On the Representations of Sensitivity Coefficients for Nonlinear Delay Functional Differential Equations with the Discontinuous Initial Condition

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Let $\mathbb{R}^n$ be the n-dimensional vector space; suppose that $O \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^r$ are open sets. Let $0 < \tau_1 < \tau_2$ and $a < t_{01} < t_{02} < t_1 < b$ be given numbers with $t_{02} + \tau_2 < t_1$; let the n-dimensional function $f(t,x,y,u)$ be continuous on $I \times O^2 \times U$ and continuously differentiable with respect to (x,y,u), where $I = [a,b]$. Furthermore, Φ is the set of continuous initial functions $\varphi : I_1 \to O$, where $I_1 = [\widehat{\tau}, b], \widehat{\tau} = a - \tau_2$ and Ω is the set of measurable control functions $u : I \to U$ with $\operatorname{cl} u(I)$ is compact set and $\operatorname{cl} u(I) \subset U$.

To each initial data $\mu = (t_0, \tau, x_0, \varphi(t), u(t)) \in \Lambda = (t_{01}, t_{02}) \times (\tau_1, \tau_2) \times O \times \Phi \times \Omega$ we assign the delay functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau), u(t)\big) \tag{1}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0), \ \ x(t_0) = x_0. \tag{2}$$

Definition 1. Let $\mu = (t_0, \tau, x_0, \varphi(t), u(t)) \in \Lambda$. A function $x(t) = x(t;\mu) \in O$, $t \in [\widehat{\tau}, t_1]$ is called a solution of equation (1) with the initial condition (2) or a solution corresponding to μ and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Let $\mu_0 = (t_{00}, \tau_0, x_{00}, \varphi_0(t), u_0(t)) \in \Lambda$ be a fixed initial data. Introduce the following notations: $\delta\mu = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi(t), \delta u(t)) \in \Lambda - \mu_0 = \{\delta\mu = \mu - \mu_0 : \ \mu \in \Lambda\}$, $\delta\mu$ is called variation of the initial data μ_0 and $\Lambda - \mu_0$ is called the set of variations. Next,

$$\|\delta\mu\| = |\delta t_0| + |\delta\tau| + |\delta x_0| + \|\delta\varphi\| + \|\delta u\|,$$

where

$$\|\delta\varphi\| = \sup\big\{|\delta\varphi(t)| : \ t \in I_1\big\}, \quad \|\delta u\| = \sup\big\{|\delta u(t)| : \ t \in I\big\}.$$

Let the solution $x(t;\mu_0)$ is defined on $[\widehat{\tau}, t_1]$. Then there exists number $\varepsilon_1 > 0$ such that for any $\delta\mu \in \Lambda_{\varepsilon_1} = \{\delta\mu \in \Lambda - \mu_0 : \ \|\delta\mu\| \le \varepsilon_1\}$ there exists solution $x(t;\mu_0 + \delta\mu)$ defined on the interval $[\widehat{\tau}, t_1]$, [1].

Theorem 1. *Let the solution $x(t;\mu_0)$ be defined on $[\widehat{\tau}, t_1]$ and let the function $\varphi_0(t)$ be absolutely continuous. Moreover, let there exist the finite limits*

$$\lim_{w \to w_0} f(w, u_0(t)) = f_0^-, \ \ w = (t,x,y) \in (t_{01}, t_{00}] \times O^2$$

and

$$\lim_{(w_1,w_2)\to(w_{11},w_{12})}\big[f(w_1,u_0(t))-f(w_2,u_0(t))\big]=f_1^-,\ \ w_1,w_2\in(t_{00},t_{00}+\tau_0]\times O^2,$$

where

$$w_0=\big(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_0)\big),$$
$$w_{11}=\big(t_{00}+\tau_0,x_0(t_{00}+\tau_0),x_{00}\big),\ \ w_{12}=\big(t_{00}+\tau_0,x_0(t_{00}+\tau_0),\varphi_0(t_{00})\big).$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta>0$ such that on the interval $[t_1-\delta,t_1]\subset(t_{00}+\tau_0,t_1]$ for arbitrary $\delta\mu\in\Lambda^-_{\varepsilon_2}=\{\delta\mu\in\Lambda_{\varepsilon_2}:\ \delta t_0\le 0,\ \delta\tau\le 0\}$ we have

$$x(t;\mu_0+\delta\mu)=x(t;\mu_0)+\delta x^-(t;\delta\mu)+o(t;\delta\mu),$$

where

$$\delta x^-(t;\delta\mu)=Y(t_{00};t)\delta x_0-\big[Y(t_{00};t)f_0^-+Y(t_{00}+\tau_0;t)f_1^-\big]\delta t_0-Y(t_{00}+\tau_0;t)f_1^-\delta\tau+\gamma(t;\delta\mu)$$

and

$$\gamma(t;\delta\mu)=Y(t_{00};t)\delta x_0-\bigg[\int\limits_{t_{00}}^{t}Y(s;t)f_{0y}[s]\dot x_0(s-\tau_0)\,ds\bigg]\delta\tau+ \\ +\int\limits_{t_{00}-\tau_0}^{t_{00}}Y(s+\tau_0;t)f_{0y}[s+\tau_0]\delta\varphi(s)\,ds+\int\limits_{t_{00}}^{t}Y(s;t)f_{0u}[s]\delta u(s)\,ds. \quad (3)$$

Here

$$\dot x_0(t)=\dot\varphi_0(t),\ \ t\in(t_{00}-\tau_0,t_{00}),\quad f_{0y}[s]=f_y\big(s,x_0(s),x_0(s-\tau_0),u_0(s)\big);$$

$Y(\xi;t)$ is the $n\times n$-matrix function satisfying the linear functional differential equation with advanced argument

$$Y_s(s;t)=-Y(s;t)f_{0x}[s]-Y(s+\tau_0;t)f_{0y}[s+\tau_0],\ \ s\in[t_{00},t]$$

and the condition

$$Y(s;t)=\begin{cases}E & \text{for } s=t,\\ \Theta & \text{for } s>t,\end{cases}$$

E is the identity matrix and Θ is the zero matrix.

Theorem 2. *Let the solution $x(t;\mu_0)$ be defined on $[\widehat\tau,t_1]$ and let the function $\varphi_0(t)$ be absolutely continuous. Moreover, let there exist the finite limits*

$$\lim_{w\to w_0}f(w,u_0(t))=f_0^+,\ \ w\in[t_{00},t_{00}+\tau_0)\times O^2$$

and

$$\lim_{(w_1,w_2)\to(w_{11},w_{12})}\big[f(w_1,u_0(t))-f(w_2,u_0(t))\big]=f_1^+,\ \ w_1,w_2\in[t_{00}+\tau_0,t_1)\times O^2.$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta>0$ such that on the interval $[t_1-\delta,t_1]$ for arbitrary $\delta\mu\in\Lambda^+_{\varepsilon_2}=\{\delta\mu\in\Lambda_{\varepsilon_2}:\ \delta t_0\ge 0,\ \delta\tau\ge 0\}$ we have

$$x(t;\mu_0+\delta\mu)=x(t;\mu_0)+\delta x^+(t;\delta\mu)+o(t;\delta\mu),$$

where

$$\delta x^+(t;\delta\mu)=-\big[Y(t_{00};t)f_0^++Y(t_{00}+\tau_0;t)f_1^+\big]\delta t_0-Y(t_{00}+\tau_0;t)f_1^+\delta\tau+\gamma(t;\delta\mu).$$

Theorem 3. *Let the assumptions of Theorems 1 and 2 be fulfilled. Moreover, let*

$$f_0^- = f_0^+ := f_0, \ \ f_1^- = f_1^+ := f_1.$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta > 0$ such that on the interval $[t_1 - \delta, t_1]$ for arbitrary $\delta\mu \in \Lambda_{\varepsilon_2}$ we have

$$x(t; \mu_0 + \delta\mu) = x(t; \mu_0) + \delta x(t; \delta\mu) + o(t; \delta\mu), \tag{4}$$

where

$$\delta x(t; \delta\mu) = -\big[Y(t_{00}; t)f_0 + Y(t_{00} + \tau_0; t)f_1\big]\delta t_0 - Y(t_{00} + \tau_0; t)f_1\delta\tau + \gamma(t; \delta\mu). \tag{5}$$

Some Comments

Theorems 1 and 2 correspond to cases when the variations at the point t_{00} are performed on the left and on the right, respectively. Theorem 3 corresponds to the case when at the point t_{00} two-sided variation is performed. The function $\delta x(t; \delta\mu)$ in the formula (4) is called the coefficient of sensitivity. The expression (5) is called representation of the sensitivity coefficient. The summand

$$-\bigg[Y(t_{00} + \tau_0; t)f_1 + \int_{t_{00}}^{t} Y(s : t)f_{0y}[s]\dot{x}_0(s - \tau_0)\, ds\bigg]\delta\tau$$

in formula (5) (see (3)) is the effect of perturbation of the delay τ_0.The expression

$$-\big[Y(t_{00}; t)f_0 + Y(t_{00} + \tau_0; t)f_1\big]\delta t_0$$

in formula (5) (see again (3)) is the effect of the discontinuous initial condition (2) and perturbation of the initial moment t_{00}. The expression

$$Y(t_{00}; t)\delta x_0 + \int_{t_{00}-\tau_0}^{t_{00}} Y(s + \tau_0; t)f_{0y}[s + \tau_0]\delta\varphi(s)\, ds + \int_{t_{00}}^{t} Y(s; t)f_{0u}[s]\delta u(s)\, ds$$

in formula (5) (see (3)) is the effect of perturbations of initial vector x_{00}, initial function $\varphi_0(t)$ and control function $u_0(t)$. It is clear that (5) can be rewrite in the form

$$\delta x(t; \delta\mu) = \delta x_1(t; \delta\mu) + \delta x_2(t; \delta\mu),$$

where

$$\begin{aligned}\delta x_1(t; \delta\mu) &= Y(t_{00}; t)[\delta x_0 - f_0\delta t_0] + \\ &+ \int_{t_{00}-\tau_0}^{t_{00}} Y(s + \tau_0; t)f_{0y}[s + \tau_0]\delta\varphi(s)\, ds + \int_{t_{00}}^{t} Y(s; t)\big[f_{0u}[s]\delta u(s) - f_{0y}[s]\dot{x}_0(s - \tau_0)\delta\tau\big]\, ds\end{aligned}$$

and

$$\delta x_2(t; \delta\mu) = -Y(t_{00} + \tau_0; t)f_1[\delta\tau + \delta t_0].$$

On the basis of the Cauchy formula on representation of solutions of the linear delay functional differential equation we get that the function $\delta x_1(t; \delta\mu)$ on the interval $[t_{00}, t_1]$ satisfies the equation

$$\dot{\delta x}(t) = f_{0x}[t]\delta x(t) + f_{0y}[t]\delta x(t - \tau_0) - f_{0y}[t]\dot{x}_0(t - \tau_0)\delta\tau + f_{0u}[t]\delta u(t)$$

with the initial condition

$$\delta x(t) = \delta\varphi(t), \ \ t \in [\widehat{\tau}, t_{00}), \ \ \delta x(t_{00}) = \delta x_0 - f_0\delta t_0,$$

and $\delta x_2(t;\delta\mu)$ on the interval $[t_{00}+\tau_0, t_1]$ satisfies the equation

$$\dot{\delta} x(t) = f_{0x}[t]\delta x(t) + f_{0y}[t]\delta x(t-\tau_0)$$

with the initial condition

$$\delta x(t) = 0, t \in [t_{00}, t_{00}+\tau_0), \delta x(t_{00}+\tau_0) = -f_1(\delta\tau + \delta t_0).$$

Thus, if $\delta x_1(t;\delta\mu)$ and $\delta x_2(t;\delta\mu)$ are solutions of the above considered linear differential equations with the corresponding initial conditions, then their sum will the coefficient of sensitivity on the interval $[t_1-\delta, t_1]$. Sensitivity analysis for various classes of functional differential equations are considered in [2–4].

References

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