continuum mechanics the problem can be formulated in terms of boundary value problem for a system of 2-nd order differential equations, see [1, Chapter 2],

$$\left|\begin{aligned} &\rho^N \frac{\partial^2 u_3}{\partial t^2} = c_{44}^N \Delta u_3^N + e_{15}^N \Delta u_4^N, \\ &e_{15}^N \Delta u_3^N - \varepsilon_{15}^N \Delta u_4^N = 0, \end{aligned}\right. \tag{1}$$

where $x = (x_1, x_2)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is Laplace operator with respect to t, $N = M$ for $x \in M$ and $N = I$ for $x \in I$; u_3^N is mechanical displacement, u_4^N is electric potential, ρ^N is the mass density, $c_{44}^N > 0$ is the shear stiffness, $e_{15}^N \neq 0$ is the piezoelectric constant and $\varepsilon_{11}^N > 0$ is the dielectric permittivity.

We shall consider the case, when I is a nano-hole or nano-inclusion and boundary conditions on S are

$$t_j^M = \frac{\partial \sigma_{lj}^S}{\partial l} \ \text{ on } S, \ \text{ or } \ \tau_3^I + t_3^M = \frac{\partial \sigma_{l3}^S}{\partial l}\,, \quad \tau_4^I + t_4^M = \frac{\partial \sigma_{l4}^S}{\partial l}\,, \tag{2}$$

where σ_{lj}^S is generalized stress [1], $j = 3, 4$, l is the tangential vector. Then we shall study boundary value problem (BVP) (1) with boundary conditions (2).

2 Integro-differential CNN model

BVP $(1), (2)$ is reduced in [1] to integro-differential equation (IDE) using the Fourier transform and then applying the Gauss theorem [6]. In this paper we shall study the general form of IDE obtained in [1]. Let us consider the following system of IDE:

$$\frac{\partial u(x)}{\partial \tau} = D \frac{\partial^2 u}{\partial x^2} - C_1 \int\limits_S G(u(x))\, dx, \tag{3}$$

where C_1 is a constant depending on the ρ^M, $c_{44}^M > 0$, $e_{15}^M \neq 0$ and $\varepsilon_{11}^M > 0$, D is diffusion coefficient, $u = (u_3, u_4)$, function $G(x)$ is a function of the displacement vectors $u_{3,4}$ and the traction $\tau_{3,4}$.

It is known [5] that some autonomous CNN represent an excellent approximation to nonlinear partial differential equations (PDEs). The intrinsic space distributed topology makes the CNN able to produce real-time solutions of nonlinear PDEs. There are several ways to approximate the Laplacian operator in discrete space by a CNN synaptic law with an appropriate A-template. In our case the CNN model of IDE (3) is:

$$\frac{du_{ij}}{dt} = DA_1 * u_{ij} - C_1 \int\limits_S G(u_{ij})\, dt, \ \ 1 \leq i \leq n, \ \ j = 3, 4, \tag{4}$$

where A_1 is 1-dimensional discretized Laplacian template [5] $A_1 : \ (1, -2, 1)$, $*$ is convolution operator, $n = M \times M$ is the number of cells of the CNN architecture.

We develop the following algorithm for studying the dynamical behavior of CNN model (4) via describing function method [4]:

1. First, we apply double Fourier transform $F(s, z)$ to IDE CNN model (4)

   $$F(s, z) = \sum_{k=-\infty}^{k=\infty} z^{-k} \int\limits_{-\infty}^{\infty} f_k(t) \exp(-st)\, dt$$

   from continuous time t and discrete space k to continuous temporal frequency ω, and continuous spatial frequency Ω such that $z = exp(I\Omega)$, $s = I\omega$, I is the imaginary identity and therefore we obtain:

   $$sU(s, z) = D\big[z^{-1}U(s, z) - 2U(s, z) + zU(s, z)\big] - C_1 s^{-1} G(U(s, z)).$$

2. We express $U(s,z)$ as a function of $G(U(s,z))$:

$$U(s,z) = \frac{C_1}{sD(z^{-1} - 2 + z) - s^2}\, G(U)$$

and obtain the transfer function $H(s,z)$:

$$H(s,z) = \frac{C_1}{sD(z^{-1} - 2 + z) - s^2}\,.$$

According to the describing function technique [4], the transfer function can be expressed in terms of temporal frequency ω and spatial frequency Ω:

$$H_\Omega(\omega) = \frac{C_1}{I\omega D(2\cos\Omega - 2) + \omega^2}\,.$$

3. We are looking for possible periodic solutions of our CNN model (4) in the form:

$$u_{ij}(t) = \xi(i\Omega + \omega t), \quad 1 \le i \le n, \quad j = 3,4,$$

for some function $\xi : \mathbb{R} \to \mathbb{R}$ and for some spatial frequency $0 \le \Omega \le 2\pi$ and temporal frequency $\omega = \frac{2\pi}{T}$, where $T > 0$ is the minimal period.

4. According to the describing function technique [4] the following constraints hold:

$$\begin{aligned} \mathcal{R}(H_\Omega(\omega)) &= \frac{U_m}{Y_m}\,, \\ \mathcal{I}(H_\Omega(\omega)) &= 0. \end{aligned} \tag{5}$$

5. Thus (5) give us necessary set of equations for finding the unknowns U_m, Ω and ω. As we mentioned before we are looking for a periodic wave solution of (4), therefore U_m will determine approximate amplitude of the wave, and $T = \frac{2\pi}{\omega}$ will determine the wave speed. Now according to the describing function technique, if for a given value of Ω we can find the unknowns (U_m, ω), then we can predict the existence of a periodic solution of our CNN IDE (4) with an amplitude U_m and period of approximately $T = \frac{2\pi}{\omega}$.

Following the above algorithm the next theorem has been proved:

Theorem 1. *CNN IDE* (4) *of the BVP* (1), (2) *with circular array of* $n = L \times L$ *cells has periodic solutions* $u_{ij}(t)$ *with a finite set of spatial frequencies* $\Omega = \frac{2\pi k}{n}$, $0 \le k \le n-1$ *and a period* $T = \frac{2\pi}{\omega}$.

Let us consider the square domain of piezoelectric solid $G_1G_2G_3G_4$ with a side a. For heterogeneities at nano–scale we have: the side of the square is $a = 10^{-7}m$; material parameters inside I for hole are 0; material parameters on $S = \partial I$ for hole and for an inclusion are: $c_{44}^S = 0.1\, c_{44}^M$, $e_{15}^S = 0.1\, e_{15}^M$, $\varepsilon_{11}^S = 0.1\, \varepsilon_{11}^M$, $\rho^S = \rho^M$.

Then simulating our CNN IDE model (4) we obtain the following periodic wave solutions (see Figure 1).

The simulations of IDE CNN model are obtained by simulation system MATCNN applying 4th- order Runge–Kutta integration. In order to minimize the computational complexity and to maximize the significance of the mean square error only outputs of 4 cells are taken into account.

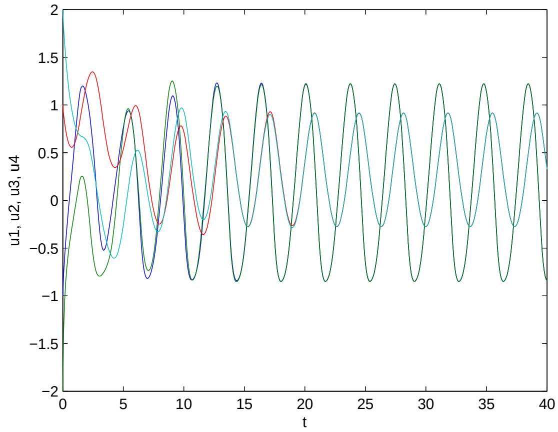

Figure 1. Simulation of IDE CNN model (4) with 4 cells

3 Stabilizing feedback control for IDE CNN model

Let us extend the IDE CNN model (4) by adding to each cell the local linear feedback:

$$\frac{du_{ij}}{dt} = D(u_{i-1j} - 2u_{ij} + u_{i+1j}) - C_1 \int\limits_S G(u_{ij})\,dt - ku_{ij}, \tag{6}$$

where k is the feedback controls coefficient which is assumed to be equal for all cells. The problem is to prove that this simple and available for the implementation feedback can stabilize the IDE CNN model (4). In the following we present a proof of this statement and give sufficient condition on the feedback coefficient values which provide stability of the CNN nonlinear model (6). The following theorem holds:

Theorem 2. *Let the parameters of IDE CNN system and feedback coefficient* k (6) *have positive values. Then its linearized model is asymptotically stable for all* $k > 0$.

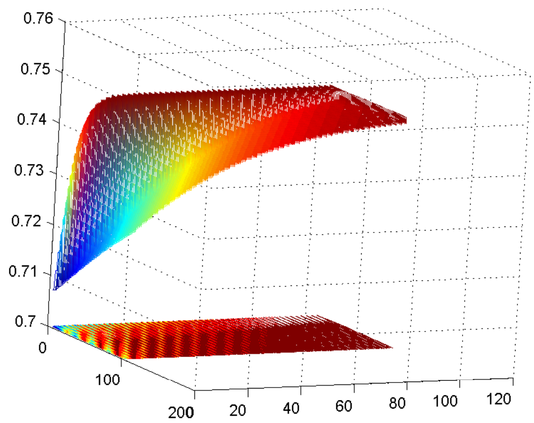

Figure 2. Simulation of stabilized IDE CNN model (6)

Proof. Define the quadratic Lyapunov function candidate $L(z) = \frac{1}{2}\, z^T z$. Then its derivative along the linearized control IDE CNN is $\frac{dL(z)}{dt} = \frac{1}{2}\, z^T (J^T(k) + J(k))z = -z^T Q(k)z$. Therefore, $\frac{dL(z)}{dt} < 0$ implies a positive definiteness of $Q(k)$. It can be shown that $Q(k)$ positive definiteness implies $k > 0$. For verification of the above statement the eigenvalues of $J(k)$ were calculated related on the values of feedback coefficient k. Stability of the linear system requires that the eigenvalues λ_j^i, $i = 1, \dots, 4$ satisfy the inequality $\max\limits_i \operatorname{Re} \lambda_j^i < 0$.

Simulations of the stabilized IDE CNN are in Figure 2. $\square$

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References

[1] P. Dineva, D. Gross, R. Müller, and T. Rangelov, Dynamic fracture of piezoelectric materials. *Solution of time-harmonic problems via BIEM. Solid Mechanics and its Applications*, 212. *Springer, Cham*, 2014.

[2] X.-Q. Fanga, J.-X. Liua, Li-H. Doub, and M.-Zh. Chena, Dynamic strength around two interacting piezoelectric nano-fibers with surfaces/interfaces in solid under electro-elastic waves. *Thin Solid Films* **520** (2012), Issue 9, 3587-3592.

[3] M. Jammes, S. G. Mogilevskaya, and S. L. Crouch, Multiple circular nano-inhomogeneities and/or nano-pores in one of two joined isotropic elastic half-planes. *Eng. Anal. Bound. Elem.* **33** (2009), no. 2, 233–248.

[4] A. I. Mees, Dynamics of feedback systems. *A Wiley-Interscience Publication. John Wiley & Sons, Ltd., Chichester*, 1981.

[5] A. Slavova, Cellular neural networks: dynamics and modelling. *Mathematical Modelling: Theory and Applications*, 16. *Kluwer Academic Publishers, Dordrecht*, 2003.

[6] V. S. Vladimirov, Equations of mathematical physics. (Russian) *Izdat. "Nauka", Moscow*, 1971.