On Solution of the Initial Value Problem for One Neutral Stochastic Differential Equation of Reaction-Diffusion Type in Hilbert Space

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The paper is devoted to the Cauchy problem for one equation of reaction-diffusion type – for a neutral stochastic integro-differential equation in Hilbert space $H = L_2^\rho(\mathbb{R}^d)$ (the space with an inner product $(f,g)_{L_2^\rho(\mathbb{R}^d)} = \int\limits_{\mathbb{R}^d} f(x)g(x)\rho(x)\,dx$ and a corresponding norm $\|f\|_{L_2^\rho(\mathbb{R}^d)} = \sqrt{\int\limits_{\mathbb{R}^d} \|f(x)\|^2\rho(x)\,dx}$) of the form

$$\begin{gathered} d\bigg(u(t,x) + \int\limits_{\mathbb{R}^d} b(t,x,\xi)u(\alpha(t))\,d\xi\bigg) = \\ = \big(\Delta_x u(t,x) + f(t,u(\alpha(t)),x)\big)\,dt + \sigma\big(t,u(\alpha(t)),x\big)\,dW(t,x),\ \ 0 < t \le T,\ \ x \in \mathbb{R}^d, \\ u(t,x) = \phi(t,x),\ \ -r \le t \le 0,\ \ x \in \mathbb{R}^d,\ \ r > 0, \end{gathered} \tag{1}$$

namely, to the investigation of existence and uniqueness of its solution. Here, $d \in \{1,2,\dots\}$ – an arbitrary positive integer, $T > 0$ – a fixed real, $\Delta_x = \sum\limits_{j=1}^{d} \partial_{x_j}^2$, $d \in \{1,2,\dots\}$, – d-measurable Laplacian, $\partial_{x_j}^2 \equiv \frac{\partial^2}{\partial x_j^2}$, $j \in \{1,\dots,d\}$, $W(t,x)$ – $L_2^\rho(\mathbb{R}^d)$-valued Wiener process, $\{f,\sigma\}\colon [0,T]\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ and $b\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$ – some given functions, $\phi\colon [-r,0]\times\mathbb{R}^d\times\Omega \to \mathbb{R}$ – an initial data and $\alpha\colon [0,T] \to [-r,\infty)$ – a delay function to be specified later. It is known [5, p. 242–244] that Δ_x is an (infinitesimal) generator of an analytic (C_0-)semigroup of operators $\{S(t),\ t \ge 0\}$ that generates the solution $u(t,x) = (S(t)g(\,\cdot\,))(x) = \int\limits_{\mathbb{R}^d} \mathscr{K}(t,x-\xi)g(\,\cdot\,)\,d\xi$ of a homogenous Cauchy problem for a heat-equation

$$\begin{gathered} \partial_t u(t,x) = \Delta_x u(t,x),\ \ t > 0,\ \ x \in \mathbb{R}^d, \\ u(0,x) = g(x),\ \ x \in \mathbb{R}^d. \end{gathered} \tag{2}$$

Throughout the paper we assume the following:

1) $(\Omega,\mathcal{F},\mathbf{P})$ – a complete probability space, equipped with a normal filtration $\{\mathcal{F}_t,\ t \ge 0\}$ that generates $L_2^\rho(\mathbb{R}^d)$-valued nuclear Q-Wiener process $W(t,x) = \sum\limits_{n=1}^{\infty} \sqrt{\lambda_n}\,e_n(x)\beta_n(t)$, where $\{\beta_n(t),\ n \in \{1,2,\dots\}\}$ – one-dimensional independent Brownian motions, $\lambda_n > 0$, $n \in$

$\{1,2,\dots\}$, and $\sum\limits_{n=1}^{\infty}\lambda_n<\infty$, $\{e_n(x),\ n\in\{1,2,\dots\}\}$ – a complete orthonormal system in $L_2^{\rho}(\mathbb{R}^d)$ such that $\sup\limits_{n\in\{1,2,\dots\}}\ \operatorname*{ess\,sup}\limits_{x\in\mathbb{R}^d}\|e_n(x)\|\le 1$;

2) $\alpha\colon[0,T]\to[-r,\infty)$ – an increasing continuously differentiable function such that $\alpha(t^*)=0$, $0<\alpha'(t)\le 1$ and $\frac{1}{\alpha'(t)}\le c$, $c>0$;

3) $\{f,\sigma\}\colon[0,T]\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$, $b\colon[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ – measurable in all their arguments functions;

4) an initial data function $\phi\colon[-r,0]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ is $\mathcal{F}_0$-measurable and such that $\mathbf{E}\sup\limits_{-r\le t\le 0}\|\phi(t)\|^2_{L_2^{\rho}(\mathbb{R}^d)}<\infty$;

5) for $\rho\in L_1(\mathbb{R}^d)$ there exists $C_\rho(T)>0$ such that $\int\limits_{\mathbb{R}^d}\mathscr{K}(t,x-\xi)\rho(\xi)\,d\xi\le C_\rho(T)\rho(x)$, $0\le t\le T$, $x\in\mathbb{R}^d$.

Our result-theorem is devoted to the existence and uniqueness for $0\le t\le T$ of so-called ***mild solution*** of (1), defined below, in the space $\mathfrak{B}_{2,T,\rho}$. Here $\mathfrak{B}_{2,T,\rho}$ is the Banach space of all $L_2^{\rho}(\mathbb{R}^d)$-valued $\mathcal{F}_t$-measurable for almost all $0\le t\le T$ stochastic random processes $\Phi\colon[0,T]\times\Omega\to L_2^{\rho}(\mathbb{R}^d)$ that are continuous in t for almost all $\omega\in\Omega$, with the norm $\|\Phi\|_{\mathfrak{B}_{2,T,\rho}}=\sqrt{\mathbf{E}\sup\limits_{0\le t\le T}\|\Phi(t)\|^2_{L_2^{\rho}(\mathbb{R}^d)}}$.

Definition. A continuous stochastic random process $u\colon[-r,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ is called a ***mild solution*** of (1) if it

1) is $\mathcal{F}_t$-measurable for almost all $-r\le t\le T$;

2) satisfies an integral equation of the form

$$
\begin{aligned}
u(t,x)=\int\limits_{\mathbb{R}^d}\mathscr{K}(t,x-\xi)\Big(\phi(0)+\int\limits_{\mathbb{R}^d}b(0,\xi,\zeta)\phi(-r)\,d\zeta\Big)\,d\xi-\int\limits_{\mathbb{R}^d}b(t,x,\xi)u(\alpha(t))\,d\xi-\\
-\int\limits_0^t\Big(\Delta_x\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)\Big(\int\limits_{\mathbb{R}^d}b(s,\xi,\zeta)u(\alpha(s))\,d\zeta\Big)\,d\xi\Big)\,ds+\\
+\int\limits_0^t\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)f(s,u(\alpha(s)),\xi)\,d\xi\,ds+\\
+\int\limits_0^t\sum_{n=1}^{\infty}\sqrt{\lambda_n}\Big(\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)\sigma(s,u(\alpha(s)),\xi)e_n(\xi)\,d\xi\Big)\,d\beta_n(s),\\
0\le t\le T,\ \ x\in\mathbb{R}^d,
\end{aligned}
\tag{3}
$$

$$
u(t,x)=\phi(t,x),\ \ -r\le t\le 0,\ \ x\in\mathbb{R}^d,\ \ r>0;
\tag{4}
$$

3) satisfies the condition $\mathbf{E}\int\limits_0^T\|u(t)\|^2_{L_2^{\rho}(\mathbb{R}^d)}\,dt<\infty$.

Theorem (existence and uniqueness of a mild solution in the space $\mathfrak{B}_{2,T,\rho}$)**.** *Let there exist $L \geq 0$ such that for $\{f,\sigma\}$ the following conditions of linear-growth and Lipschitz by the second argument are fulfilled:*

$$f^2(t,u,x)+\sigma^2(t,u,x)\leq L^2(1+u^2),$$
$$\big(f(t,u,x)-f(t,v,x)\big)^2+\big(\sigma(t,u,x)-\sigma(t,v,x)\big)^2\leq L^2(u-v)^2,$$
$$0\leq t\leq T,\ \ \{u,v\}\subset\mathbb{R},\ \ x\in\mathbb{R}^d,$$

the function b is such that $\sup\limits_{0\leq t\leq T}\frac{\|b(t,x,\,\cdot\,)\|}{\sqrt{\rho(\,\cdot\,)}}\in L_2(\mathbb{R}^d)$, $0\leq t\leq T$, $x\in\mathbb{R}^d$, and

$$\sup_{0\leq t\leq T}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(t,x,\xi)\|^2}{\rho(\xi)}\,d\xi\bigg)\rho(x)\,dx<\infty,$$

and for $\partial_x b$ there exists a majorizing function $\varphi\colon[0,T]\times\mathbb{R}^d\to[0,\infty)$ such that $\|\partial_x b(t,x,\xi)\|\leq\varphi(t,\xi)$, $0\leq t\leq T$, $\{x,\xi\}\subset\mathbb{R}^d$, with $\sup\limits_{0\leq t\leq T}\frac{\varphi(t,\,\cdot\,)}{\sqrt{\rho(\,\cdot\,)}}\in L_2(\mathbb{R}^d)$. Then the problem (1) *has a unique for $0\leq t\leq T$ mild solution $u\in\mathfrak{B}_{2,T,\rho}$ if*

$$\sup_{0\leq t\leq T}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(t,x,\xi)\|^2}{\rho(\xi)}\,d\xi\bigg)\rho(x)\,dx<\frac14\,.$$

Proof. The method of the proof is taken from [2], where authors have proved uniqueness of a fixed point for a certain operator with the help of the classical theorem of Banach on a contractive mapping. Our goal is to check execution of conditions of this theorem for the operator $\Psi\colon\mathfrak{B}_{2,T,\rho}\to\mathfrak{B}_{2,T,\rho}$, whose action is given by the rule

$$\begin{aligned}\big(\Psi(t)u(\,\cdot\,)\big)(x)=&\int\limits_{\mathbb{R}^d}\mathscr{K}(t,x-\xi)\bigg(\phi(0)+\int\limits_{\mathbb{R}^d}b(0,\xi,\zeta)\phi(-r)\,d\zeta\bigg)\,d\xi-\int\limits_{\mathbb{R}^d}b(t,x,\xi)u(\alpha(t))\,d\xi-\\&-\int\limits_0^t\bigg(\Delta_x\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)\bigg(\int\limits_{\mathbb{R}^d}b(s,\xi,\zeta)u(\alpha(s))\,d\zeta\bigg)\,d\xi\bigg)\,ds+\\&+\int\limits_0^t\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)f(s,u(\alpha(s)),\xi)\,d\xi\,ds+\\&+\int\limits_0^t\sum_{n=1}^{\infty}\sqrt{\lambda_n}\bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)\sigma(s,u(\alpha(s)),\xi)e_n(\xi)\,d\xi\bigg)\,d\beta_n(s)=\\=&\sum_{i=0}^4 I_i(t)(x),\ \ 0\leq t\leq T,\ \ x\in\mathbb{R}^d.\end{aligned}$$

Due to it, we need to prove that $\Psi(u)\in\mathfrak{B}_{2,T,\rho}$ for all $u\in\mathfrak{B}_{2,\mathrm{T},\rho}$ and to find out a condition of contraction. In order to prove the first item, we need to show that $\|I_j(s)\|^2_{\mathfrak{B}_{2,t,\rho}}=\mathbf{E}\sup\limits_{0\leq s\leq t}\|I_j(s)\|^2_{L_2^\rho(\mathbb{R}^d)}$, $j\in\{0,\dots,4\}$: indeed, a chain of computations, involving application of the inequality of Cauchy-Schwartz and the theorem of Fubini, yields

$$\|I_0(s)\|^2_{\mathfrak{B}_{2,t,\rho}}=\mathbf{E}\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(s,x-\xi)\bigg(\phi(0,\xi)+\int\limits_{\mathbb{R}^d}b(0,\xi,\zeta)\phi(-r,\zeta)\,d\zeta\bigg)\,d\xi\bigg)^2\rho(x)\,dx\leq$$

$$\leq 2\mathbf{E}\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\sqrt{\mathscr{K}(s,x-\xi)}\,\sqrt{\mathscr{K}(s,x-\xi)}\phi(0,\xi)\,d\xi\bigg)^2\rho(x)\,dx+$$

$$+2\mathbf{E}\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\sqrt{\mathscr{K}(s,x-\xi)}\,\sqrt{\mathscr{K}(s,x-\xi)}\bigg(\int\limits_{\mathbb{R}^d}b(0,\xi,\zeta)\phi(-r,\zeta)\,d\zeta\bigg)\,d\xi\bigg)^2\rho(x)\,dx\leq$$

$$\leq 2\mathbf{E}\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(s,x-\xi)\rho(\xi)\,d\xi\bigg)\phi^2(0,x)\,dx+$$

$$+2\mathbf{E}\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(s,x-\xi)\rho(\xi)\,d\xi\bigg)\bigg(\int\limits_{\mathbb{R}^d}\|b(0,x,\zeta)\|\|\phi(-r,\zeta)\|\,d\zeta\bigg)^2\,dx\leq$$

$$\leq 2C_\rho(T)\mathbf{E}\int\limits_{\mathbb{R}^d}\phi^2(0,x)\rho(x)\,dx+2C_\rho(T)\mathbf{E}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(0,x,\zeta)\|}{\sqrt{\rho(\zeta)}}\,\|\phi(-r,\zeta)\|\sqrt{\rho(\zeta)}\,d\zeta\bigg)^2\rho(x)\,dx\leq$$

$$\leq 2C_\rho(T)\mathbf{E}\|\phi(0)\|^2_{L_2^\rho(\mathbb{R}^d)}+2C_\rho(T)\bigg(\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(0,x,\zeta)\|^2}{\rho(\zeta)}\,d\zeta\bigg)\rho(x)\,dx\bigg)\mathbf{E}\|\phi(-r)\|^2_{L_2^\rho(\mathbb{R}^d)}<\infty, \tag{5}$$

$$\|I_1(s)\|^2_{\mathfrak{B}_{2,t,\rho}}=\mathbf{E}\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}b(s,x,\xi)u(\alpha(s),\xi)\,d\xi\bigg)^2\rho(x)\,dx\leq$$

$$\leq \mathbf{E}\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\|b(s,x,\xi)\|\|u(\alpha(s),\xi)\|\,d\xi\bigg)^2\rho(x)\,dx=$$

$$=\mathbf{E}\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(s,x,\xi)\|}{\sqrt{\rho(\xi)}}\,\|u(\alpha(s),\xi)\|\sqrt{\rho(\xi)}\,d\xi\bigg)^2\rho(x)\,dx\leq$$

$$\leq\bigg(\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(s,x,\xi)\|^2}{\rho(\xi)}\,d\xi\bigg)\rho(x)\,dx\bigg)\cdot\mathbf{E}\sup_{0\leq s\leq t}\|u(\alpha(s))\|^2_{L_2^\rho(\mathbb{R}^d)}=$$

$$=\bigg(\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(s,x,\xi)\|^2}{\rho(\xi)}\,d\xi\bigg)\rho(x)\,dx\bigg)\times$$

$$\times\bigg(\mathbf{E}\sup_{0\leq s\leq t^*}\|u(\alpha(s))\|^2_{L_2^\rho(\mathbb{R}^d)}+\mathbf{E}\sup_{t^*\leq s\leq t}\|u(\alpha(s))\|^2_{L_2^\rho(\mathbb{R}^d)}\bigg)\leq$$

$$\leq\bigg(\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(s,x,\xi)\|^2}{\rho(\xi)}\,d\xi\bigg)\rho(x)\,dx\bigg)\times$$

$$\times\bigg(\mathbf{E}\sup_{-r\leq s\leq 0}\|\phi(s)\|^2_{L_2^\rho(\mathbb{R}^d)}+\mathbf{E}\sup_{0\leq s\leq \alpha(t)}\|u(s)\|^2_{L_2^\rho(\mathbb{R}^d)}\bigg)\leq$$

$$\leq\bigg(\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(s,x,\xi)\|^2}{\rho(\xi)}\,d\xi\bigg)\rho(x)\,dx\bigg)\times$$

$$\times\bigg(\mathbf{E}\sup_{-r\leq s\leq 0}\|\phi(s)\|^2_{L_2^\rho(\mathbb{R}^d)}+\mathbf{E}\sup_{0\leq s\leq t}\|u(s)\|^2_{L_2^\rho(\mathbb{R}^d)}\bigg)<\infty, \tag{6}$$

$$\|I_2(s)\|^2_{\mathfrak{B}_{2,t,\rho}} =$$

$$= \mathbf{E}\sup_{0\le s\le t}\int\limits_{\mathbb{R}^d}\bigg(\sum_{i=1}^d\int\limits_0^s\bigg(\partial^2_{x_i}\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\bigg(\int\limits_{\mathbb{R}^d}b(\tau,\xi,\zeta)u(\alpha(\tau),\zeta)\,d\zeta\bigg)\,d\xi\bigg)\,d\tau\bigg)^2\rho(x)\,dx \le$$

$$\le d\int\limits_{\mathbb{R}^d}\sum_{i=1}^d\mathbf{E}\sup_{0\le s\le t}\bigg(\int\limits_0^s\bigg(\partial^2_{x_i}\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\bigg(\int\limits_{\mathbb{R}^d}b(\tau,\xi,\zeta)u(\alpha(\tau),\zeta)\,d\zeta\bigg)\,d\xi\bigg)\,d\tau\bigg)^2\rho(x)\,dx =$$

$$= d\int\limits_{\mathbb{R}^d}\sum_{i=1}^d\mathbf{E}\sup_{0\le s\le t}\bigg(\int\limits_0^s\bigg(\bigg(\int\limits_{\mathbb{R}^d}b(\tau,x,\zeta)u(\alpha(\tau),\zeta)\,d\zeta\bigg)\int\limits_{\partial B}\partial_{x_i}\mathscr{K}(s-\tau,x-\xi)\cos(\upsilon,\xi_i)\,dS_\xi +$$

$$+\bigg(\int\limits_{\mathbb{R}^d}b(\tau,x,\zeta)u(\alpha(\tau),\zeta)\,d\zeta\bigg)\int\limits_{\overline{B}}\partial^2_{x_i}\mathscr{K}(s-\tau,x-\xi)\,d\xi +$$

$$+\int\limits_{\mathbb{R}^d}\partial^2_{x_i}\mathscr{K}(s-\tau,x-\xi)\bigg(\int\limits_{\mathbb{R}^d}\big(b(\tau,\xi,\zeta)-b(\tau,x,\zeta)\big)u(\alpha(\tau),\zeta)\,d\zeta\bigg)\,d\xi\bigg)\,d\tau\bigg)^2\rho(x)\,dx =$$

$$= d\int\limits_{\mathbb{R}^d}\sum_{i=1}^d\mathbf{E}\sup_{0\le s\le t}\bigg(\int\limits_0^s\int\limits_{\mathbb{R}^d}\partial^2_{x_i}\mathscr{K}(s-\tau,x-\xi)\bigg(\int\limits_{\mathbb{R}^d}\big(b(\tau,\xi,\zeta)-b(\tau,x,\zeta)\big)u(\alpha(\tau),\zeta)\,d\zeta\bigg)\,d\xi\,d\tau\bigg)^2\rho(x)\,dx \le$$

$$\le d\int\limits_{\mathbb{R}^d}\sum_{i=1}^d\mathbf{E}\sup_{0\le s\le t}\bigg(\int\limits_0^s\int\limits_{\mathbb{R}^d}\frac{C}{(s-\tau)^\mu\|x-\xi\|^{d+2-2\mu}}\times$$

$$\times\bigg(\sup_{y\in\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\big|\partial_y b(\tau,y,\zeta)\|x-\xi\|u(\alpha(\tau),\zeta)\big|\,d\zeta\bigg)\,d\xi\,d\tau\bigg)^2\rho(x)\,dx \le$$

$$\le d\int\limits_{\mathbb{R}^d}\sum_{i=1}^d\mathbf{E}\sup_{0\le s\le t}\bigg(\int\limits_{\mathbb{R}^d}\frac{C}{\|x-\xi\|^{d+1-2\mu}}\times$$

$$\times\bigg(\int\limits_0^s\frac{\int_{\mathbb{R}^d}\frac{\varphi(\tau,\zeta)}{\sqrt{\rho(\zeta)}}\|u(\alpha(\tau),\zeta)\|\sqrt{\rho(\zeta)}\,d\zeta}{(s-\tau)^{\frac{\mu}{2}}}\,\frac{1}{(s-\tau)^{\frac{\mu}{2}}}\,d\tau\bigg)\,d\xi\bigg)^2\rho(x)\,dx \le$$

$$\le d\int\limits_{\mathbb{R}^d}\sum_{i=1}^d\bigg(\int\limits_{\mathbb{R}^d}\frac{d\xi}{\|x-\xi\|^{d+1-2\mu}}\bigg)^2\bigg(\sup_{0\le s\le t}\int\limits_0^s\frac{d\tau}{(s-\tau)^\mu}\bigg)\times$$

$$\times\bigg(\mathbf{E}\sup_{0\le s\le t}\int\limits_0^s\frac{1}{(s-\tau)^\mu}\bigg(\int\limits_{\mathbb{R}^d}\frac{\varphi^2(\tau,\zeta)}{\rho(\zeta)}\,d\zeta\bigg)\|u(\alpha(\tau))\|^2_{L_2^\rho(\mathbb{R}^d)}\,d\tau\bigg)\rho(x)\,dx \le$$

$$\le d\int\limits_{\mathbb{R}^d}\sum_{i=1}^d\bigg(\int\limits_{\mathbb{R}^d}\frac{d\xi}{\|x-\xi\|^{d+1-2\mu}}\bigg)^2\bigg(\sup_{0\le s\le t}\int\limits_0^s\frac{d\tau}{(s-\tau)^\mu}\bigg)^2\times$$

$$\times\bigg(\mathbf{E}\sup_{0\le\tau\le s}\|u(\alpha(\tau))\|^2_{L_2^\rho(\mathbb{R}^d)}\bigg)\bigg(\sup_{0\le\tau\le s}\int\limits_{\mathbb{R}^d}\frac{\varphi^2(\tau,\zeta)}{\rho(\zeta)}\,d\zeta\bigg)\rho(x)\,dx \le$$

$$\le d^2C^2\bigg(\int\limits_{\mathbb{R}^d}\frac{d\xi}{\|x-\xi\|^{d+1-2\mu}}\bigg)^2\frac{t^{2-2\mu}}{(1-\mu)^2}\bigg(\int\limits_{\mathbb{R}^d}\rho(x)\,dx\bigg)\bigg(\sup_{0\le\tau\le t}\int\limits_{\mathbb{R}^d}\frac{\varphi^2(\tau,\zeta)}{\rho(\zeta)}\,d\zeta\bigg)\times$$

$$\times\Big(\mathbf{E}\sup_{-r\le\tau\le 0}\|\phi(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}+\mathbf{E}\sup_{0\le\tau\le t}\|u(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}\Big)<\infty,\ \ \frac{1}{2}<\mu<1,\ \ C>0, \tag{7}$$

$$\|I_3(s)\|^2_{\mathfrak{B}_{2,t,\rho}}=\mathbf{E}\sup_{0\le s\le t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_0^s\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)f(\tau,u(\alpha(\tau),\xi),\xi)\,d\xi\,d\tau\bigg)^2\rho(x)\,dx\le$$

$$\le t\,\mathbf{E}\sup_{0\le s\le t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_0^s\bigg(\int\limits_{\mathbb{R}^d}\sqrt{\mathscr{K}(s-\tau,x-\xi)}\,\sqrt{\mathscr{K}(s-\tau,x-\xi)}\,f\big(\tau,u(\alpha(\tau),\xi),\xi\big)\,d\xi\bigg)^2d\tau\bigg)\rho(x)\,dx\le$$

$$\le t\mathbf{E}\sup_{0\le s\le t}\int\limits_0^s\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\rho(\xi)\,d\xi\bigg)f^2\big(\tau,u(\alpha(\tau),x),x\big)\,dx\,d\tau\le$$

$$\le C_\rho(T)t\mathbf{E}\int\limits_0^t\int\limits_{\mathbb{R}^d}f^2\big(\tau,u(\alpha(\tau),x),x\big)\rho(x)\,dx\,d\tau\le$$

$$\le L^2C_\rho(T)t\bigg(t\int\limits_{\mathbb{R}^d}\rho(x)\,dx+\mathbf{E}\int\limits_0^t\|u(\alpha(\tau))\|^2_{L_2^\rho(\mathbb{R}^d)}\,d\tau\bigg)=$$

$$=L^2C_\rho(T)t\bigg(t\int\limits_{\mathbb{R}^d}\rho(x)\,dx+\mathbf{E}\int\limits_0^{t^*}\big\|u(\alpha(\tau))\big\|^2_{L_2^\rho(\mathbb{R}^d)}\,\frac{1}{\alpha'(\tau)}\,\alpha'(\tau)\,d\tau+$$

$$+\mathbf{E}\int\limits_{t^*}^t\big\|u(\alpha(\tau))\big\|^2_{L_2^\rho(\mathbb{R}^d)}\,\frac{1}{\alpha'(\tau)}\alpha'(\tau)\,d\tau\bigg)\le$$

$$\le L^2C_\rho(T)t\bigg(t\int\limits_{\mathbb{R}^d}\rho(x)\,dx+c\mathbf{E}\int\limits_{-r}^0\|\phi(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}\,d\tau+c\mathbf{E}\int\limits_0^{\alpha(t)}\|u(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}\,d\tau\bigg)\le$$

$$\le L^2C_\rho(T)t^2\bigg(\int\limits_{\mathbb{R}^d}\rho(x)\,dx+c\mathbf{E}\sup_{-r\le\tau\le 0}\|\phi(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}+c\mathbf{E}\sup_{0\le\tau\le t}\|u(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}\bigg)<\infty, \tag{8}$$

$$\|I_4(s)\|^2_{\mathfrak{B}_{2,t,\rho}}=$$

$$=\int\limits_{\mathbb{R}^d}\mathbf{E}\sup_{0\le s\le t}\bigg(\int\limits_0^s\sum_{n=1}^\infty\sqrt{\lambda_n}\bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\sigma(\tau,u(\alpha(\tau),\xi),\xi)e_n(\xi)\,d\xi\bigg)\,d\beta_n(\tau)\bigg)^2\rho(x)\,dx\le$$

$$\le 4\sum_{n=1}^\infty\lambda_n\times$$

$$\times\mathbf{E}\int\limits_0^t\bigg(\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\sqrt{\mathscr{K}(s-\tau,x-\xi)}\,\sqrt{\mathscr{K}(s-\tau,x-\xi)}\sigma(\tau,u(\alpha(\tau),\xi),\xi)e_n(\xi)\,d\xi\bigg)^2\rho(x)\,dx\bigg)\,d\tau\le$$

$$\le 4\sum_{n=1}^\infty\lambda_n\mathbf{E}\int\limits_0^t\bigg(\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\rho(\xi)\,d\xi\bigg)\sigma^2(\tau,u(\alpha(\tau),x),x)e_n^2(x)\,dx\bigg)\,d\tau\le$$

$$\le 4C_\rho(T)\sum_{n=1}^\infty\lambda_n\cdot\mathbf{E}\int\limits_0^t\bigg(\int\limits_{\mathbb{R}^d}\sigma^2(\tau,u(\alpha(\tau),x),x)e_n^2(x)\rho(x)\,dx\bigg)\,d\tau\le$$

$$\leq 4L^2 C_\rho(T)\bigg(\sum_{n=1}^{\infty}\lambda_n\bigg)t\bigg(\int\limits_{\mathbb{R}^d}\rho(x)\,dx + c\mathbf{E}\sup_{-r\leq\tau\leq 0}\|\phi(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)} + c\mathbf{E}\sup_{0\leq\tau\leq t}\|u(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}\bigg) < \infty. \quad (9)$$

Thus estimates (5)–(9) imply that for $u \in \mathfrak{B}_{2,T,\rho}$, $\|\Psi(u)\|^2_{\mathfrak{B}_{2,T,\rho}} \leq 5\sum\limits_{i=0}^{4}\|I_i(t)\|^2_{\mathfrak{B}_{2,T,\rho}} < \infty$, – that is Ψ is well defined. The second step is to prove that the operator under consideration has a unique fixed point. Indeed, taking into account estimates (6)–(9), for any $\{u,v\} \subset \mathfrak{B}_{2,t,\rho}$ we conclude

$$\begin{aligned}
\|\Psi(u)-\Psi(v)\|^2_{\mathfrak{B}_{2,t,\rho}} &= \Big\|\sum_{i=1}^{4} I_i(s)(u) - \sum_{i=1}^{4} I_i(s)(v)\Big\|^2_{\mathfrak{B}_{2,t,\rho}} = \\
&= \Big\|\sum_{i=1}^{4}\big(I_i(s)(u)-I_i(s)(v)\big)\Big\|^2_{\mathfrak{B}_{2,t,\rho}} \leq 4\sum_{i=1}^{4}\big\|I_i(s)(u)-I_i(s)(v)\big\|^2_{\mathfrak{B}_{2,t,\rho}} \leq \\
&\leq 4\bigg(\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(s,x,\xi)\|^2}{\rho(\xi)}\,d\xi\bigg)\rho(x)\,dx + \\
&+ d^2C^2\bigg(\int\limits_{\mathbb{R}^d}\frac{d\xi}{\|x-\xi\|^{d+1-2\mu}}\bigg)^2\frac{t^{2-2\mu}}{(1-\mu)^2}\bigg(\int\limits_{\mathbb{R}^d}\rho(x)\,dx\bigg)\bigg(\sup_{0\leq\tau\leq t}\int\limits_{\mathbb{R}^d}\frac{\varphi^2(\tau,\zeta)}{\rho(\zeta)}\,d\zeta\bigg) + \\
&+ L^2C_\rho(T)ct^2 + 4L^2C_\rho(T)\bigg(\sum_{n=1}^{\infty}\lambda_n\bigg)ct\bigg)\|u-v\|^2_{\mathfrak{B}_{2,t,\rho}} = \gamma(t)\|u-v\|^2_{\mathfrak{B}_{2,t,\rho}}.
\end{aligned}$$

Because of the assumption of the theorem, the first term of γ is less than one. Therefore, by choosing small $0 \leq t_1 \leq T$, we conclude that $0 \leq \gamma(t_1) \leq 1$. It means that Ψ, defined in the Banach space $\mathfrak{B}_{2,t_1,\rho}$, is contractive, and therefore, by the theorem of Banach on a contractive mapping, has a unique fixed point – the solution $u \in \mathfrak{B}_{2,t_1,\rho}$ of the equation $\Psi(u) = u$ that can be obviously presented in the form (3) and satisfies (4), that is a mild solution in $\mathfrak{B}_{2,t_1,\rho}$ of (1) on the interval $[0,t_1]$. This procedure can be repeated finitely many steps on other sufficiently small intervals $[t_1,t_2],[t_2,t_3],\dots,[t_{n-2},t_{n-1}],[t_{n-1},T]$ – components of the entire interval $[0,T]$ – and, as a result, we get the solution as a union of the solutions on these intervals. $\square$

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